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EQUIDISTANT LOCI AND THE MINKOWSKIAN GEOMETRIES

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1. The space R of this paper is a metrization, with a not necessarily symmetric distance xy, of an open convex set D in the n-dimensional affine space A^n such that $xy + yz = xz$ if and only if x, y, z lie on an affine line with y between x and z and such that all the balls $px \leqq \rho$ are compact. These spaces are called straight desarguesian G-spaces or sometimes open projective metric spaces. The hyperbolic geometry is an example; a large variety of other examples is studied by contributors to Hilbert's problem IV. When $D = A^n$ and all the affine translations are isometries for the metric xy, the space is called a Minkowskian space or sometimes a finite dimensional Banach space, the (not necessarily symmetric) distance of a Minkowskian space being a (positive homogeneous) norm. In this paper geometric conditions in terms of equidistant loci are given for the space R to be a Minkowskian space.

More precisely and in detail, denote for a set M and a point p by Mp and pM, respectively, the numbers $\inf\{mp|m \in M\}$ and $\inf\{pm|m \in M\}$ and define the equidistant loci of M to be the sets $E(M, \alpha^{\rightarrow}) = \{x|Mx = \alpha\}$ and $E(M, {}^{\leftarrow}\alpha) = \{x|xM = \alpha\}$ for $\alpha \geqq 0$. Denoting the two sides of a hyperplane H by $\sigma^{\pm}(H)$, define $E_{\pm}(H, {}^{\leftarrow}\alpha^{\rightarrow}) = E(H, {}^{\leftarrow}\alpha^{\rightarrow}) \cap \sigma^{\pm}(H)$. In this paper the following theorems are proved:

THEOREM 1. *The space R is Minkowskian if and only if for all hyperplanes H the equidistant loci $E_{\pm}(H, {}^{\leftarrow}\alpha^{\rightarrow})$ are also hyperplanes.*

THEOREM 2. *The space R is Minkowskian if and only if for all lines L the equidistant loci $E(L, {}^{\leftarrow}\alpha^{\rightarrow})$ are convex sets which are unions of lines.*

THEOREM 3. *The space R is Minkowskian if and only if all the balls $xp \leqq \rho$ are also compact, the sets $E(L, {}^{\leftarrow}\alpha^{\rightarrow})$ are convex, and for hyperplanes H either all $E_{\pm}(H, {}^{\leftarrow}\alpha)$ or all $E_{\pm}(H, \alpha^{\rightarrow})$ are hyperplanes.*

Examples are also given at the end to show that weaker hypotheses on the equidistant loci do not suffice to single out interesting geometries.

For $n = 2$ and under strong differentiability and regularity assumptions (the latter in the sense of calculus of variations), our Theorem 1 is a special case of a theorem of Funk [**2**; **3**], who shows that we obtain a Minkowskian or his "Geometrie der spezifischen Massbestimmung" when the equidistant loci of lines ($n = 2$!) are lines locally; i.e., an affine segment S in the space has

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a convex neighborhood U such that the points x in U at a constant distance from the line containing S lie on lines. We make no assumption other than convexity on the domain D of definition of the geometry and give a purely synthetic proof of all our theorems in n dimensions. In fact our proof would reduce to only a few lines (see § 3) if D is assumed to be either the entire affine plane or the interior of a strictly convex closed curve in the plane (which is the case considered by Funk).

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2. Notation and preliminaries. In this paper we use the following notation. The balls $px < \rho$ and $xp < \rho$ are denoted by $S^+(p, \rho)$ and $S^-(p, \rho)$, respectively, while the spheres $px = \rho$ and $xp = \rho$ are written $K^+(p, \rho)$ and $K^-(p, \rho)$. When $Mp = qp$ we say that q is an initial foot (i-foot) of p on M and when $pM = pq$ we say that q is a terminal foot (t-foot) of p on M. Straight lines are always parametrized as $x(t)$, $\alpha < t < \infty$, so that $x(t)x(s) = s - t$ for $s > t$. A segment joining p to q is denoted $T(p, q)$ and the symbol (pqr) stands for collinear points p, q, r with q between p and r. We put $C\ (M, {}^{\rightarrow}\alpha) = \{x | Mx \leqq \alpha\}$ and define the sets $C(M, {}^{\leftarrow}\alpha)$ analogously. When all $E_{\pm}(H, {}^{\leftarrow}\alpha^{\rightarrow})$ are hyperplanes we say that the equidistant loci are flat or that the space has the equidistant locus property and when all tubes $E(L, {}^{\leftarrow}\alpha^{\rightarrow})$ are convex unions of lines we say that the tubes are cylindrical or that the space has the cylindrical tubes property. When considering the case of symmetric distance we shall sometimes omit the arrows or the signs $+$ or $-$ when they indicate the direction of distance.

We collect below some results on the convexity of spheres and on perpendiculars which will be useful later.

Since the convex sets in our space are affine convex sets it is easy to prove that if every supporting hyperplane of a compact set M meets M in a convex set then M contains the boundary of its convex hull. Consequently, if such a set M is star-shaped then it is convex. An immediate application of this fact is that if the set of t-feet (respectively, i-feet) of every point on every line is connected, then the balls $px \leqq \rho$ (respectively, compact $xp \leqq \rho$) are convex. An argument similar to that of [**1**, § 20] then shows that if all $px \leqq \rho$ are convex then they are strictly convex and that if every point has at most one i-foot on every hyperplane then compact $xp \leqq \rho$ are strictly convex.

The perpendiculars are defined as follows: Let H and G be two straight lines in a 2-dimensional space meeting at a point p. Let $\sigma^{\pm}(G)$ denote the

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two sides of G. If for all $x \in H \cap \sigma^+(G)$ p is the unique terminal foot of x on G, and for all $x \in H \cap \sigma^-(G)$ p is the unique initial foot of x on G, then we say that H is perpendicular to G incoming on the side $\sigma^+(G)$ of G. Using the methods of [**4**, § 8] and [**1**, § 20], we can show: In a 2-dimensional space, if all compact circles are convex, then given a line L and a point $p \notin L$, there exists exactly one perpendicular to L through p incoming on the side of L containing p. There exists exactly one perpendicular through p incoming on the other side of L if p has an initial foot on L. In a higher dimensional space perpendiculars to hyperplanes are similarly defined. An analogue of the above result then holds true.

3. The 2-dimensional case. We will first prove the first theorem in the 2-dimensional case with symmetric distance. The details of the proof in the n-dimensional case are given in §§ 4 and 5 with § 5 giving the proof in the general case of a not necessarily symmetric distance. We then prove Theorem 2 by showing that the equidistant locus property is implied by the cylindrical tubes property. The proof of Theorem 3 is then given. The paper is concluded with some examples of spaces which satisfy weaker conditions regarding the equidistant loci.

