Can. J. Math. Vol. **50** (1), 1998 pp. 193–209

INTERTWINING OPERATOR AND *h*-HARMONICS ASSOCIATED WITH REFLECTION GROUPS

YUAN XU

ABSTRACT. We study the intertwining operator and h-harmonics in Dunkl's theory on h-harmonics associated with reflection groups. Based on a biorthogonality between the ordinary harmonics and the action of the intertwining operator V on the harmonics, the main result provides a method to compute the action of the intertwining operator V on polynomials and to construct an orthonormal basis for the space of h-harmonics.

0. **Introduction.** With respect to a family of measures on S^{d-1} that are invariant under a finite reflection group, a theory analogous to spherical harmonics has been developed by Dunkl [2–6] recently. The key ingredient of the theory is a family of commutative differential-difference operators, which play the role of the partial differentials in the classical theory. These Dunkl's operators lead to a structure based on the connection between a Laplacian operator and orthogonality with respect to the group-invariant measure. Among its many applications, this structure offers a way to study orthogonal polynomials on S^{d-1} with respect to a large family of measures. One important tool in the theory is the linear isomorphism on polynomials that intertwines the algebra generated by Dunkl's operators with the algebra of partial differential operators. This intertwining operator allows the transfer of results about ordinary harmonic polynomials to the h-harmonics associated to reflection groups. Closed formula of the intertwining operator is known only in a few cases; to find such a formula is a challenging problem.

The purpose of this paper is to study the relation between the intertwining operator and the h-harmonics. The study is based on a biorthogonality, previously unnoticed, between the ordinary harmonics $S_{n,i}$ and $VS_{n,i}$, the action of the intertwining operators on the ordinary harmonics. This relation allows one to compute the inner product of $VS_{n,i}$ and leads to a method to compute $VS_{n,i}$ in terms of an expansion in $S_{n,i}$. The results offer a simple formula for the action of V on polynomials when an orthonormal basis of the h-harmonic polynomials is known.

The paper is organized as follows. In Section 1 we review the basic definitions and present the preliminary materials. In Section 2 we prove the fundamental formula and study the action of the intertwining operator V on ordinary harmonic polynomials. In Section 3 we consider the action of V when an orthonormal basis of the h-harmonics is known. In Section 4 we discuss examples that illustrate the results.

Received by the editors October 28, 1996; revised March 31, 1997.
Supported by the National Science Foundation under Grant DMS-9500532.
AMS subject classification: 33C50, 33C45.
Key words and phrases: h-harmonics, intertwining operator, reflection group.

https://doi.org/10.4153/CJM-1998-010-9 Published online by Cambridge University Press

1. **Background and preliminary.** For $\mathbf{x},\mathbf{y}\in\mathbb{R}^d$ we let $\langle \mathbf{x},\mathbf{y}\rangle$ denote the usual inner product of $\mathbb{R}^d$ and $|\mathbf{x}| = \langle \mathbf{x},\mathbf{x}\rangle^{1/2}$ the Euclidean norm. Let $B^d = \{\mathbf{x} : |\mathbf{x}| \le 1\}$ be the unit ball in $\mathbb{R}^d$ and let $S^{d-1} = \{\mathbf{x} : |\mathbf{x}| = 1\}$ be the unit sphere in $\mathbb{R}^d$. We denote by $d\omega$ the surface measure on S^{d-1} and write $\omega_{d-1} = \int_{S^{d-1}} d\omega = 2\pi^{d/2}/\Gamma(d/2)$.

For a nonzero vector $\mathbf{v}\in\mathbb{R}^d$ define the reflection $\sigma_{\mathbf{v}}$ by

$$\mathbf{x}\sigma_{\mathbf{v}} := \mathbf{x} - 2(\langle \mathbf{x},\mathbf{v}\rangle/|\mathbf{v}|^2)\mathbf{v}, \quad \mathbf{x}\in\mathbb{R}^d.$$

Suppose that G is a finite reflection group on $\mathbb{R}^d$ with the set $\{\mathbf{v}_i : i = 1,2,\ldots,m\}$ of positive roots; assume that $|\mathbf{v}_i| = |\mathbf{v}_j|$ whenever σ_i is conjugate to σ_j in G, where we write $\sigma_i = \sigma_{\mathbf{v}_i}$, $1\le i\le m$. Then G is a subgroup of the orthogonal group generated by the reflections $\{\sigma_i : 1 \le i \le m\}$. We consider weight functions of the form $h_\alpha^2 d\omega$ on S^{d-1}, where

$$h_\alpha(\mathbf{x}) := \prod_{i=1}^m |\langle \mathbf{x},\mathbf{v}_i\rangle|^{\alpha_i}, \quad \alpha_i \ge 0, \tag{1.1}$$

with $\alpha_i = \alpha_j$ whenever σ_i is conjugate to σ_j in G. Then h_α is a positively homogeneous G-invariant function of degree $|\alpha|_1 = \alpha_1 + \cdots + \alpha_m$. We denote by H_α the normalization constant defined by $H_\alpha^{-1} = \int_{S^{d-1}} h_\alpha^2\, d\omega$. Notice that if $\alpha = 0$, then $h_\alpha = 1$; in particular, $H_0^{-1} = \omega_{d-1}$.

The h-harmonics are orthogonal homogeneous polynomials on S^{d-1} with respect to $h_\alpha^2 d\omega$. The key ingredient of the theory is a family of commuting first-order differential-difference operators, $\mathcal{D}_i$ (Dunkl's operators), defined by

$$\mathcal{D}_i f(\mathbf{x}) := \partial_i + \sum_{j=1}^m \alpha_j \frac{f(\mathbf{x}) - f(\mathbf{x}\sigma_j)}{\langle \mathbf{x},\mathbf{v}_j\rangle}\langle \mathbf{v}_j,\mathbf{e}_i\rangle, \quad 1\le i\le d, \tag{1.2}$$

where ∂_i is the ordinary partial derivative with respect to x_i and $\mathbf{e}_1,\ldots,\mathbf{e}_d$ are the standard unit vectors of $\mathbb{R}^d$. The h-Laplacian, which plays the role similar to that of the ordinary Laplacian, is defined by

$$\Delta_h = \mathcal{D}_1^2 + \cdots + \mathcal{D}_d^2. \tag{1.3}$$

We keep the notation Δ for the ordinary Laplacian. The fundamental relation between the h-Laplacian and the orthogonality is as follows. Let $\mathcal{P}_n^d$ denote the space of homogeneous polynomials of degree n in $\mathbf{x} = (x_1,\ldots,x_d)$. If $P\in\mathcal{P}_n^d$, then

$$\int_{S^{d-1}} PQh_\alpha^2\, d\omega = 0, \quad \forall Q \in \bigcup_{k=0}^{n-1} \mathcal{P}_k^d$$

if and only if $\Delta_h P = 0$. The polynomials P in $\mathcal{P}_n^d$ that satisfy $\Delta_h P = 0$ are called *h-harmonic* polynomials. When $h_\alpha = 1$ the h-harmonics become the ordinary harmonics, which satisfy the classical Laplacian equation $\Delta P = 0$. We denote by

$$\mathcal{H}_n = \mathcal{P}_n^d \cap \ker\Delta \quad \text{and} \quad \mathcal{H}_n^h = \mathcal{P}_n^d \cap \ker\Delta_h,$$

respectively, the space of ordinary harmonic polynomials and the spaces of h-harmonic polynomials of degree n. Here and in the following we take the dimension d as fixed, and we omit the parameter d from the notation of $\mathcal{H}_n$ and $\mathcal{H}_n^h$. We shall also write $\mathcal{P}_n$ in place of $\mathcal{P}_n^d$ whenever there is no danger of confusion. The dimension of $\mathcal{H}_n^h$ is the same as that of $\mathcal{H}_n$, which we denote by $N_n = N(n,d)$; thus,

$$N_n = \dim \mathcal{H}_n^h = \dim \mathcal{P}_n - \dim \mathcal{P}_{n-2} = \binom{n+d-1}{n} - \binom{n+d-3}{n-2}.$$

Analogous to the classical theory, it is shown in [2, p. 37] that there is a decomposition $\mathcal{P}_n = \oplus_{k=0}^{[n/2]} |\mathbf{x}|^{2k} \mathcal{H}_{n-2k}^h$; that is, if $P \in \mathcal{P}_n$, then there is a unique decomposition

$$P(\mathbf{x}) = \sum_{k=0}^{[n/2]} |\mathbf{x}|^{2k} P_{n-2k}(\mathbf{x}), \quad P_{n-2k} \in \mathcal{H}_{n-2k}^h. \tag{1.4}$$

The intertwining operator V is a linear operator uniquely defined by ([5])

$$V\mathcal{P}_n \subset \mathcal{P}_n,\ V1 = 1,\ \mathcal{D}_i V = V\partial_i, \quad 1 \le i \le d. \tag{1.5}$$

Note that $V\mathcal{H}_n \subset \mathcal{H}_n^h$. In particular, if $\{S_{n,1}, \ldots, S_{n,N_n}\}$ is an orthonormal basis of $\mathcal{H}_n$, then $\{VS_{n,1}, \ldots, VS_{n,N_n}\}$ is a basis of $\mathcal{H}_n^h$, although no longer an orthonormal one in general. A closed form of V is known only for $h_\alpha(\mathbf{x}) = |x_1|^{\alpha_1} \cdots |x_d|^{\alpha_d}$, associated with the group $\mathbb{Z}^2 \times \cdots \times \mathbb{Z}^2$ ([13], the case $d = 1$ appeared early in [5]), and for $h_\alpha(\mathbf{x}) = |(x_1 - x_2)(x_2 - x_3)(x_1 - x_3)|^\alpha$, associated with the symmetric group S_3 ([6]). The formula of V in the first case will be given later in Section 4.1. The formula in [6] for S_3 is rather complicated. In general, the problem of finding a closed formula of V is very difficult. In [14], it is shown that one can get rid of V if one takes the integral of Vf with respect to $h_\alpha^2 d\omega$.