Let therefore the dimension of R be 2 and assume that the equidistants of lines are lines. We first collect several geometric properties of the space implied by the equidistant locus property. We use the easily proved fact that if L_1 is equidistant to L_2 then L_2 is equidistant to L_1 to show that every point x has a unique foot on every line L. For if x had two feet f_1 and f_2 on L, draw $L_1 = E(L, xf_1)$ and L_2 equidistant to L_1 on the side of L_1 not containing L. Let $yx = L_2x$, and if y, x, f_1 are not collinear let the line joining y and f_1 meet L_1 in $g \neq x$. Then we have $yx \leqq yg$, $xf_1 \leqq gf_1$ but, by the triangle inequality, $yg + gf_1 = yf_1 < yx + xf_1$ which is a contradiction. Thus y, x, f_1 must be collinear. So also y, x, f_2 are collinear, showing that $f_1 = f_2$. This argument also shows that if L_α is equidistant to L_β and L_β is equidistant to L_γ then the successive feet of points are collinear, so the distances add up and hence L_α and L_γ are also equidistants. Also, since the successive feet are collinear, a family of equidistants has common perpendiculars. In other words a family of equidistants and their perpendiculars form a *net* in the space. Since the interior of a strictly convex closed curve in the plane cannot admit a net we see that in the case considered by Funk [**3**], our theorem follows at once because in this case D can only be the entire plane and then from [**1**, Theorem 24.1] the space is Minkowskian.

We resume the discussion when D need not be the interior of a strictly convex closed curve. From the uniqueness of feet we see that the circles are convex as noted in § 2. Therefore at every point on the boundary of the circle there exists a supporting line and the line joining the center to this point is perpendicular to the supporting line. Thus given a line H we can always find a line L such that H is perpendicular to L.

We use the above geometric properties to show that the domain D in which the geometry is defined is either a triangle or a strip or the entire affine plane. We observe the fact that if D contains an entire affine line then it must be a half plane or a strip which are projectively equivalent. If it does not contain an entire affine line, then it has a supporting line L which meets the affine closure of D in an affine compact set. Choosing L' parallel to L as the line at infinity we can transform $\bar{D}$ into a compact set. If $\bar{D}$ is compact it has at least three extreme points p_1, p_2, p_3 say. We prove that $\bar{D}$ is in fact the triangle $p_1p_2p_3$. For, if it is not the triangle then one of the sides of the triangle, say p_1p_2, separates D. Let p and H_1 lie on opposite sides of the line H which joins p_1 and p_2. Since the line H_1 and the point p lie on opposite sides of H, the equidistant H_2 through p of H_1 cannot contain both p_1 and p_2 because $p \notin H$. However, every point x of the region between H_1 and H_2 is collinear with its feet f_i on H_i since H_i, being equidistants, have a common perpendicular through x. Thus the region between H_1 and H_2 is the convex hull of $H_1 \cup H_2$. Since either p_1 or p_2 must belong to this region either p_1 or p_2 is not an extreme point, which is a contradiction to the hypothesis. Thus $\bar{D}$ is the triangle $p_1p_2p_3$.

Hence to prove that the geometry is defined in the entire affine plane it suffices to prove that it cannot be defined inside a triangle or in a strip. To show that it cannot be defined inside a triangle observe first that the equidistants of a line through a vertex of the triangle must again pass through that vertex. To prove this, it is first clear that the equidistants must pass through the vertex at least on one side since the vertex is an extreme point of the triangle and hence cannot lie between two equidistants as observed above. So we get two equidistants L_1 and L_2 passing through the vertex p, say. It suffices to show that we obtain an equidistant to L_1 on the side $\sigma(L_1)$ of L_1 which does not contain L_2. If L_1 and L_2 are at a distance $\eta > 0$, choose y with $yx < \eta$, $y \in \sigma(L_1)$, $x \in L_1$, and let L_3 join y and p. We assert that the equidistant L_3' to L_3 through x must be L_1 because, otherwise, L_3' intersects L_2 and contains points at a distance $> \eta$. Hence all lines through a vertex are mutually equidistant.

Now let L be a line not passing through a vertex and let H be a perpendicular to L at w. We show that H must pass through a vertex. For if not, we can find a side of the triangle which intersects H and L say in p and q, respectively. Then we have equidistants L_1 and L_2, at a distance $\alpha > 0$ say, which join a vertex to p and q. Then for each x on the ray of H containing p, if the perpendicular through x to L_i meets them in u and v, respectively, and meets L in z, we have $xw < xz < uv < \alpha$ showing that an outgoing ray of H has finite length. This is a contradiction, so H must pass through a vertex. However, this also leads to a contradiction as can be seen by revolving L about w and noting that the perpendiculars must vary continuously. Thus the geometry cannot be defined inside a triangle.

Next we show that the geometry cannot be defined inside a strip. For, if it is defined inside a strip bounded by Λ and Λ', then find a line L and its perpendicular G such that L and G meet Λ and Λ' in q, s and p, r, respectively. Let L and G meet in u and let the perpendicular to N from u meet N, M in v and w, respectively, where N and M are lines joining p, q and s, r. Let M be represented by $x(t)$ and let $x(t_0) = w$. Since G is perpendicular to L we have $\min x(t)N = wN$ and $x(t)N \to \infty$ as $t \to \infty$ and as $t \to -\infty$. But then the equidistant through w to N can neither intersect nor coincide with M. Thus the geometry cannot be defined inside a strip.

It is now easy to prove that the geometry, defined in all of A^2, is Minkowskian. In fact our circles are now convex curves such that a pair of equidistant supporting lines at the ends of a diameter are parallel. This implies, from the theory of convex curves in the affine plane (see [**1**, 16.7, p. 89]), that the metric center of a circle is its affine center. Thus the metric midpoint of a segment coincides with its affine midpoint which shows (see [**1**, p. 94]) that the geometry is Minkowskian.

4. The n-dimensional case (symmetric distance). All the geometric properties proved for the 2-dimensional case carry over without much change for the higher dimensional spaces satisfying the equidistant locus property. Thus for example, the relation of being equidistant is an equivalence relation among the hyperplanes of the space, every point has a unique foot on every hyperplane, the spheres are strictly convex, and perpendiculars to hyperplanes exist.