Next let us recall a bilinear form on $\mathcal{P}_n$ that plays an important role in the study of the intertwining operator. It is defined by ([5, p. 1220]),

$$[p,q]_h = p(\mathcal{D})q(\mathbf{x}), \quad p,q \in \mathcal{P}_n, \tag{1.6}$$

where $\mathcal{D} = (\mathcal{D}_1, \ldots, \mathcal{D}_d)$ is a tuple of Dunkl's operators and $p(\mathcal{D})$ is acted on $q(\mathbf{x})$. In case $h_\alpha = 1$, the bilinear form is written as $[p,q]$, which is classical (*cf.* [8, p. 139]), and $\mathcal{D}$ in its definition is replaced by $\partial = (\partial_1, \ldots, \partial_d)$. The bilinear form is symmetric; that is, $[p,q]_h = [q,p]_h$. We state one of its important property in the following lemma ([5, Theorem 3.8, p. 1222]).

LEMMA 1.1. *If $p \in \mathcal{P}_n$ and $q \in \mathcal{H}_n^h$, then*

$$[p,q]_h = 2^n(|\alpha|_1 + d/2)_n H_\alpha \int_{S^{d-1}} pq h_\alpha^2 \, d\omega. \tag{1.7}$$

In [5] the formula (1.7) is stated for both p and q in $\mathcal{H}_n^h$, but the proof of the theorem there shows that one of the polynomial only has to be in $\mathcal{P}_n$.

https://doi.org/10.4153/CJM-1998-010-9 Published online by Cambridge University Press

The ordinary harmonics have been under extensive studies because their many distinguished applications (*cf.* [8, 10]). An orthonormal basis of $\mathcal{H}_n$ has been given explicitly. For later use we present the formulae here. Let $C_n^{(\lambda)}$ denote the standard Gegenbauer polynomial of degree n (*cf.* [9, p. 80], where the notation $P_n^{(\lambda)}$ is used). The special cases of $\lambda = 0$ and $\lambda = 1$ correspond to the Chebyshev polynomials of the first and the second kind, usually denoted by T_n and U_n, respectively; these polynomials are defined by $T_n(x) = \cos n\theta$ and $U_n(x) = \sin(n+1)\theta / \sin\theta$, where $x = \cos\theta$ in $[-1, 1]$. For $d = 2$, the ordinary harmonic polynomials are given as

$$Y_n^{(1)}(x_1, x_2) = r^n T_n(x_1/r) \quad \text{and} \quad Y_n^{(2)}(x_1, x_2) = r^n x_2 U_{n-1}(x_1/r), \tag{1.8}$$

where $r = (x_1^2 + x_2^2)^{1/2}$. For $d \ge 2$ and each $n \in \mathbb{N}_0$, an orthonormal basis of $\mathcal{H}_n$ is given by (*cf.* [10, p. 466])

$$Y_{\mathbf{k},n}^{(i)}(\mathbf{x}) = A_{\mathbf{k}}^n \prod_{j=0}^{d-3} |\mathbf{x}_{d-j}|^{k_j - k_{j+1}} C_{k_j - k_{j+1}}^{(k_{j+1} + \frac{d-j-2}{2})}\left(\frac{x_{d-j}}{|\mathbf{x}_{d-j}|}\right) Y_{k_{d-2}}^{(i)}(x_1, x_2) \tag{1.9}$$

where $|\mathbf{x}_{d-j}|^2 = x_1^2 + \cdots + x_{d-j}^2$, $n = k_0 \ge k_1 \ge \cdots \ge k_{d-2}$. The symbol $\mathbf{k}$ here denotes the sequence $\mathbf{k} = (k_1, \ldots, k_{d-2})$, and $A_{\mathbf{k}}^n$ denotes a normalization constant. It is shown in [15] that $Y_{\mathbf{k},n}^{(i)}$, $i = 1, 2$, are related to the orthogonal polynomials with respect to $1/\sqrt{1 - |\mathbf{x}|^2}$ and $\sqrt{1 - |\mathbf{x}|^2}$, respectively, on the unit ball B^{d-1}. We will discuss this connection in Section 3.

2. **Action of intertwining operator on ordinary harmonics.** Throughout this section we assume that an orthonormal basis for the space $\mathcal{H}_n$ of ordinary harmonics of degree n is given by $\{S_{n,1}, \ldots, S_{n,N_n}\}$. For example, we can give an order among the ordinary harmonics in (1.9) and rename them as $S_{n,i}$.

We start with a simple formula which is fundamental to the forthcoming development in the entire paper.

THEOREM 2.1. *For $p \in \mathcal{P}_n$ and $q \in \mathcal{H}_n$,*

$$\int_{S^{d-1}} p(Vq) h_\alpha^2 \, d\omega = E_{\alpha,n} \int_{S^{d-1}} pq \, d\omega. \tag{2.1}$$

where

$$E_{\alpha,n} = E_{\alpha,n}^d = \frac{(d/2)_n H_0}{(|\alpha|_1 + d/2)_n H_\alpha}.$$

PROOF. We use the pairing (1.6) and Lemma 1.1. Since $q \in \mathcal{H}_n$ implies that $Vq \in \mathcal{H}_n^h$, the assumption of Lemma 1.1 is satisfied with Vq in place of q. Hence,

$$\begin{aligned}
2^n(|\alpha|_1 + d/2)_n H_\alpha \int_{S^{d-1}} p(Vq) h_\alpha^2 \, d\omega &= p(\mathcal{D}) Vq(\mathbf{x}) \\
&= Vp(\partial) q(\mathbf{x}) \\
&= p(\partial) q(\mathbf{x}) \\
&= 2^n (d/2)_n H_0 \int_{S^{d-1}} pq \, d\omega
\end{aligned}$$

where the first equality follows from (1.6) and (1.7), the second one follows from the intertwining property of V, the third follows from the fact that $p(\partial)q(\mathbf{x})$ is a constant and $V1 = 1$, and the fourth follows from (1.6) and (1.7) with $\alpha = 0$. ∎

An immediate consequence of the formula (2.1) is the following biorthogonal relation.

COROLLARY 2.2. *Let $\{S_{n,i}\}$ be an orthonormal basis of $\mathcal{H}_n$. Then $\{VS_{n,i}\}$ and $\{S_{n,i}\}$ are biorthogonal. More precisely,*

$$\int_{S^{d-1}} (VS_{n,i})S_{n,j}h_\alpha^2\, d\omega = E_{\alpha,n}\delta_{i,j}, \quad 1 \le i,\ j \le N_n. \tag{2.2}$$

The equation (2.2) follows from (2.1) by setting $p = S_{n,j}$ and $q = S_{n,i}$. It should be pointed out that the biorthogonality is restricted to elements of $\mathcal{H}_n$ and $\mathcal{H}_n^h$ for the same n. In general, $S_{n,i}$ is not orthogonal to $VS_{m,i}$ with respect to $h_\alpha^2 d\omega$ if $m < n$ since $S_{n,i}$ is not an element of $\mathcal{H}_n^h$.

The following analog of the fundamental formula (2.1) turns out to be useful as well.

THEOREM 2.3. *For $p \in \mathcal{P}_n$ and $q^h \in \mathcal{H}_n^h$,*

$$\int_{S^{d-1}} (Vp)q^h h_\alpha^2\, d\omega = E_{\alpha,n} \int_{S^{d-1}} pq^h\, d\omega. \tag{2.3}$$

The proof of (2.3) follows in the same line as that of (2.1), we leave the detail to the reader. We note that the requirement $q^h \in \mathcal{H}_n^h$ is needed in order to use Lemma 1.1.

From the intertwining property of V it follows that $VS_{n,i} \in \mathcal{H}_n^h$ provided that $S_{n,i} \in \mathcal{H}_n$. Hence, the biorthogonality may help us to construct a basis for $\mathcal{H}_n^h$ using $VS_{n,j}$. For that purpose, it is essential to be able to compute the inner product of $VS_{n,i}$ and $VS_{n,j}$. In order to do so we need some notations first.