However, to prove the theorem in this case we will need some additional results. We first prove that the equidistant loci $E(L, \alpha)$ are convex. It suffices to show that for every x with $xL = \alpha$ we have a supporting hyperplane H to $E(L, \alpha)$ at x. Since L supports the sphere $K(x, \alpha)$, L can be imbedded in a supporting hyperplane H' to $K(x, \alpha)$. Then the equidistant H to H' through x supports $E(L, \alpha)$ which proves that $E(L, \alpha)$ are convex. We use this result to prove that the map ϕ which sends each point x on H to its foot on an equidistant hyperplane H' is a projectivity, i.e., an incidence preserving bijective map. Since H and H' are mutually equidistant, at a distance α say, and since the feet of points on hyperplanes are unique, it follows that ϕ is a bijection. To show that a line $L \subset H$ is sent into a line, we observe first that $\phi L = H' \cap E(L, \alpha)$ as is easy to prove. But since H' and $E(L, \alpha)$ are convex this implies that the curve ϕL is a 1-dimensional convex set which implies that it must be a straight line. This proves that ϕ is a projectivity. Hence ϕ can be written as a linear transformation in terms of suitable affine co-ordinates and can be extended to the affine boundary of $D \cap H$. Thus ϕ carries boundary into boundary and extreme points into extreme points. Now denote by ϕ_α the projectivity between H and H_α, equidistant to H at a distance α, on a given side of H. Then the images $p_\alpha = \phi_\alpha p$, $p \in H$ are all collinear since they lie on the perpendicular to H at p. The same result holds, by continuity, for points p on the boundary of $D \cap H$.

We will need one more preliminary result. We prove the following:

If an entire affine line L is contained in D and if H is a hyperplane containing L, then perpendiculars to H along L are coplanar.

Proof. Suppose that p is a point on a perpendicular to H at a point q of L. Let P denote the two-plane containing p and the line L. Then if $pq = \alpha$, the hyperplane $E(H, \overleftarrow{\alpha})$ through p meets P in a line L', say. Since the set D is convex and since L is an entire affine line contained in it, the plane P meets D in a strip or a half plane bounded by a line parallel to L or else P meets D in an entire two-plane. Hence L' is the only line through p in P which does not meet L and hence L' is the affine parallel of L in P passing through p.

Now if for a certain $x \in L'$, $\beta = xL > \alpha$ then consider the supporting hyperplane H' to $E(L, \overleftarrow{\beta})$ at $x \in E(L, \overleftarrow{\beta})$. Then H' intersects P in a line L'' and since every point of H' is at least xL units away from L, L'' and L are also nonintersecting and are affine parallels of each other. Since the point x is common to both L' and L'', we have $L' = L''$. Then $\alpha = pL \geqq xL > \alpha$, which is a contradiction. Hence for all $x \in L'$, $xL = \alpha$ or this implies that L and L' are equidistant. Since $L' = P \cap E(H, \overleftarrow{\alpha})$ we have proved that the perpendiculars to H along L all meet L' and hence they are coplanar.

We return to the proof of Theorem 1 in the n-dimensional case. By using arguments similar to those in the 2-dimensional case we can show that the set D in which the geometry can be defined is projectively equivalent to either a simplex, a cylinder with r-dimensional generators, a strip between two hyperplanes, or the entire affine space. We show that the geometry cannot be defined in the simplex or the cylinder, the proof of the impossibility of the strip is similar to that in the 2-dimensional case. See also § 5 for a more detailed proof of the case of the strip.

In the case of the simplex also, we need consider only the 3-dimensional case because the higher dimensional case is similar. We first show that whenever a hyperplane passes through a vertex all its equidistants also pass through the same vertex. For a proof, let the tetrahedron be $OABC$ and assume that the hyperplane H passes through A. Since the images of A under the projectivities are collinear, either all equidistants on the side $\sigma^-(H)$ or on the side $\sigma^+(H)$ pass through A. We show that both these events happen. Otherwise, suppose that all equidistants on $\sigma^-(H)$ pass through A and that there is a last equidistant H_0 on $\sigma^+(H)$ which passes through A. Let $\sigma^-(H_0)$ contain H and draw $H_1 \subset \sigma^+(H_0)$ through A. Then all equidistants of H_1 on $\sigma^+(H_1)$, $\sigma^+(H_1) \not\supset H_0$, pass through A. Let H_2 be such an equidistant; say $H_2 = E_+(H_1, \alpha^{\rightarrow})$. Then since H_0 and H_2 are separated by H_1 from which H_2 is equidistant we have

$$\beta = \inf\{x_0x_2 | x_i \in H_i\} > 0.$$

Now $H^* = E_+(H_0, \beta/2)$ must pass through A because otherwise H^* would contain points on $\sigma^+(H_2)$ and all points x on $\sigma^+(H_2)$ satisfy $H_0x > \beta$. However, that H^* passes through A is a contradiction to the hypothesis that H_0 is the

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last of the associated equidistants in a family of equidistants of H which pass through A.

We use the above result to prove that the geometry cannot be defined in a simplex. For an indirect proof, assume that the geometry is defined inside a simplex $OABC$. Then if H is a plane AQP with (CQB) and (CPO), all $E(H, \alpha)$ pass through A and therefore the projectivities between H and its equidistants induce perspectivities on all lines in H which pass through A because A is self-corresponding. The center of perspectivity of these perspectivities must be the point C because otherwise an interior point x and its successive images would not be collinear. This implies that the projectivity between H and its equidistant must be a perspectivity from C because the projectivity coincides with the perspectivity for four suitable points of H, no three of which are collinear.

Now, since the projectivities depend continuously on H and since the plane AOR with (CRB), say H_0, is a limit of planes H for different P and Q, we see that the projectivities between H_0 and its associated equidistants are also perspectivities from C. But then the plane H_0 can also be approached through planes $H^* = AQP$ with (CQB) and (OPB), and by the same arguments as given above for planes H, the projectivities between H^* and its associated equidistants are also perspectivities, this time from the point B. This implies that the projectivities between H_0 and its associated equidistants are perspectivities from B. Putting the above results together, the projectivities between H_0 and its associated equidistants are perspectivities from C as well as from B. This is impossible and hence the geometry cannot be defined inside a simplex.

Assume next that the geometry is defined inside a cylinder D with r-dimensional generators. We introduce the notion of an extreme generator. We say that a generator G is an extreme generator if it is not contained in the interior of a portion of an $(r + 1)$-dimensional flat on the boundary. For example, if D is a cylinder in the 3-space based on a quadrilateral then the generators which pass through the vertices of the quadrilateral are extreme generators. A generator is extreme if and only if an $(n - r)$-dimensional section of the cylinder meets the generator in an extreme point of the cross section.

A section of the cylinder by an $(n - r)$-dimensional flat which does not contain a generator, i.e., which intersects the generators in points only, is a compact section and we can prove, by a method exactly similar to that in § 3 that this section has only $(n - r + 1)$ extreme points. Consequently, there are only a finite number of extreme generators. Hence if G is an extreme generator and H is a hyperplane, $G \not\subset H$, then the set $G \cap H$ goes into $G \cap E(H, {}^{\leftarrow}\alpha^{\rightarrow})$ under the projectivities. This is because points of an extreme generator must go into points of an extreme generator only, by linearity, while they cannot go into points of a different extreme generator because the projectivities change continuously and there are only a finite number of extreme generators.