Let us denote by $\mathbb{S}_n$ and $V\mathbb{S}_n$, respectively, the column vectors defined by

$$\mathbb{S}_n = (S_{n,1}, \ldots, S_{n,N_n})^T \quad \text{and} \quad V\mathbb{S}_n = (VS_{n,1}, \ldots, VS_{n,N_n})^T.$$

The use of vector notation is suggested by the recent study of orthogonal polynomials in several variables (*cf.* [11]). We also define matrices $M_{i,j}$ whose elements are inner products of the ordinary harmonics with respect to $h_\alpha^2 d\omega$,

$$M_{i,j} := \int_{S^{d-1}} \mathbb{S}_i \mathbb{S}_j^T h_\alpha^2\, d\omega,$$

where $\mathbb{S}_j^T$ means the transpose of $\mathbb{S}_j$ so that $\mathbb{S}_i\mathbb{S}_j^T$ is a matrix of $N_i \times N_j$ and the integral is acted entry by entry. Similarly we define

$$M_n^V = \int_{S^{d-1}} V\mathbb{S}_n (V\mathbb{S}_n)^T h_\alpha^2\, d\omega = \left(\int_{S^{d-1}} VS_{n,i} VS_{n,j} h_\alpha^2\, d\omega \right)$$

whose elements are inner products of $VS_{n,i}$. Furthermore, we use $M_{i,j}$ as building blocks to define a square matrix $\mathcal{M}_n$ of size $\sum_{i=0}^{[n/2]} N_{n-2i}$,

$$\mathcal{M}_n = (M_{n-2i,n-2j})_{i,j=0}^{[n/2]}, \quad \text{and} \quad \mathcal{M}_n^{-1} = (\tilde{M}_{n-2i,n-2j})_{i,j=0}^{[n/2]}$$

https://doi.org/10.4153/CJM-1998-010-9 Published online by Cambridge University Press

where $\tilde{M}_{i,j}$ are matrices of the same size as $M_{i,j}$. Evidently, $\mathcal{M}_n$ is symmetric and positive definite; hence it is invertible.

Since $S_{n,i}$, as elements of an orthonormal basis for the ordinary harmonics of degree n, can be written down explicitly as in (1.9), the matrix M_{ij} and $\mathcal{M}_n$ can be considered as known. The matrix M_n^V whose elements are inner products of $VS_{n,i}$ is what we need to figure out. Since $VS_{n,i}$ is homogeneous of degree n, by the unique decomposition of $\mathcal{P}_n$ in terms of $\mathcal{H}_n$ we can write it as

$$VS_{n,i} = \sum_{j=1}^{N_n} a_{ij}^n S_{n,j} + |\mathbf{x}|^2 \sum_{j=1}^{N_{n-2}} a_{ij}^{n-2} S_{n-2,j} + \cdots,$$

where $a_{i,j}^k$ are real numbers. Using the notation $\mathbb{S}_k$ we can write the above expansion in vector-matrix form as

$$V\mathbb{S}_n = A_{n,n}\mathbb{S}_n + A_{n-2,n}|\mathbf{x}|^2\mathbb{S}_{n-2} + \cdots = \sum_{k=0}^{[n/2]} A_{n-2k,n}|\mathbf{x}|^{2k}\mathbb{S}_{n-2k}, \tag{2.4}$$

where $A_{n-2k,n}$ are matrices of the size $N_n \times N_{n-2k}$. The basic formula (2.1) allows us to determine these matrices as follows.

THEOREM 2.4. *The coefficient matrices $A_{n,n-2j}$ in (2.4) are given by*

$$A_{n,n-2j} = E_{\alpha,n}\tilde{M}_{n,n-2j}, \quad 0 \le j \le [n/2]. \tag{2.5}$$

Moreover, $M_n^V = A_{n,n} = E_{\alpha,n}\tilde{M}_{n,n}$.

PROOF. Since $VS_{n,i} \in \mathcal{H}_n^h$ is orthogonal to all polynomials of lower degree with respect to $h_\alpha^2 d\omega$, we have by (2.2) that

$$\int_{S^{d-1}} (VS_{n,i}) S_{n-2k,j} h_\alpha^2 \, d\omega = \delta_{k,0} E_{\alpha,n}$$

for $k \ge 0$. Therefore, multiplying (2.4) by $(\mathbb{S}_{n-2k})^T$ and integrating with respect to $h_\alpha^2 d\omega$, it follows that

$$\delta_{k,0} E_{\alpha,n} = \sum_{j=0}^{[n/2]} A_{n-2j,n} M_{n-2j,n-2k}$$

for $0 \le k \le [n/2]$. We can rewrite these equations as

$$(A_n, A_{n-2,n}, \ldots, A_{n-2[n/2],n})\mathcal{M}_n = (E_{\alpha,n}I, 0, \ldots, 0),$$

from which (2.5) follows upon using $\mathcal{M}_n^{-1}$. Moreover, by the basic formula (2.1),

$$M_n^V = \int_{S^{d-1}} V\mathbb{S}_n (V\mathbb{S}_n)^T h_\alpha^2 \, d\omega = E_{\alpha,n} \int_{S^{d-1}} \mathbb{S}_n (V\mathbb{S}_n)^T \, d\omega.$$

Therefore, using (2.4) to replace $V\mathbb{S}_n$ it follows from the orthogonality of $S_{n,i}$ that $M_n^V = E_{\alpha,n}A_n$. The last statement on A_n is evident from (2.5). ∎

The importance of this result lies in the fact that it shows a way to compute $VS_{n,i}$ and the inner product of $VS_{n,i}$ and $VS_{n,j}$, even though the closed formula of V is not known in general. Once $VS_{n,i}$ is known, an orthonormal basis for $\mathcal{H}_n^h$ can be constructed from them right away. Indeed, since $(M_n^V)^{-1}$ is positive definite, we can define its unique positive definite square root matrix $(M_n^V)^{-1/2}$. Then, we have

https://doi.org/10.4153/CJM-1998-010-9 Published online by Cambridge University Press

THEOREM 2.5. *Let* $\mathbb{S}_n^h := (S_{n,1}^h, \ldots, S_{n,N_n}^h)^T$ *be defined by* $\mathbb{S}_n^h = (M_n^V)^{-1/2} V\mathbb{S}_n$. *Then the homogeneous polynomials* $S_{n,i}^h$ *form an orthonormal basis for* $\mathcal{H}_n^h$.

PROOF. From the definition

$$\int_{S^{d-1}} \mathbb{S}_n^h (\mathbb{S}_n^h)^T h_\alpha^2 \, d\omega = (M_n^V)^{-1/2} \int_{S^{d-1}} V\mathbb{S}_n (V\mathbb{S}_n)^T h_\alpha^2 \, d\omega (M_n^V)^{-1/2}$$
$$= (M_n^V)^{-1/2} M_n^V (M_n^V)^{-1/2} = I,$$

which gives the desired result. ∎

The Theorem 2.4 deals with the action of V on the ordinary harmonic polynomials. In order to understand the action of V on other polynomials, we need to know the action of V on polynomials $|\mathbf{x}|^{2k} S_{n-2k,i}$, according to the unique decomposition (1.4). Using the formula (2.3), one may find the expansion of $V(|\cdot|^{2k}\mathbb{S}_{n-2k})$ in terms of $\mathbb{S}_k$ as in (2.4). In stead of deriving a formula for the coefficients in such an expansion, which will be of little use in actual computation, we will deal with a more practical case in the following section.

Together, Theorems 2.4 and 2.5 offer a way to compute the action of V on the ordinary harmonics and an orthonormal basis for $\mathcal{H}_n^h$. It should be pointed out, however, that the formula (2.5) may not be very useful in practical computation, since it may be difficult to compute the integrals in $M_{i,j}$, not to say the inverse of $\mathcal{M}_n$, for even moderate size of n. For a given reflection group, one may use these formulae to generate, perhaps with the help of a computer, h-harmonics of lower degree.

3. **Intertwining operator and h-harmonics.** In the previous section we write $VS_{n,i}$ in terms of the ordinary harmonics $S_{n,i}$ and use $VS_{n,i}$ to construct an orthonormal basis for h-harmonics. For some weight functions h_α^2, one may be able to find a basis for $\mathcal{H}_n^h$ by some other means. For example, if h_α^2 is also S-symmetric, then an orthonormal basis for $\mathcal{H}_n^h$ may be given in terms of orthonormal polynomials on B^{d-1} (see discussion after Corollary 3.3 below). In such a case, the basic formula (2.1) allows us to write down the action of V on the ordinary harmonics rather easily. We discuss the related formula in this section.

Let us assume that $\{S_{n_1}^h, \ldots, S_{n,N_n}^h\}$ is an orthonormal basis of $\mathcal{H}_n^h$. We also use $\mathbb{S}_n^h$ to denote the column vector with components $S_{n,i}^h$.