Now unless the set D is a strip enclosed by two parallel hyperplanes, there is at least one extreme generator, G say. We have already considered the case of the strip, so we assume the existence of an extreme generator G. Let F be the portion of an $(r+1)$-dimensional flat contained in D which is spanned by G and an r-flat in D parallel to G. Then F is actually an $(r+1)$-dimensional strip or a half-space but not an entire $(r+1)$-dimensional affine flat. However, we will show that F has the equidistant locus property and this will provide the necessary contradiction because, since $(r+1) < n$, the induction hypothesis implies that if F has the equidistant locus property then F is an entire affine flat.

To show that F has the *ELP*, let G_0 be an r-dimensional flat, i.e., a hyperplane in F and let p be a point in $F - G_0$, q be the terminal (or initial) foot of p on G_0, and let L be the line joining p and q. By using the convexity of balls we can find a hyperplane H containing G_0 such that L is perpendicular to H at q. We show that all the perpendiculars to H at points of G_0 lie in F. Suppose then that $x \in G_0$. If the line L_1 joining q and x does not meet the generator G then it is an entire affine line and all the perpendiculars along it are coplanar. On the other hand if the line L_1 meets the generator G in a point r say, then since G is an extreme generator, under the projectivities r goes again into a point s of G only as observed above. Then the line L_1 goes into the line L_2, say, joining p and s and since p and s are in F the entire line L_2 is also in F. Since $x \in L_1$, this proves that the perpendicular to H at x lies in F. We have thus shown that the perpendiculars to H at points of G_0 lie in F. Therefore $E(G_0, {}^{\leftarrow}\alpha^{\rightarrow})$ are precisely $E(H, {}^{\leftarrow}\alpha^{\rightarrow}) \cap F$, thus proving that F has the equidistant locus property. This provides the necessary contradiction to our geometry being defined in a cylinder. Thus the geometry can only be defined in the entire affine space A^n.

That the geometry is Minkowskian now follows from [**1**, Theorem 24.1, p. 144].

5. Nonsymmetric distance. When the distance of the space is not necessarily symmetric and when the balls $xp \leqq \rho$ are not assumed to be compact, the straight lines of the space are maps $x(t)$, $\alpha < t < \infty$, $x(s)x(t) = t - s$ for $t > s$, where α need not equal $-\infty$. Also, initial feet of points on closed sets need not exist. Our proof of Theorem 1 therefore needs important changes. The main parts are the proof that the equidistant locus property implies that every point has a unique initial foot on every hyperplane and the proof that the geometry cannot be defined inside a strip.

Since a complete proof would repeat many of the methods of the proof of the previous sections, we only outline the modifications necessary. Thus by using the compactness of the balls $px \leqq \rho$ and a continuity argument, we can show that the relation of "being equidistant" is an equivalence relation, the terminal foot (and the initial foot when existing) of a point on a hyperplane is unique, and that all compact spheres are convex.

Since all the balls $px \leqq \rho$ are compact, the terminal foot of a point on a hyperplane always exists. We show that the initial foot also exists. Since $H = E_\pm(H', \alpha^\rightarrow)$ if and only if $H' = E_\pm(H, {}^\leftarrow\alpha)$, it suffices to show that the map which takes $p' \in H' = E_+(H, {}^\leftarrow\alpha)$ into its t-foot p on H is a bijective topological map. The proof is quite long and is based on families of equidistant hyperplanes. We put $H_{-\alpha} = E_+(H, {}^\leftarrow\alpha)$, $H_\alpha = E_-(H, \alpha^\rightarrow)$ and $H_0 = H$. Each point of the space lies on exactly one H_α. If $\beta > \alpha$, $H_\beta \subset \sigma^+(H_\alpha)$, then $H_\beta = E_-(H_\alpha, (\beta - \alpha)^\rightarrow)$ and $H_\alpha = E_+(H_\beta, {}^\leftarrow(\beta - \alpha))$. Since successive t-feet of a point on H_α are collinear, a perpendicular corresponding to the family $\{H_\alpha\}$ is a line $y(t), \gamma_y < t < \infty$, with $y(t_1)y(t_2) = t_2 - t_1$, $y(\alpha) \in H_\alpha$ such that for $\beta > \alpha$, the point $y(\beta)$ is the t-foot of $y(\alpha)$ on H_β and $y(\alpha)$ is the i-foot of $y(\beta)$ on H_α. We prove that γ_y is independent of the perpendicular $y(t)$. This implies that each perpendicular meets each H_α and this in turn implies the existence of an initial foot on every hyperplane of every point.

We show in fact that $\gamma_y = \operatorname{Sup}\{xH | x \in \sigma^-(H_\alpha)\}$. It suffices to show this: If $y(\alpha) = p$ and H_β is defined for $\beta < \alpha$ then $y(t)$ is defined for $t < \beta$. (We use here the fact that $\beta = \operatorname{Sup} xH$ is impossible.)

If p has an i-foot on H_β then there is nothing to prove. We show that we reach a contradiction if p does not have an i-foot on H_β. Choose first $\beta < \gamma < \alpha$ and so close to α that p has an initial foot q on H_γ. Since $H_\beta p = \alpha - \beta$, we have a sequence $\{f_\nu\}$ in H_β such that $f_\nu p \to \alpha - \beta$. We show that $f_\nu \to f$, a point on the boundary of D (since p is assumed not to have an initial foot on H_β), and that p, q, f are collinear. We show later on that this leads to a contradiction.

By uniqueness of feet we have $xp > \alpha - \gamma$ in $x \in H_\gamma - \{q\}$. Hence $T(f_\nu, p)$ intersects H_γ in a point q_ν. Then $q_\nu p \geqq \alpha - \gamma$ with the equality holding only for $q_\nu = q$. Since $f_\nu p \geqq \gamma - \beta$ we have $q_\nu p \to \alpha - \gamma$, $f_\nu p \geqq \gamma - \beta$. Hence if $\{q_\nu\}$ has an accumulation point in H_γ, it can only be q. To show that $q_\nu \to q$ it must be shown that no subsequence $\{q_\mu\}$ of $\{q_\nu\}$ can tend to a point of the boundary of H_γ or to a point at infinity in A^n.

The set $C = \{x | qx = 1, x \in H_\gamma\}$ is compact and hence $\min\{xp | x \in C\} = \delta > \alpha - \gamma$. Consider the line L_μ through q and q_μ where μ is so large that $\alpha - \gamma < q_\mu p < \delta$. The line L_μ intersects C in a_μ and b_μ, say, and $q_\mu \notin T(a_\mu, b_\mu)$ because, otherwise, $\{q_\mu\}$ would have an accumulation point.