THEOREM 3.1. *If* $\{S_{n_1}^h, \ldots, S_{n,N_n}^h\}$ *forms an orthonormal basis of* $\mathcal{H}_n^h$, *then the action of* V *on the ordinary harmonics is given by the formula*

$$V\mathbb{S}_n = M_n^h \mathbb{S}_n^h, \quad \text{where} \quad M_n^h = E_{\alpha,n} \int_{S^{d-1}} \mathbb{S}_n (\mathbb{S}_n^h)^T \, d\omega, \tag{3.1}$$

and M_n^h *is a matrix of size* $N_n \times N_n$. *Furthermore,*

$$(M_n^h)^{-1} = E_{\alpha,n}^{-1} \int_{S^{d-1}} \mathbb{S}_n^h (\mathbb{S}_n)^T h_\alpha^2 \, d\omega. \tag{3.2}$$

https://doi.org/10.4153/CJM-1998-010-9 Published online by Cambridge University Press

PROOF. Since the components of $\mathbb{S}_n^h$ form an orthonormal basis for $\mathcal{H}_n^h$, we can write $VS_{n,i}$ in terms of them. Hence, there exists a matrix M_n^h such that $V\mathbb{S}_n = M_n^h \mathbb{S}_n^h$. The orthonormality of $\mathbb{S}_n^h$ implies that

$$M_n^h = \int_{S^{d-1}} V\mathbb{S}_n (\mathbb{S}_n^h)^T h_\alpha^2 \, d\omega = E_{\alpha,n} \int_{S^{d-1}} \mathbb{S}_n (\mathbb{S}_n^h)^T \, d\omega$$

where the second equality follows from (2.1). Multiplying (3.1) by $\mathbb{S}_n^T$ and integrating with respect to $h_\alpha^2 d\omega$, it follows from the biorthogonality (2.3) that

$$E_{\alpha,n} I = M_n^h \int_{S^{d-1}} \mathbb{S}_n^h (\mathbb{S}_n)^T h_\alpha^2 \, d\omega,$$

from which the desired formula (3.2) follows. ∎

As we mentioned in the end of the previous section, in order to understand the action of V on other polynomials than ordinary harmonics, it is essential to know the action of V on polynomials of the form $|\mathbf{x}|^{2k} S_{n-2k,i}$. If a basis for $\mathcal{H}_n^h$ is known, then this action can be computed rather easily.

THEOREM 3.2. *If $\{S_{n_1}^h, \ldots, S_{n,N_n}^h\}$ forms an orthonormal basis of $\mathcal{H}_n^h$, then*

$$V(|\cdot|^{2k} \mathbb{S}_{n-2k}) = B_{n,n}^k \mathbb{S}_n^h + B_{n-2,n}^k |\mathbf{x}|^2 \mathbb{S}_{n-2}^h + \cdots + B_{n-2k,n}^k |\mathbf{x}|^{2k} \mathbb{S}_{n-2k}^h \tag{3.3}$$

where $B_{n-2i,n}^k$ are matrices of the size $N_{n-2k} \times N_{n-2i}$; moreover, these matrices are given by

$$B_{n-2j,n}^k = \binom{k}{j} \frac{(n-k-j+d/2)_j}{(|\alpha_1| + n - 2j + d/2)_j} E_{\alpha,n-2j} \int_{S^{d-1}} \mathbb{S}_{n-2k} (\mathbb{S}_{n-2j}^h)^T \, d\omega.$$

PROOF. Since $V(|\cdot|^{2k} S_{n-2k,i})$ is a homogeneous polynomial of degree n, it follows from the decomposition (1.4) that there exist matrices $B_{n,n-2j}^k$ such that

$$V(|\cdot|^{2k} \mathbb{S}_{n-2k}) = B_{n,n}^k \mathbb{S}_n^h + B_{n-2,n}^k |\mathbf{x}|^2 \mathbb{S}_{n-2}^h + \cdots = \sum_{j=0}^{[n/2]} B_{n-2j,n}^k |\mathbf{x}|^{2j} \mathbb{S}_{n-2j}^h$$

by the unique decomposition (1.4). We will prove that $B_{n-2j,n}^k$ are given by the stated formulae for $j \le k$ and that $B_{n-2j,n} = 0$ for $j > k$. From the orthogonality of $\mathbb{S}_n^h$ and the formula (2.3), the first coefficient matrix is determined by the formula

$$B_{n,n}^k = \int_{S^{d-1}} V(|\cdot|^{2k} \mathbb{S}_{n-2k}) (\mathbb{S}_n^h)^T h_\alpha^2 \, d\omega = E_{\alpha,n} \int_{S^{d-1}} \mathbb{S}_{n-2k} (\mathbb{S}_n^h)^T \, d\omega \tag{3.4}$$

which is the desired formula for $j = 0$. Moreover, it follows from the orthogonality of $S_{n-2j,l}^h$ that

$$B_{n-2j,n}^k = \int_{S^{d-1}} V(|\cdot|^{2k} \mathbb{S}_{n-2k}) (\mathbb{S}_{n-2j}^h)^T h_\alpha^2 \, d\omega.$$

We note that the formula (2.3) cannot be used to remove the intertwining operator V in the above integral, since $j > 0$ means that the degree of $S_{n-2j,l}^h$ is less than the degree of

$V(|\cdot|^{2k}S_{n-2k,i})$. To evaluate the last integral, let $\mathrm{proj}_{\mathcal{H}_n^h}$ be the projection operator from $\mathcal{P}_n$ to $\mathcal{H}_n^h$. From [2, p. 38] we have that for $g \in \mathcal{P}_n$

$$g(\mathbf{x}) = \sum_{p=0}^{[n/2]} |\mathbf{x}|^{2p} \frac{1}{4^p p!\,(|\alpha|_1 + n - 2p + d/2)_p} \mathrm{proj}_{\mathcal{H}_{n-2p}^h} \Delta_h^p g.$$

Using this formula with $g = V(|\cdot|^{2k}S_{n-2k,i})$ and the orthogonality of $S_{n-2j,l}^h$ with respect to $h_\alpha^2 d\omega$, we obtain that

$$\int_{S^{d-1}} V(|\cdot|^{2k}S_{n-2k,i})S_{n-2j,l}^h h_\alpha^2\, d\omega = \frac{1}{4^j j!\,(|\alpha|_1 + n - 2j + d/2)_j} \int_{S^{d-1}} \mathrm{proj}_{\mathcal{H}_{n-2j}^h}\big(\Delta_h^j V(|\cdot|^{2k}S_{n-2k,i})\big)S_{n-2j,l}^h h_\alpha^2\, d\omega$$

Moreover, from [2, p. 38] we have that for $f \in \mathcal{P}_m$

$$\mathrm{proj}_{\mathcal{H}_m^h} f = \sum_{p=0}^{[n/2]} \frac{1}{4^p p!\,(-|\alpha|_1 - m + 2 - d/2)_p} |\mathbf{x}|^{2p}\Delta_h^p f(\mathbf{x}).$$

Therefore, using this formula with $m = n - 2j$ and $f = \Delta_h^j V(|\cdot|^{2k}S_{n-2k,i})$ and using the orthogonality of $S_{n-2j,l}$ again, we obtain

$$\int_{S^{d-1}} V(|\cdot|^{2k}\mathbb{S}_{n-2k})(\mathbb{S}_{n-2j}^h)^T h_\alpha^2\, d\omega = \frac{1}{4^j j!\,(|\alpha|_1 + n - 2j + d/2)_j} \int_{S^{d-1}} \Delta_h^j V(|\cdot|^{2k}\mathbb{S}_{n-2k})(\mathbb{S}_{n-2j}^h)^T h_\alpha^2\, d\omega,$$

where we have changed back to the vector notation. Using the identity

$$\Delta(|\mathbf{x}|^{2k}P_n) = 4k(n + k - 1 + d/2)|\mathbf{x}|^{2k-2}P_n + |\mathbf{x}|^{2k}\Delta P_n$$

for $P_n \in \mathcal{P}_n$ and the intertwining property of V it follows that

$$\Delta_h^j V(|\cdot|^{2k}\mathbb{S}_{n-2k}) = V\{\Delta^j(|\cdot|^{2k}\mathbb{S}_{n-2k})\} = 4^j \frac{k!}{(k-j)!}(n - k - j + d/2)_j V(|\cdot|^{2k-2j}\mathbb{S}_{n-2k})\sigma_{k,j},$$

where $\sigma_{k,j} = 1$ if $j \le k$ and $\sigma_{k,j} = 0$ if $j > k$. Therefore, we conclude that

$$\begin{aligned}\int_{S^{d-1}} V(|\cdot|^{2k}\mathbb{S}_{n-2k})(\mathbb{S}_{n-2j}^h)^T h_\alpha^2\, d\omega &= \binom{k}{j}\frac{(n-k-j+d/2)_j}{(|\alpha|_1 + n - 2j + d/2)_j}\int_{S^{d-1}} V(|\cdot|^{2k-2j}\mathbb{S}_{n-2k})(\mathbb{S}_{n-2j}^h)^T h_\alpha^2\, d\omega\sigma_{k,j} \\ &= \binom{k}{j}\frac{(n-k-j+d/2)_j}{(|\alpha|_1 + n - 2j + d/2)_j} B_{n-2j,n-2j}^{k-j}\sigma_{k,j}\end{aligned}$$

using the first equation of formula (3.4) for $B_{n,n}^k$; the desired result follows from the second equation of (3.4). ∎

https://doi.org/10.4153/CJM-1998-010-9 Published online by Cambridge University Press

From the unique decomposition (1.4) of $\mathcal{P}_n$ in terms of $\mathcal{H}_n^h$, the result in Theorem 3.2 can be used to get the action of V on any polynomial $P \in \mathcal{P}_n$. In particular, we can take $n = 2k$ in the formula (3.3) so that it gives a formula for $V(|\cdot|^{2k})$.