Now $a_\mu p \geqq \delta > \alpha - \gamma = qp$. Hence u with $(a_\mu u q)$ and $up = q_\mu p$ exists. Similarly, v with (qvb_μ) and $vp = q_\mu p$ exists. But then $\bar{S}^-(p, q_\mu p)$ would not be convex, since it meets L_μ in three points. Therefore $q_\nu \to q$ and the line through f_ν and p converges. Set $f = \lim f_\nu$ where f may lie at infinity or on the boundary of D. Then the above argument shows, for $\beta > \beta_1$, that $f \in H_{\beta_1}$ also. We show that this leads to a contradiction. Let $f \in F = H_\beta \cap H_{\beta_1}$ and put $L = L(f, q)$. Let H^* be the hyperplane containing L and F. We show that H^* is equidistant from H_{β_1}. Since H_β separates H^* and H_{β_1}, $\inf\{yx | y \in H_{\beta_1}, x \in H^*\} = \delta > 0$. Take p with $\delta_1 = H_{\beta_1}p < \delta$ and $\delta_2 = pH^* < \delta$. Such a p is, for example, the midpoint of $T(y, x)$, $y \in H_{\beta_1}$, $x \in H^*$, $yx < 3\delta/2$. Then $E_+(H_{\beta_1}, {}^\rightarrow\delta_1)$ must pass through F and p because, otherwise, it contains points

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at a distance greater than δ_1 which lie on the side of H^* containing H_γ. Similarly, $E_-(H^*, {}^{\leftarrow}\delta_2)$ passes through F and p showing $E_-(H^*, {}^{\leftarrow}\delta_2) = E_+(H_{\beta_1}, \delta_1{}^{\rightarrow})$. By transitivity, H^* and H_{β_1}, and hence H_γ, are equidistant. But H^* cannot be equidistant to H_γ since $L \subset H^*$ and L intersects H_γ in q. This completes the proof that γ_y is independent of y.

The existence of initial feet of points on hyperplanes now follows as oberved above. Thus if ϕ denotes the map which sends a point $x \in H_1$ to its terminal foot on $H_2 = E_+(H_1, \alpha^{\rightarrow})$ then ϕ is a bijective map. That ϕ is continuous follows from the compactness of $px \leqq \delta$ while the continuity of ϕ^{-1} can be proved using ideas similar to those in the proof of the independence of γ_y and y.

The existence of initial feet clears the way to extend other geometric properties and the classification of domains to the nonsymmetric case with methods similar to those in the symmetric case. Thus the geometry can be defined inside a simplex, a cylinder with r-dimensional generators, the strip between two hyperplanes, or the entire affine space. The impossibility of the simplex and the cylinder with r-dimensional generators can be proved in the same manner as the symmetric case. We outline the proof of the impossibility of the strip in the 2-dimensional case since the proof in the n-dimensional case is similar.

In the following we denote, for a line L and a point p, by $\mathscr{R}^+(L, p)$ a ray of L bounded by p and oriented in a direction going away from p, and by $\mathscr{R}^-(L, p)$ a ray of L bounded by p and oriented in a direction coming towards p. Since the closed positive balls are compact, the rays $\mathscr{R}^+(L, p)$ always have infinite length, while the rays $\mathscr{R}^-(L, p)$ may have finite length. We denote by Λ and Λ' the lines which bound the strip D. For a line L with sides $\sigma^\pm(L)$ we define $\beta^+(L) = \operatorname{Sup}\{xL, x \in \sigma^+(L)\}$, $\beta^-(L) = \{xL, x \in \sigma^-(L)\}$.

1. *Let P be a line affinely parallel to Λ and let $p \in P$. If a ray $\mathscr{R}^-(P, p)$ has finite length, then we get a contradiction.*

Proof. Let length $\mathscr{R}^-(P, p) = \rho < \infty$. We can assume without loss of generality that $\rho < \rho' < \beta^\pm(P) = \operatorname{Sup}\{xP | x \in R\}$, because if not, we can move the point p for a suitable distance. Then the disc $S^-(p, \rho')$ contains all of $R^-(P, p)$ since $\rho < \rho'$ while because $\rho' < \beta^\pm(P)$, there exist points q in the space such that $qP > \rho'$; in particular, $qp > \rho'$. Let $\bar{\rho} = qP$ and draw through q the equidistant P' to P at a distance $\bar{\rho}$. Then P' is the affine parallel to P through q. Therefore all lines L which meet Λ have their rays oriented towards p of length at least ρ. By convexity of spheres, we find a line H such that its incoming perpendicular at p say L meets Λ and lies on the same side of H as contains $\mathscr{R}^-(P, p)$. This leads to a contradiction because then we can draw $H_1 = E(H, {}^{\leftarrow}\rho)$ meeting $\mathscr{R}^-(P, p)$ in r, say. Then $\rho =$ length of $\mathscr{R}^-(P, p) > rp \geqq rH = \rho$, which is a contradiction. This proves 1.

2. *If there exists a line L meeting Λ and a perpendicular G of L also meeting Λ with, say, $\beta^+(L) = \infty$, then also we get a contradiction.*

Proof. Suppose $x = L \cap \Lambda$, $y = G \cap \Lambda$, $x' = L \cap \Lambda'$, and $y' = G \cap \Lambda'$. Let N be the line joining x' to y oriented towards x' and represented by $x(t)$. Let M join x to y'. Let the incoming perpendicular to M through p meet N in r, let $r = x(t_0)$, and assume $r \in \sigma^+(M)$.

We know that the function $x(t)M$ is strictly monotonic. We prove that it decreases in $(\alpha(N), t_0)$ and increases in (t_0, ∞). This will give us the necessary contradiction.

If $x(t)$ does not decrease in $(\alpha(N), t_0]$, then it increases there and so $\beta = x(t_0)M = \max\{x(t)M | t \in (\alpha(N), t_0]\}$. Then if $E_+(M, {}^{\leftarrow}\beta) = M_1$, the ray of N joining y to r is contained between M_1 and M. Hence if u is any point on $\mathscr{R}^+(y, p)$ and if the incoming perpendicular to M from u meets L in v, then we have $up < uv < \beta$. This is a contradiction to the hypothesis that the length of the perpendicular joining y to p is ∞. Thus $x(t)M$ decreases in $(\alpha(N), t_0]$. We can similarly prove that $x(t)M$ increases in $[t_0, \infty)$. These two conclusions are incompatible with $x(t)M$ being strictly monotonic. This proves the assertion 2.

3. *If the hypotheses of* 1 *and* 2 *do not hold, then also we get a contradiction. Thus our geometry cannot be defined in a strip.*

Proof. Let P be parallel to Λ and $p \in P$. Let L be a transversal to P through p. Since not all lines through p can be transversal to P, we have line L_1 and its perpendicular P_1 through p such that L_1 and P_1 both meet Λ. Let $\sigma^+(L)$ contain the outgoing rays of P and P_1.

Then since the hypotheses of 1 and 2 do not hold, length $\mathscr{R}^-(P, p)$ is equal to ∞ and $\beta^-(L_1)$ is a finite number. Put $q = L_1 \cap \Lambda$, $r = L \cap \Lambda$. Let $x \in \mathscr{R}^-(P, p)$ and x_1 be the terminal foot of x on L_1 and x_2 be the terminal foot of x_1 on L. Then $xp \leqq xx_2 \leqq xx_1 + x_1x_2 < \beta^-(L_1) + x_1x_2$. Since $xp \to \infty$ as x recedes on $\mathscr{R}^-(P, p)$ we see that $x_1x_2 \to \infty$ as x recedes on $\mathscr{R}^-(P, p)$, or which is the same, as $x_1 \to q$ in the affine sense.

But the lines joining x_1 and x_2 are parallel to Λ and so they are perpendicular to L. Thus $x_1x_2 \to \infty$ implies that the triangular region Δpqr contains portions which have arbitrarily large lengths. Therefore every $E_-(L, {}^{\leftarrow}\alpha)$ meets the domain Δpqr. Hence the length of the ray joining q to p has infinite length. So also all rays joining s to p have infinite length where s is any point outside Δpqr and $s \in \sigma^-(L)$. We can therefore choose such a ray which is perpendicular to a line meeting Λ. This is a contradiction to the assumption that the hypothesis of 2 does not hold.

This completes the proof that our geometry cannot be defined inside a strip.

The proof that the geometry is Minkowskian is now easy to complete. As in the case of the strip we can prove that every receding ray of every line has infinite length; consequently, all the balls $xp \leqq \rho$ must also be compact. Thus our space is defined in the entire affine space, it is finitely compact, and all the balls are convex. Hence by [**4**, Theorem 10.3], the space is Minkowskian.

6. The cylindrical tubes property. In this section we prove Theorem 2 by showing that the cylindrical tubes property implies the equidistant locus property. The proof is quite long and the first step is to prove that the cylindrical tubes property implies the convexity of spheres.

Observe first that the convexity of the tubes implies the convexity of $C(H, {}^{\leftarrow}\alpha^{\rightarrow})$. In fact if (pqr) and $\alpha = \max(Hp, Hr)$, then for every integer n there exist points p_n and r_n in H such that $p_np, r_nr < \alpha + 1/n$. Then $p, r \in C(L_n, (\alpha + 1/n)^{\rightarrow})$ where L_n denotes the line joining p_n and r_n. By convexity of $C(L_n, \alpha + 1/n)$ we have $q \in C(L_n, (\alpha + 1/n)^{\rightarrow})$. Therefore $Hq \leqq L_nq \leqq \alpha + 1/n$ for each n which shows that $Hq \leqq \alpha$. This proves the convexity of $C(H, \alpha^{\rightarrow})$. We can similarly prove that $C(H, {}^{\leftarrow}\alpha)$ are also convex. We use this result to show that the set of terminal feet of a point on a line is a connected set. To show this let $pL = pq = pr = \alpha$ and $pu = \max\{px|x \text{ with } (qxr)\} = \beta$. We show that $\alpha = \beta$. Since $pL = \alpha$, we know that $\beta \geqq \alpha$. We show that $\beta = \alpha + 2\epsilon$ with $\epsilon > 0$ leads to a contradiction. Let H support $E(L, {}^{\leftarrow}\alpha)$ at p and put $L_1 = H \cap P$, where P is the plane of the triangle pqr.

Then $L_1u \leqq \max(L_1q, L_1r) \leqq \alpha$. Therefore there exists a point $v \in L_1$, $vu < \alpha + \epsilon$. Let x be the point in which $T(u, v)$ meets, say, $T(p, r)$. We distinguish two cases.

Case 1. If $px \leqq vx$, we have $\alpha + \epsilon < pu < px + xu \leqq vx + xu = vu < \alpha + \epsilon$, which is a contradiction.

Case 2. If $px > vx$, we have $vr < vx + xr < px + xr = pr = \alpha$. But $vr < \alpha$ is a contradiction to v belonging to the supporting plane of $E(L, {}^{\leftarrow}\alpha)$.

Thus in either case we get a contradiction. This proves that $pu = \beta = \alpha$. Hence u is also a foot of p. Therefore the set of terminal feet of a point on a line is connected and so the balls $px \leqq \rho$ are convex as observed in § 2. We can prove similarly that the set of initial feet of a point on a line is a connected set and hence the balls $xp \leqq \rho$ are convex whenever they are compact.

We need in fact a stronger result that the balls are strictly convex. To show this it suffices to show that every point has a unique terminal foot and at most one initial foot on every line. We give the proof for the initial feet. Suppose p has two feet q and r on L and let $qp = rp = \alpha$. Let L_1 be the intersection of a supporting plane H to $E(L, \alpha^{\rightarrow})$ and the plane of the triangle pqr. We can find a point x on a curve C in this 2-plane, joining two points on L_1, such that p is an initial foot of x on L_1 and such that x lies on the side of L_1 in the 2-plane pqr which does not contain L. Then if, say, x, p, q are not collinear we have $qx < qp + px$, while if $T(q, x)$ meets L_1 in y, $yx \geqq px$ and $qy \geqq \alpha = qp$ shows that $qx = qy + yx \geqq qp + px$. Thus x, p, q must be collinear. Similarly x, p, r are collinear. This implies $q = r$. Thus every point has at most one initial foot on every line. Consequently, all compact $xp \leqq \rho$ are strictly convex. Similarly, all $px \leqq \rho$ are strictly convex.