The formula (3.3) shows, in particular, that $V(|\cdot|^{2k}S_{n-2k,i})$ is orthogonal to $\mathcal{H}_l$ with respect to $h_\alpha^2 d\omega$ for $l < n-2k$. Therefore, it follows from the unique decomposition of $\mathcal{P}_n$ in (1.4) that $V(|\cdot|^{2k}S_{n-2k,i})$ is orthogonal to all polynomials of degree $\le n-2k-1$. We formulate this fact as a corollary since it seems to be of independent interest.

COROLLARY 3.3. *For each k, $2k \le n$, and $1 \le i \le N_{n-2k}$,*

$$\int_{S^{d-1}} V(|\cdot|^{2k}S_{n-2k,i})qh_\alpha^2\,d\omega = 0, \quad q \in \bigcup_{j=0}^{n-2k-1} \mathcal{P}_j.$$

For a large family of weight functions on S^{d-1}, including many reflection invariant ones, it is shown in [15] that an orthonormal basis can be expressed in terms of orthogonal polynomials on the unit ball B^{d-1}. Comparing to orthogonal structure on the spheres, orthogonal structure on balls seems to be better understood at present time and it is relatively simple. For example, an orthonormal basis on B^{d-1} can be constructed by the standard Gram-Schmidt process. Hence, this connection provides a possible way to obtain a basis of $\mathcal{H}_n^h$. For results about orthogonal polynomials in several variables, including some recent developments, we refer to the survey [12]. In the following we describe the construction of h-harmonics in terms of orthogonal polynomials on B^{d-1} which appears in [15].

A weight function H defined on $\mathbb{R}^d$ is called *S-symmetric* if it is even in y_d and is centrally symmetric with respect to variables $\mathbf{y}' = (y_1, \ldots, y_{d-1})$; that is, H satisfies

$$H(\mathbf{y}', y_d) = H(\mathbf{y}', -y_d) \quad \text{and} \quad H(\mathbf{y}', y_d) = H(-\mathbf{y}', y_d) \quad \mathbf{y} = (\mathbf{y}', y_d) \in \mathbb{R}^d.$$

Examples of S-symmetric functions are h_α^2 whenever h_α is even in each of its variables. We shall restrict our discussion to the case that $H = h_\alpha^2$ is reflection invariant as well as S-symmetric. In associate to an S-symmetric weight function h_α^2 on $\mathbb{R}^d$ we define a weight function W_h on B^{d-1} by

$$W_h(\mathbf{x}) = h_\alpha^2(\mathbf{x}, \sqrt{1-|\mathbf{x}|^2}), \quad \mathbf{x} \in B^{d-1}.$$

The assumption on h_α implies that W_h is centrally symmetric on B^{d-1}. We denote by $\{P_k^n\}$ and $\{Q_k^n\}$ systems of orthonormal polynomials with respect to the weight functions

$$W_h^{(1)}(\mathbf{x}) = 2W_h(\mathbf{x})\Big/\sqrt{1-|\mathbf{x}|^2} \quad \text{and} \quad W_h^{(2)}(\mathbf{x}) = 2W_h(\mathbf{x})\sqrt{1-|\mathbf{x}|^2}, \tag{3.5}$$

respectively, where we adopt the convention that the superscript n means that P_k^n and Q_k^n are polynomials of degree n; the subindex k has the range $1 \le k \le r_n^d$ so that $\{P_k^n\}$, or $\{Q_k^n\}$, forms an orthonormal basis for orthogonal polynomials of degree n.

In this connection we fix the following notation: For $\mathbf{y} \in \mathbb{R}^d$, we write

$$\mathbf{y} = (y_1, \ldots, y_{d-1}, y_d) = (\mathbf{y}', y_d) = r\mathbf{x} = r(\mathbf{x}', x_d), \quad \mathbf{x} \in S^{d-1},\ \mathbf{x}' \in B^{d-1}, \tag{3.6}$$

where $r = |\mathbf{y}| = \sqrt{y_1^2 + \cdots + y_d^2}$ and $\mathbf{x}' = (x_1, \ldots, x_{d-1})$. Keeping in mind this notation we define

$$Y_{k,n}^{(1,h)}(\mathbf{y}) = r^n P_k^n(\mathbf{x}') \quad \text{and} \quad Y_{j,n}^{(2,h)}(\mathbf{y}) = r^n x_d Q_j^{n-1}(\mathbf{x}'), \tag{3.7}$$

where $1 \le k \le r_n^{d-1}$, $1 \le j \le r_{n-1}^{d-1}$ and we define $Y_{j,0}^{(2,h)}(\mathbf{y}) = 0$. The following theorem is proved in [15] for all S-symmetric weight functions.

LEMMA 3.4. *Let h_α^2 be a S-symmetric weight function on $\mathbb{R}^d$. Then the functions $Y_{k,n}^{(1,h)}(\mathbf{y})$ and $Y_{k,n}^{(2,h)}(\mathbf{y})$ defined in (3.7) are homogeneous polynomials of degree n on $\mathbb{R}^d$ and they form an orthonormal basis for $\mathcal{H}_n^h$.*

When $h_\alpha = 1$ we are back to the ordinary harmonics. In fact, since the polynomials T_n and U_n are orthogonal polynomials on $[-1, 1]$ with respect to the weight function $1/\sqrt{1-x^2}$ and $\sqrt{1-x^2}$, respectively, the formula (1.8) provides an illustrating example for this construction. More generally, we can derive formulae for orthogonal polynomials with respect to $1/\sqrt{1-|\mathbf{x}|^2}$ and $\sqrt{1-|\mathbf{x}|^2}$, respectively, from the ordinary harmonic polynomials in (1.9).

For the S-symmetric weight function, we can simplify Theorem 3.1 by taking into account the additional symmetry. We need some notations first. Let h_α^2 be S-symmetric, and $Y_{n,k}^{(i,h)}$ be the h-harmonics given in terms of orthogonal polynomials on B^{d-1} in (3.7). We denote by $\mathbb{Y}_n^{(1,h)}$ and $\mathbb{Y}_n^{(2,h)}$, respectively, the vectors

$$\mathbb{Y}_n^{(1,h)} = (Y_{n,1}^{(1,h)}, \ldots, Y_{n,r_n^{d-1}}^{(1,h)})^T, \quad \text{and} \quad \mathbb{Y}_n^{(2,h)} = (Y_{n,1}^{(2,h)}, \ldots, Y_{n,r_{n-1}^{d-1}}^{(2,h)})^T.$$

When $h_\alpha^2 = 1$, we write $\mathbb{Y}_n^{(i,h)}$ as $\mathbb{Y}_n^{(i)}$, which consists of the ordinary harmonic polynomials. We note that for $d = 2$ the vector $\mathbb{Y}_n^{(i)}$ becomes a scalar, and we have $\mathbb{Y}_n^{(i)} = Y_n^{(i)}$. So the notation agrees with that in (1.8). In view of the previous notation $\mathbb{S}_n$ and $\mathbb{S}_n^h$, we have $\mathbb{S}_n^T = (\mathbb{Y}_n^{(1)}, \mathbb{Y}_n^{(2)})^T$.