We need to show that the sets $E_{\pm}(H, {}^{\leftarrow}\alpha^{\rightarrow})$ are hyperplanes. The proof that $E_{\pm}(H, {}^{\leftarrow}\alpha)$ are hyperplanes is shorter. We show that $p, q \in E_{\pm}(H, {}^{\leftarrow}\alpha)$ implies that $T(p, q)$ is contained in $E_{\pm}(H, {}^{\leftarrow}\alpha)$. An argument similar to that of [**1**, 24.14] then shows that $E_{\pm}(H, {}^{\leftarrow}\alpha)$ are hyperplanes. Suppose then that $pH = pr = qH = qs = \alpha$ and (pxq). We have to show $xH = \alpha$. Let $\min\{xH | (pxq)\} = x_0H = x_0y_0 = \beta$. By convexity, $\beta \leqq \alpha$. Hence it suffices to show that $\beta < \alpha$ leads to a contradiction. Suppose there is a γ such that $\beta < \gamma < \alpha$. Let L_1 and L be the lines joining p, q and r, s, respectively. Then since the supporting hyperplane to $E(H, {}^{\leftarrow}\alpha)$ at p cannot meet $T(q, s)$ at a point other than q for all $x \in L_1$ to the left of p, we have $xH \geqq \alpha$. Similarly, for all y with y to the right of q we have $yH \geqq \alpha$. Hence H supports $E(L_1, \beta^{\rightarrow})$ at y_0. Therefore there exists a line $L_{y_0} \subset H$ such that for all $u \in L_{y_0}$, we have $L_1u = \beta$. Hence for each $u \in L_{y_0}$, there exists a $p_u \in L_1$ such that $p_uu < \gamma$. Also, because of the observation above, $p_u \in T(p, q)$. We show that this leads to a contradiction. Now we have $y_0u \leqq y_0x_0 + x_0p_u + p_uu \leqq y_0x_0 + \max(x_0p, x_0q) + \gamma$ = a number δ independent of u. But $y_0u \to \infty$ as u traverses L_{y_0}, and hence the contradiction.

This shows that $p, q \in E_{\pm}(H, {}^{\leftarrow}\alpha)$ implies that $T(p, q) \subset E_{\pm}(H, {}^{\leftarrow}\alpha)$ and this in turn implies, as observed above, that the sets $E_{\pm}(H, {}^{\leftarrow}\alpha)$ are hyperplanes.

It remains to show that $E(H, \alpha^{\rightarrow})$ are hyperplanes. We break the complicated proof into several parts.

1. (Transitivity of equidistants.) *Let H_1, H_2, H_3 be three hyperplanes with H_1, H_3 lying on opposite sides of H_2. Suppose that $H_2 = E(H_3, {}^{\leftarrow}\beta)$ and that $H_1 = E(H_2, {}^{\leftarrow}\alpha)$. Then $H_1 = E(H_3, {}^{\leftarrow}\gamma), \gamma = \alpha + \beta$.*

Proof. For any $x \in H_1$, let y be the t-foot of x on H_2 and z the t-foot of y on H_3. Then $xy = \alpha$, $yz = \beta$. On the other hand, for any two points w, w_1 in H_1, H_3, respectively, we have if $T(w, w_1)$ meets H_2 in w_2, $ww_2 \geqq \alpha$, while $w_2w_1 \geqq \beta$. Thus x, y, z are collinear and $xz = xy + yz = \gamma$. This proves 1.

2. *If $Hp = qp = \alpha$ and H' is a supporting hyperplane to $E(H, \alpha^{\rightarrow})$, then $H = E(H', {}^{\leftarrow}\alpha)$.*

Proof. Since $qH' = qp = \alpha$, we draw $E(H', {}^{\leftarrow}\alpha) = H''$ through q. Then H'' is a hyperplane. If $H'' \neq H$ then H'' intersects H and there exists a point $x \in H''$ such that x lies on the side $\sigma^+(H)$ of H which does not contain H'. Then if y is the terminal foot of x on H' and if $T(x, y)$ meets H in z, then $xy = \alpha$ while $xz > 0$ so that $zy < \alpha$. This is a contradiction to $y \in H'$ and H' being a supporting hyperplane of $E(H, \alpha^{\rightarrow})$.

3. *If $up = Hp = Hq = vq = \alpha$, then for all x with (pxq) we have $Hx = \alpha$.*

Proof. Let H' support $E(H, \alpha^{\rightarrow})$ at p. If $q \in H'$ we have nothing to prove, because then $x \in H'$ and since (pxq), $Hx \leqq \alpha$ while $x \in H'$ gives $Hx \geqq \alpha$. Then $H = E(H', {}^{\leftarrow}\alpha)$, and since $q \notin H'$, there exists $r \in H'$ with $vr = \alpha$, $r \neq q$. Let L be the line in which the plane of triangle rvq meets the hyperplane H. Then there exists a point w on the side of H which does not contain

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p such that the t-foot of w on L is v. Joins $T(w, q)$ and $T(w, r)$ meet L in t and s respectively. Since $wv \leqq ws$ and $wv \leqq wt$, we have either $vr > sr$ or $vq > tq$. However, $vr > sr$ implies $sr < \alpha$ which contradicts $r \in H'$ and H' supporting $E(H, \alpha^{\rightarrow})$, while $vq > tq$ contradicts v being the initial foot of q on H. This proves 3.

4. *Let* $H = E(H_1, {}^{\leftarrow}\alpha)$ *and* $C = \{x | x \in H_1, x$ *is the t-foot of some point on* $H\}$. *Then the set* C *is convex, open, and* $(n - 1)$*-dimensional.*

Proof. The proof of this can be obtained along the same lines as in § 5. The convexity of C then follows from 3 and from the fact that if x is t-foot of y, then y is i-foot of x.

5. *Suppose* L_1 *and* L_2 *are two lines and* $L_2p = qp = \alpha$ *and* $L_2u = \alpha$ *for all* $u \in L_1$. *Then either*: (*i*) *a ray* $\mathscr{R}$ *of* L_2 *from* q *is equidistant from* L_1, *i.e., for each* $v \in \mathscr{R}$, $vL_1 = \alpha$, *or* (ii) *each point* $u \in L_1$ *has an initial foot on* L_2.

Proof. If (i) does not hold, then there exist two points r, s on L_2 one on each side of q such that rL_1, $sL_1 > \alpha$. Suppose that rL_1, $sL_1 > \alpha + \epsilon$. Draw through r a supporting hyperplane H to $E(L_1, {}^{\leftarrow}rL_1)$. Since $qp = \alpha < \alpha + \epsilon$, the plane H cannot contain q. Hence for every x with (xrq), we have $xu > \alpha + \epsilon$. Similarly, we can prove that for every $u \in L_1$ and y with (qsy), $yu > \alpha + \epsilon$. Denote by T the segment joining r and s. Now if $u \in L_1$, then $L_2u = \alpha$; hence there exists a sequence $\{u_n\}$ $u_n \in L_2$ such that $u_nu < \alpha + 1/n$. Thus for all large n, $u_n \in T$. Since T is compact, $\operatorname{Sup}\{ut | t \in T\} < \infty$, so there exists a ρ such that for all sufficiently large n, $u_n \in S^+(u, \rho)$. Hence $\{u_n\}$ has an accumulation point u_0, say. Since $u_0u = \alpha$, u_0 is the initial foot of u on L_2. Since u is any point of L_1 we have proved the assertion.

6. *Let* H *and* H_1 *be two hyperplanes such that* $E(H_1, {}^{\leftarrow}\alpha) = H$. *Then* $H_1 = E(H, \alpha^{\rightarrow})$.