THEOREM 3.5. *If h_α^2 is S-symmetric in addition, then*

$$V\mathbb{Y}_n^{(i)} = M_n^{(i,h)} \mathbb{Y}_n^{(i,h)}, \; M_n^{(i,h)} = E_{\alpha,n} \int_{S^{d-1}} \mathbb{Y}_n^{(i)} (\mathbb{Y}_n^{(i,h)})^T \, d\omega, \quad i = 1, 2, \tag{3.8}$$

where $M_n^{(1,h)}$ is a matrix of size $r_n^{d-1} \times r_n^{d-1}$ and $M_n^{(2,h)}$ is a matrix of size $r_{n-1}^{d-1} \times r_{n-1}^{d-1}$. Furthermore,

$$(M_n^{(i,h)})^{-1} = E_{\alpha,n}^{-1} \int_{S^{d-1}} \mathbb{Y}_n^{(i)} (\mathbb{Y}_n^{(i,h)})^T h_\alpha^2 \, d\omega, \quad i = 1, 2. \tag{3.9}$$

PROOF. Since h_α^2 is S-symmetry, it is even with respect to y_d. By the definition in (3.7), $Y_{n,i}^{(1,h)}$ is even in y_d and $Y_{n,i}^{(2,h)}$ is odd in y_d; moreover, the same holds for the ordinary harmonics. Therefore, it follows that

$$\int_{S^{d-1}} \mathbb{Y}_n^{(1)} (\mathbb{Y}_n^{(2,h)})^T \, d\omega = 0 \quad \text{and} \quad \int_{S^{d-1}} \mathbb{Y}_n^{(1)} (\mathbb{Y}_n^{(2,h)})^T h_\alpha^2 \, d\omega = 0.$$

Hence, writing $\mathbb{S}_n^T = \big((\mathbb{Y}_n^{(1)})^T, (\mathbb{Y}_n^{(2)})^T\big)$ and a similar formula for $\mathbb{S}_n^h$, we see that the matrix M_n^h in (3.1) takes the form of the block diagonal matrix, from which the desired result follows from Theorem 3.1. ∎

Similarly, we have the following simplified version of Theorem 3.2.

https://doi.org/10.4153/CJM-1998-010-9 Published online by Cambridge University Press

THEOREM 3.6. *If h_α^2 is S-symmetric in addition, then*

$$V(|\cdot|^{2k}\mathbb{Y}_{n-2k}^{(i)}) = B_n^{k,i}\mathbb{Y}_n^{(i,h)} + B_{n-2,n}^{k,i}|\mathbf{x}|^2\mathbb{Y}_{n-2}^{(i,h)} + \cdots + B_{n-2k,n}^{k,i}|\mathbf{x}|^{2k}\mathbb{Y}_{n-2k}^{(i,h)} \tag{3.10}$$

where $B^{k,i}n = B_{n,n}^{k,i}$ and the matrices $B_{n-2i,n}^{k,i}$ are given by

$$B_{n-2j,n}^{k,i} = \binom{k}{j}\frac{(n-k-j+d/2)_j}{(|\alpha_1|+n-2j+d/2)_j}E_{\alpha,n-2j}\int_{S^{d-1}}\mathbb{Y}_{n-2k}^{(i)}(\mathbb{Y}_{n-2j}^{(i,h)})^T\,d\omega.$$

From the formula for B_n^k in Theorem 3.2, it is evident that the same argument as in the proof of Theorem 3.6 works in this case. We omit the details. As an special case of this theorem, we formulate the the following corollary.

COROLLARY 3.7. *If h_α^2 is S-symmetric in addition, then*

$$\begin{aligned} &V(|\cdot|^{2k})(\mathbf{x}) \\ &= \frac{H_0}{H_\alpha}\sum_{j=0}^{k}|\mathbf{x}|^{2k-2j}\binom{k}{j}\frac{(d/2)_k(d/2)_{2j}}{(d/2)_j(|\alpha|_1+d/2)_{k+j}}\Big(\int_{S^{d-1}}(\mathbb{Y}_{2j}^{1,h})^T\,d\omega\Big)\mathbb{Y}_{2j}^{1,h}(\mathbf{x}). \end{aligned} \tag{3.11}$$

To get (3.11), we substitute the formula of B_{n-2j}^{k-j} into (3.10) and take $n = 2k$. The constants are rewritten using the formula for $E_{\alpha,n-2j}$.

4. **Examples.** In this section we use examples to illustrate the results in the previous sections. Because of the difficulties in computation, the examples are mostly given in the case $d = 2$. We note that $\mathbb{Y}_n^{(i,h)}$ becomes a scalar for $d = 2$ so that Theorems 3.5 and 3.6 become particularly simple.

4.1 *Product weight function.* This is the weight function associated to $\mathbb{Z}_2\times\cdots\times\mathbb{Z}_2$; it is defined by

$$h_\alpha(\mathbf{x}) = |x_1|^{\alpha_1}\cdots|x_d|^{\alpha_d}, \quad \alpha_i \ge 0,\ \mathbf{x}\in S^{d-1},$$

where $\alpha_i \ge 0$. The h-harmonics and the intertwining operator are known in this case; they are studied as examples of the general theory in [3] for $d = 2$ and later in [13] in more detail. The intertwining operator turns out to be an integral operator given by

$$Vf(\mathbf{x}) = \int_{[-1,1]^d} f(x_1t_1,\ldots,x_dt_d)\prod_{i=1}^{d}(1+t_i)\prod_{i=1}^{d}c_{\alpha_i}(1-t_i^2)^{\alpha_i-1}\,d\mathbf{t}, \tag{4.1}$$

where the constant c_λ is defined by $c_\lambda^{-1} = \int_{-1}^{1}(1-t^2)^{\lambda-1}\,dt$. To describe the h-harmonics, we introduce the orthonormal polynomials with respect to the weight function

$$w^{(\lambda,\mu)}(x) = w_{\lambda,\mu}(1-x^2)^{\lambda-\frac12}|x|^{2\mu}, \quad -1\le x\le 1,\ \lambda,\mu > -1/2, \tag{4.2}$$

where $w_{\lambda,\mu}$ is the normalization constant so that the integral of $w^{(\lambda,\mu)}$ on $[-1,1]$ is 1;

$$w_{\lambda,\mu} = \frac{\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+1/2)\Gamma(\mu+1/2)}.$$

https://doi.org/10.4153/CJM-1998-010-9 Published online by Cambridge University Press

The orthonormal polynomial of degree n with respect to $w^{(\lambda,\mu)}$ is denoted by $D_n^{(\lambda,\mu)}$; it can be given in terms of the classical Jacobi polynomials $P_n^{(\alpha,\beta)}$ (*cf.* [9]) as

$$D_{2n}^{(\lambda,\mu)}(x) = c_n(\lambda,\mu)P_n^{(\lambda-\frac{1}{2},\mu-\frac{1}{2})}(2x^2-1), \tag{4.3a}$$

$$D_{2n+1}^{(\lambda,\mu)}(x) = c_n(\lambda,\mu+1)\Big(\frac{\lambda+\mu+1}{\mu+1/2}\Big)^{1/2} xP_n^{(\lambda-\frac{1}{2},\mu+\frac{1}{2})}(2x^2-1), \tag{4.3b}$$

where

$$c_n(\lambda,\mu) = \Big(\frac{\Gamma(\mu+\frac{1}{2})\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda+\mu+1)}\Big)^{1/2}\Big(\frac{(2n+\lambda+\mu)\Gamma(n+\lambda+\mu)\Gamma(n+1)}{\Gamma(n+\mu+\frac{1}{2})\Gamma(n+\lambda+\frac{1}{2})}\Big)^{1/2}. \tag{4.4}$$

We note that if $\mu = 0$, then $D_n^{(\lambda,0)} = \tilde{C}_n^{(\lambda)}$ is the orthonormal Gegenbauer polynomial, which is a constant multiple of $C_n^{(\lambda)}$. For $d = 2$, an orthonormal basis for $\mathcal{H}_n^h$ with respect to the normalized $H_\alpha h_\alpha^2 d\omega$ is given by

$$Y_n^{(1,h)}(\mathbf{x}) = r^n D_n^{(\alpha_1,\alpha_2)}(\cos\theta), \quad Y_n^{(2,h)}(\mathbf{x}) = r^n\Big(\frac{\alpha_1+\alpha_2+1}{\alpha_1+1/2}\Big)^{1/2}\sin\theta D_{n-1}^{(\alpha_1+1,\alpha_2)}(\cos\theta),$$

where we use the polar coordinates $\mathbf{x} = (r\sin\theta, r\cos\theta)$.

Since the intertwining operator is an integral transform in this case, the relation between $VY_n^{(i)}$ and the h-harmonics leads to an integral formula of the Gegenbauer polynomials, which includes a classical formula of Feldheim and Vilenkin. Indeed, using (1.8) and applying Theorem 3.5 to $Y_n^{(1,h)}$ implies that

(4.5)

$$D_n^{(\alpha_1,\alpha_2)}(\cos\theta) = b_n\int_{-1}^1\int_{-1}^1 (t_1^2\sin^2\theta + t_2^2\cos^2\theta)^{n/2} \times T_n\Big(\frac{t_2\cos\theta}{(t_2^2\cos^2\theta+t_1^2\sin^2\theta)^{1/2}}\Big)(1+t_2)(1-t_1^2)^{\alpha_1-1}(1-t_2^2)^{\alpha_2-1}\,dt_1\,dt_2,$$

where b_n is a constant which can be determined by setting $\theta = 0$. In particular, if we let $\alpha_2 \longrightarrow 0$ by using the relation

$$\lim_{\mu\to 0} c_\mu\int_{-1}^1 f(t)(1-t^2)^{\mu-1}\,dt = \frac{f(1)+f(-1)}{2}$$

and write $\alpha_1 = \alpha$, then (4.5) becomes

$$\tilde{C}_n^{(\alpha)}(\cos\theta) = b_n\int_{-1}^1 (t^2\sin^2\theta+\cos^2\theta)^{n/2} T_n\Big(\frac{\cos\theta}{(\cos^2\theta+t^2\sin^2\theta)^{1/2}}\Big)(1-t^2)^{\alpha-1}\,dt.$$

By setting $\theta = 0$ the constant b_n is seen to equal to $c_\alpha\tilde{C}_n^{(\alpha)}$. Hence, changing variable $t_1 = \cos\psi$ and then $\psi \longmapsto (\pi/2)-\phi$, we conclude that (4.5) with $\alpha_2 = 0$ and $\alpha_1 = \alpha$ is equivalent to

$$\frac{C_n^{(\alpha)}(\cos\theta)}{C_n^{(\alpha)}(1)} = 2c_\alpha\int_0^{\pi/2}(1-\sin^2\theta\sin^2\phi)^{n/2} T_n\Big(\frac{\cos\theta}{(1-\sin^2\phi\sin^2\theta)^{1/2}}\Big)(\sin\phi)^{2\alpha-1}\,d\phi.$$

This formula is a special case of a formula due to Feldheim and Vilenkin (*cf.* [1, p. 24]), which has Gegenbauer polynomial $C_n^{(\lambda)}$ in place of T_n in its general form.

The same consideration with the explicit formula of V allows us to write down the h-harmonics on $\mathbb{R}^d$ in terms of an integral transform of the ordinary harmonics. Let us consider a special case of $h_\alpha = H_\alpha |x_d|^\alpha$ on S^{d-1} with $d \ge 3$; that is, we take $\alpha_1 = \cdots = \alpha_{d-1} = 0$ and $\alpha_d = \alpha$. From the formula in [13], an orthonormal basis of $\mathcal{H}_n^h$ for this h_α is given by formulae similar to those of the ordinary harmonics; in fact, we only need to replace $C_{n-k_1}^{(k_1+\frac{d-1}{2},\alpha)}$ in (1.9) by $D_{n-k_1}^{(k_1+\frac{d-1}{2},\alpha)}$ (of course, the normalization constant changes as well). It follows from Theorem 3.4 and the explicit formula of V (which reduces to only one fold integral in this case) that we have

$$D_{n-k}^{(k+\frac{d-2}{2},\alpha)}(\cos\theta) = b_n \int_{-1}^{1} (\sin^2\theta + t^2\cos^2\theta)^{n/2} C_{n-k}^{(k+\frac{d-2}{2})}\left(\frac{t\cos\theta}{(\sin^2\theta + t^2\cos^2\theta)^{1/2}}\right) \times (1+t)(1-t^2)^{\alpha-1}\,dt,$$

where b_n is a constant. In particular, let $k = 0$, $n = 2m$ and use (4.3a), we end up with the following interesting formula

$$P_m^{(\frac{d-3}{2},\alpha-\frac{1}{2})}(\cos 2\theta) = a_m \int_0^{\pi} (1-\cos^2\theta\sin^2\phi)^m C_{2m}^{(\frac{d-2}{2})}\left(\frac{\cos\theta\cos\phi}{(1-\cos^2\theta\sin^2\phi)^{1/2}}\right) \times (\sin\phi)^{2\alpha-1}\,d\phi,$$

where the constant a_m can be determined by setting $\theta = 0$. Using quadratic transform to change $C_{2m}^{(\frac{d-2}{2})}$ to $P_m^{(\frac{d-3}{2},-\frac{1}{2})}$, it can be seen that the above integral is a special case of a formula on Jacobi polynomials first derived by Askey and Fitch (see [1, p. 20, (3.10)]), which also follows from a formula of hypergeometric function of Bateman (*cf.* [1, (3.5)]).

4.2 *Dihedral group D_4*. The weight function is

$$h_\alpha(\mathbf{x}) = |2x_1x_2|^\lambda |x_1^2 - x_2^2|^\mu, \quad H_\alpha = w_{\lambda,\mu}/2, \tag{4.6}$$

where $w_{\lambda,\mu}$ is given in (4.2) and we write $\alpha_1 = \lambda$ and $\alpha_2 = \mu$. In this case, a closed formula of the intertwining operator V is not known.

An orthogonal basis for $\mathcal{H}_n^h$ can be derived using Lemma 3.4. It is easy to verify that

$$H_\alpha W_h^{(1)}(t) = 2^{2\lambda+2\mu} w_{\lambda,\mu} |t|^{2\lambda} (1-t^2)^\lambda |t^2 - 1/2|^{2\mu} := w_4^{(\lambda,\mu)}(t),$$

and $\int_{-1}^{1} w_4^{(\lambda,\mu)}(t)\,dt = 1$. Hence, by Lemma 3.4, to derive an orthonormal basis for $\mathcal{H}_n^h$, we need to find orthonormal polynomials for $w_4^{(\lambda,\mu)}$ and $(1-t^2)w_4^{(\lambda,\mu)}$, respectively.

It turns out that an orthonormal basis for $\mathcal{H}_{2n}^h$ can be given rather easily. Indeed, in terms of polar coordinates $\mathbf{x} = (x_1, x_2) = r(\cos\theta, \sin\theta)$, it is easy to verify that

$$Y_{2n}^{(1,h)}(\mathbf{x}) = r^{2n} D_n^{(\lambda,\mu)}(\cos 2\theta), \quad Y_{2n}^{(2,h)}(\mathbf{x}) = r^{2n}\left(\frac{\lambda+\mu+1}{\lambda+1/2}\right)^{1/2} \sin 2\theta D_{n-1}^{(\lambda+1,\mu)}(\cos 2\theta).$$

The formulae for $Y_{2n+1}^{(i,h)}$ are much more involved; for example,

$$Y_{2n+1}^{(1,h)}(\mathbf{x}) = a_n r^{2n+1} \cos\theta [D_n^{(\lambda,\mu)}(\cos 2\theta) - b_n \cos^2\theta D_{n-1}^{(\lambda+1,\mu)}(\cos 2\theta)],$$

where a_n and b_n are constants and their formulae are different for n being even and odd. The formula for $Y_{2n+1}^{(2,h)}$ takes a similar form. We will not give them explicitly but refer the reader to [2], where a complete basis for $\mathcal{H}_n^h$ is derived by solving $\Delta_h P = 0$ and the basis is given in terms of Jacobi polynomials (see (4.3)).

Using Theorem 3.5 or by elementary consideration it is easy to conclude that $VY_n^{(i)}$ is a constant multiple of $Y_n^{(i,h)}$. The importance of the results in the previous section is that Theorem 3.5 shows a way to compute the constant. Indeed, we have that

$$VY_n^{(i)}(\mathbf{x}) = M_n^{(i,h)} Y_n^{(i,h)}, \quad M_n^{(i,h)} = E_{\alpha,n} \int_{S^{d-1}} Y_n^{(i,h)} Y_n^{(i)}\, d\omega.$$

Therefore, let $\gamma_n^{(\lambda,\mu)}$ denote the leading coefficient of $D_n^{(\lambda,\mu)}$, we derive easily that

$$VY_{2n}^{(1)}(\mathbf{x}) = \frac{\Gamma(2n+1)}{2^n(1+\lambda+\mu)_{2n}} \gamma_n^{(\lambda,\mu)} D_n^{(\lambda,\mu)}(\cos 2\theta),$$

$$VY_{2n}^{(2)}(\mathbf{x}) = \frac{\Gamma(2n+1)}{2^n(1+\lambda+\mu)_{2n}} \frac{\lambda+\mu+1}{\lambda+1/2} \gamma_{n-1}^{(\lambda+1,\mu)} \sin 2\theta D_n^{(\lambda+1,\mu)}(\cos 2\theta).$$

Hence, we have the explicit formula for the action of V on the ordinary harmonic polynomials of even degree; the formula in the case of odd degree can be derived in a same way, but it is cumbersome. Moreover, using (3.7) we can derive the formulae for $V(|\cdot|^{2k} Y_{n-2k}^{(i,h)})$. In particular, since by (4.3)

$$\int_{S^1} Y_{4n+2}^{(1,h)}\, d\omega = \text{const} \int_{-1}^{1} x P_n^{(\lambda-\frac{1}{2},\mu+\frac{1}{2})}(2x^2-1) \frac{dx}{\sqrt{1-x^2}} = 0,$$

it follows from (3.11) that we have

$$V(|\cdot|^{2k})(\mathbf{x}) = r^{2k}\Gamma(k+1)\frac{H_0}{H_\alpha} \sum_{j=0}^{[k/2]} \binom{k}{2j} \frac{(2j+1)_{2j}}{(\lambda+\mu+1)_{k+2j}} c_j^2(\lambda,\mu)$$
$$\times \int_0^{2\pi} P_j^{(\lambda-\frac{1}{2},\mu-\frac{1}{2})}(\cos 4\phi)\, d\phi P_j^{(\lambda-\frac{1}{2},\mu-\frac{1}{2})}(\cos 4\theta),$$

where $c_j(\lambda,\mu)$ is defined in (4.3) integral of $[P_j^{(\lambda-1/2,\mu-1/2)}]^2$ with respect to the normalized weight function $w_{\lambda,\mu}(1-x)^{\lambda-1/2}(1+x)^{\mu-1/2}$. It is conjectured by Dunkl that V is always positive. If the conjecture were proved, then the above would give a nonnegative sum of Jacobi polynomials.