Proof. Let p be the terminal foot of a point $q \in H$ on H_1. Then $qp = \alpha$. Since for any two points x, $y \in H$, H_1, respectively, we have $xy \geqq \alpha$, H_1 supports $E(H, \alpha^{\rightarrow})$ at p. Also, since $qp = \alpha$, q is the initial foot of p on H.

Given any line L_1 through q in H, $p \in E(L_1, \alpha^{\rightarrow})$. Therefore there exists L_1' through p in H_1 such that L_1' is equidistant to L_1; i.e., for all $u \in L_1'$, $L_1u = \alpha$.

We show that different lines L_1, L_2 in H correspond to different lines L_1', L_2' in H_1. This is because, if L_1, L_2 contained in H, $L_1 \neq L_2$, correspond to the same line L_1' we reach a contradiction. In fact, in that case, by 5, there exist segments T_1, T_2 (formed by initial feet of points of L_1') on L_1, L_2, respectively, such that each point of T_1 and T_2 is at a distance α from L_1'. Since H supports $E(L_1', {}^{\leftarrow}\alpha)$ and since $E(L_1', {}^{\leftarrow}\alpha)$ is convex, the convex hull of $T_1 \cup T_2$ is also equidistant from L_1'. Let U be an open 2-dimensional set in $\operatorname{Conv}(T_1 \cup T_2)$ and define $\phi: U \rightarrow L_1'$ by sending $x \in U$ to the terminal foot of x on L_1'.

Then by 4, ϕ is a homeomorphism of U onto its image. However, this is a contradiction because U is 2-dimensional and L_1' is 1-dimensional. Thus L_1, L_2 cannot both correspond to L_1'.

Hence as L_1 sweeps out H, L_1' sweeps out H_1. Consequently, every point x of H_1 satisfies $Hx = \alpha$. Hence $H_1 = E(H, \alpha^{\rightarrow})$.

7. $E(H, \alpha^{\rightarrow})$ *are hyperplanes.*

Proof. Let $p \in E(H, \alpha^{\rightarrow})$ and take any point q on H. Let $T(q, p)$ be represented by $x(t)$, $0 \leqq t \leqq \beta$. Put $C = \{Hx(t)|$ there exists H_t such that $H = E(H_t, {}^{\leftarrow}(Hx(t)))\}$. Then for small t, $x(t)$ has an initial foot on H and then H_t can be taken to be a supporting hyperplane to $E(H, Hx(t)^{\rightarrow})$. Then $H = E(H_t, {}^{\leftarrow}(Hx(t)))$. Hence $t \in C$ for small t.

If $Hx(t_n) \in C$ and $t_n \to t$, then let $H_{t_n} \to H_t$ through a subsequence if necessary. Then taking limits in the equation $H = E(H_{t_n}, {}^{\leftarrow}(Hx(t_n)))$ as $n \to \infty$, we have by continuity of distance, $H = E(H_t, {}^{\leftarrow}(Hx(t)))$. Hence the set C is closed.

We show that Sup $C = \alpha$. If Sup $C = \gamma < \alpha$, then for small ϵ there exists on the side of H_γ not containing H, a hyperplane H_ϵ such that $H_\gamma = E(H_\epsilon, {}^{\leftarrow}\epsilon)$. Then the equations $H = E(H_\gamma, {}^{\leftarrow}\gamma)$ and $H_\gamma = E(H_\epsilon, {}^{\leftarrow}\epsilon)$ imply that $H = E(H_\epsilon, {}^{\leftarrow}(\gamma + \epsilon))$. This is a contradiction to the assumption that Sup $C = \gamma$.

Thus given H, there exists H_1 such that $H = E(H_1, {}^{\leftarrow}\alpha)$. By 6, then we have $H_1 = E(H, \alpha^{\rightarrow})$. Therefore $E(H, \alpha^{\rightarrow})$ is a hyperplane for all α.

The results show that the cylindrical tubes property implies the equidistant locus property which in turn implies that the geometry is Minkowskian.

7. The proof of Theorem 3. We now assume that all the balls $px \leqq \rho$ as well as $xp \leqq \rho$ are compact, that all $E_{\pm}(H, {}^{\leftarrow}\alpha)$ are hyperplanes, and that all tubes $E(L, {}^{\leftarrow}\alpha^{\rightarrow})$ are convex. To show that the geometry is Minkowskian it suffices to show that if $H' = E(H, {}^{\leftarrow}\alpha)$, then $H = E(H', \alpha^{\rightarrow})$, because this shows that the sets $E(H, \alpha^{\rightarrow})$ are also hyperplanes, It can be proved as in the last section, using the convexity of the tubes, that all spheres are strictly convex. Hence from [**4**, § 8], the perpendiculars exist. Hence given any $x \in H$, on the incoming perpendicular to H at x we can find a point $y \in H'$. Then x is the terminal foot of $y \in H'$. Since $H' = E(H, {}^{\leftarrow}\alpha)$ we have $yx = yH = \alpha$. This implies that $H'x = \alpha$ also, because for any $u \in H'$, $ux \geqq \alpha$ while $yx = \alpha$. This proves that $H = E(H', \alpha^{\rightarrow})$. The converse can be proved in a similar manner. Thus the hypothesis of Theorem 3 implies the hypothesis of Theorem 1. This proves Theorem 3.

8. Weaker flatness conditions. In closing, we give two examples to show that if we simultaneously weaken the hypotheses regarding the compactness and the equidistant locus property, then we get a large number of solutions which are geometrically of little interest.

Consider, for example, a metrization of the upper half plane as follows. Denote by $e(p, q)$ the Euclidean distance between two points $p = (p_x, p_y)$ and $q = (q_x, q_y)$ and let $d(p, q)$ denote the number $|1/p_y - 1/q_y|$. Let $d^*(p, q) = e(p, q)$ if the oriented ray joining p to q does not meet the x-axis and $d^*(p, q) = d(p, q) + e(p, q)$ if the oriented ray joining p to q meets the x-axis. One can show that the function d^* is a metric which makes the upper half plane a nonsymmetric desarguesian plane with only the balls $px \leqq \rho$ compact for all $\rho > 0$. Also, given any line L and any number $\alpha > 0$ at least one of the four sets $E_{\pm}(L, {}^{\leftarrow}\alpha^{\rightarrow})$ is always a line because by choosing the direction suitably only the Euclidean distance enters.

Our other example is a modification of the Funk distance [**3**]. Using the same notation as above, define $f(p, q) = \log(e(p, u)/e(u, q))$ if u is the point in which the oriented ray joining p to q meets the x-axis. Then the definition $f^*(p, q) = e(p, q)$ if the oriented ray joining p to q does not meet the x-axis, and $f^*(p, q) = f(p, q) + e(p, q)$, otherwise, gives a metric which also has the same properties as the first example above.

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