Still, we do not know a closed formula of V for this weight function. It would be very interesting if V can be written as an integral operator as in the case of $\mathbb{Z}_2$. One approach is to derive an integration formula for the reproducing kernel P_n^h and make the connection to V through the general formula (see [5] and [14])

$$P_n^h(\mathbf{x},\mathbf{y}) := Y_n^{(1,h)}(\mathbf{x})Y_n^{(1,h)}(\mathbf{y}) + Y_n^{(2,h)}(\mathbf{x})Y_n^{(2,h)}(\mathbf{y})$$
$$= \frac{n+2\lambda+2\mu}{2\lambda+2\mu}[VC_n^{(2\lambda+2\mu)}(\langle\cdot,\mathbf{y}\rangle)](\mathbf{x}), \quad |\mathbf{y}| \le |\mathbf{x}| = 1.$$

Such an approach is used in [13] to derive a formula of V in (4.1) for the case of $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$. Based on the product formulae of $D_n^{(\lambda,\mu)}$ in [13] and ad hoc transforms, we can follow this approach to prove that

https://doi.org/10.4153/CJM-1998-010-9 Published online by Cambridge University Press

PROPOSITION 4.1. *For h_α in (4.6), if $\lambda+\mu \ge 3/2$ and $\lambda, \mu \ge 0$, then for all $n \ge 0$*

$$[V(\langle\cdot,\mathbf{y}\rangle)^{2n}](\mathbf{x}) = c_\lambda c_\mu (\lambda+\mu)(\lambda+\mu-1)\pi^{-3/2}\int_{B^3}(1-|\mathbf{u}|^2)^{\lambda+\mu-2}\,d\mathbf{u}$$
$$\Bigg\{\int_{-1}^1\int_{-1}^t \left(\mathbf{x}^T\begin{bmatrix}\sqrt{\frac{1+s}{2}}u_2+\sqrt{\frac{t-s}{2}}u_3 & \sqrt{\frac{1-t}{2}}u_1\\ -\sqrt{\frac{1-t}{2}}u_1 & \sqrt{\frac{1+s}{2}}u_2+\sqrt{\frac{t-s}{2}}u_3\end{bmatrix}\mathbf{y}\right)^{2n}\Phi(t,s)\,dt\,ds$$
$$+\int_{-1}^1\int_t^1 \left(\mathbf{x}^T\begin{bmatrix}\sqrt{\frac{1+t}{2}}u_1 & \sqrt{\frac{1-s}{2}}u_2+\sqrt{\frac{s-t}{2}}u_3\\ -\sqrt{\frac{1-s}{2}}u_2+\sqrt{\frac{s-t}{2}}u_3 & \sqrt{\frac{1+t}{2}}u_1\end{bmatrix}\mathbf{y}\right)^{2n}\Phi(t,s)\,dt\,ds\Bigg\},$$

where $\Phi(t,s) = (1+t)(1+s)(1-t^2)^{\mu-1}(1-s^2)^{\lambda-1}$.

This formula indicates that there should be an integral formula for the intertwining operator V in this case. Unfortunately, the obvious choice hinted by this formula does not seem to work in general. Because it is only a partial result and the proof is rather ad hoc and long, we will not give the proof here. Although the formula looks complicated, a formula of similar type with 6-fold integrals has been conjectured by Dunkl (private communication) based on some consideration using integration over the unitary group.

4.3 *Other Dihedral groups.* We can apply Theorem 3.5 and 3.6 to other Dihedral group D_k since an orthogonal basis for the h-harmonics associated with D_k can be given explicitly (*cf.* [2, 4]). To get $M_n^{(i,h)}$ in (3.8) we need to normalize the basis; that is, we need to compute $\int [Y_n^{(i,h)}]^2 h_\alpha^2\,d\omega$, which could be complicated. For example, for D_3 whose corresponding weight function is given by $h_\alpha(\cos\theta,\sin\theta) = |\sin 3\theta|^\alpha$ in polar coordinates, we know form [2, p. 52] that

$$Y_{3n-1}^{(1,h)} = r^{3n-1}[\cos 2\theta C_{n-1}^{(\alpha+1)}(\cos 3\theta) - \cos\theta C_n^{(\alpha+1)}(\cos 3\theta)].$$

The norm of this function is not easy to compute. Other than this computation, we should have no problem to write down the action of V on ordinary harmonics using (3.8).

4.4 *Remarks on the case of $d > 2$.* Naturally, we would like to get results on the intertwining operator for $d > 2$. In this regard, the results in Section 2 may not be practical; it requires to compute M_n^{-1}, where M_n contains inner product of ordinary harmonics with respect to $h_\alpha^2 d\omega$. These inner products can be difficult to compute. Take, for example, the octahedral group, which is the symmetric group of the unit cube $\{\pm1,\pm1,\pm1\}$ in $\mathbb{R}^3$, with the weight function $h_\alpha(\mathbf{x}) = |(x_1^2-x_2^2)(x_2^2-x_3^2)(x_3^2-x_1^2)|^\alpha$. For $d=3$ the ordinary harmonics are given by

$$Y_{k,n}^{(1)} = r^n C_{n-k}^{(k+1/2)}(\cos\phi)\cos k\theta, \quad Y_{k,n}^{(2)} = r^n C_{n-k}^{(k+1/2)}(\cos\phi)\sin k\theta,$$

$0 \le k \le n$, under the standard spherical coordinates $x_1 = r\sin\theta\sin\phi$, $x_2 = r\sin\theta\cos\phi$ and $x_3 = r\cos\theta$, where $0 \le \phi \le 2\pi$ and $0 \le \theta \le \pi$. In order to work out M_n, we then need to compute integrals such as

$$\int_{S^2} Y^{(1)}_{k,n} Y^{(1)}_{j,m} h^2_\alpha \, d\omega = \int_0^{2\pi} \int_0^{\pi} C^{(k+1/2)}_{n-k}(\cos\phi) C^{(j+1/2)}_{m-j}(\cos\phi) \cos k\theta \cos j\theta \times |\sin^2\theta \cos 2\phi (\sin^2\theta \sin 2\phi - \cos^2\theta)(\sin^2\theta \cos 2\phi - \cos^2\theta)|^{2\alpha} \, d\theta \, d\phi.$$

A moment reflection tells that this is rather difficult even for moderate m and n.

On the other hand, the results in Section 3 are workable provided a basis for $\mathcal{H}^h_n$ is known. So far, however, such a basis has been constructed only for the product weight functions in 4.1. We note that for S-symmetric function, we can work with orthogonal polynomials on B^{d-1} with respect to the weight functions defined in (3.5). For small n, at least, we can find the basis for orthogonal polynomials using the standard Gram-Schmidt method.

REFERENCES

1. R. Askey, *Orthogonal polynomials and special functions*. SIAM, Philadelphia, 1975.

2. C. Dunkl, *Reflection groups and orthogonal polynomials on the sphere.* Math. Z. **197**(1988), 33–60.

3. ———, *Differential-difference operators associated to reflection groups.* Trans. Amer. Math. Soc. **311** (1989), 167–183.

4. ———, *Possion and Cauthy kernels for orthogonal polynomials with dihedra symmetry.* J. Math. Anal. Appl. **143**(1989), 459–470.

5. ———, *Integral kernels with reflection group invariance.* Canad. J. Math. **43**(1991), 1213–1227.

6. ———, *Intertwining operators associated to the group S_3.* Trans. Amer. Math. Soc. **347**(1995), 3347–3374.

7. A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher transcendental functions*. McGraw-Hill **2**, New York, 1953.

8. E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*. Princeton Univ. Press, Princeton, New Jersey, 1971.

9. H. Szegő, *Orthogonal polynomials*. 4th ed., Amer. Math. Soc., Colloq. Publ. **23**, Providence, Rhode Island, 1975.

10. N. J. Vilenkin, *Special functions and the theory of group representations*. Amer. Math. Soc., Trans. Math. Monographs **22**, Providence, Rhode Island, 1968.

11. Y. Xu, *On multivariate orthogonal polynomials.* SIAM J. Math. Anal. **24**(1993), 783–794.

12. ———, *On orthogonal polynomials in several variables.* In: Special functions, *q*-series, and related topics, The Fields Institute for Research in Math. Sci., Communications Series **14**(1997), 247–270.

13. ———, *Orthogonal polynomials for a family of product weight functions on the spheres.* Canad. J. Math. **49**(1997), 175–192.

14. ———, *Integration of the intertwining operator for h-harmonic polynomials associated to reflection groups.* Proc. Amer. Math. Soc. **125**(1997), 2963–2973.

15. ———, *Orthogonal polynomials and cubature formulae on spheres and on balls.* SIAM. J. Math. Anal., to appear.

Department of Mathematics
University of Oregon
Eugene, Oregon 97403-1222
U.S.A.
e-mail: yuan@math.uoregon.edu