# ON SCHOENBERG'S RATIONAL POLYGON PROBLEM

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## 1. Introduction

A polygon is said to be rational if all its sides and diagonals are rational, and I. J. Schoenberg has posed the difficult question, 'Can any given polygon be approximated as closely as we like by a rational polygon?' Many of the known results concerning this question are contained as special cases in theorem 1 below which was proved by one of us (cf. the references).

THEOREM 1. (T. K. Sheng) *Let $CB$ be a diameter of the unit circle $\mathscr{C}$ with centre $O$, and let $D$ be a point on $CB$ (produced if necessary). Then given $\varepsilon > 0$, there exists on $CB$, a point $A$ within $\varepsilon$ of $D$ and a finite set $\mathscr{S}_\varepsilon$ of points on $\mathscr{C}$ such that:*

(i) *Given any point $E$ on $\mathscr{C}$, there exists a point $P_i$ of $\mathscr{S}_\varepsilon$ within $\varepsilon$ of $E$.*

(ii) *The set of points $\mathscr{S}_\varepsilon$ is symmetric about $CB$.*

(iii) *The polygon formed by the points of $\mathscr{S}_\varepsilon$ and the points $O$, $A$, $B$, $C$ is a rational polygon with rational area.*

The object of this paper is to extend Sheng's investigation of rational polygons on a point and a circle. Throughout we adopt the following notation. We let $AOP$ be a triangle with sides $a = OP$, $b = AO$, $c = AP$ and $2\theta$ as angle $AOP$. We denote the circle with centre $O$ and radius $a$ by $\mathscr{D}$, and we denote the set of all points $Q$ of $\mathscr{D}$ such that $AOPQ$ is a rational polygon by $\mathscr{Q}$. We allow $P$ to be in $\mathscr{Q}$. Next we let $B$, $C$ be the intersections of the circle $\mathscr{D}$ with $AO$ produced, and we let $A^*$ be the inverse image of $A$ in $\mathscr{D}$, that is to say $A^*$ is the point of $AO$ with $AO \cdot A^*O = a^2$. Finally we write $\mathscr{I}$ for the set of all square free positive integers $I$ of the form $I = (R^2+S^2)/W^2$ for some integers $R$, $S$, $W$ with $W \geqq 1$. By reduction modulo 8 one can show that if $I \in \mathscr{I}$ then $I = 1, 2, 5 \pmod 8$.

If $\mathscr{Q}$ is to have more than 2 points, then as we show in lemma 1, the set $\mathscr{Q}$ is symmetric about $OA$. There is an integer $I \in \mathscr{I}$ such that the sines and cosines of all the angles $\frac{1}{2} \angle AOQ$ with $Q \in \mathscr{Q}$ are rational multiples of $\sqrt{I}$. Moreover all the triangles formed by points of the set $\mathscr{Q} \cup \{A, A^*, O\}$ are Heron triangles, that is to say they have rational sides and area; and in the special case $I = 1$ all the triangles of the larger set

$$\mathcal{Q} \cup \{A, A^*, O, B, C\}$$

are Heron triangles. In lemma 2 we prove that $\mathcal{Q}$ can be empty. This fact is interesting because it shows that it was essential for Sheng in theorem 1 above to approximate $D$ by $A$. In lemma 3 we use the result about $\sqrt{I}$ to show that all the points of $\mathcal{Q}$ can be obtained from the rational solutions $X$, $Y$ of the equation

$$X(X+1)(X+e^2) = Y^2 \text{ where } e = (a+b)/(a-b).$$

As a particularly useful example we prove in section 6 that $\mathcal{Q}$ is dense in $\mathcal{D}$ when $b = c$. This example enables us to prove our main result which is

THEOREM 2. *Given $I \in \mathcal{I}$, the set of all Heron triangles $AOP$ for which $\sqrt{I}$ Sin $\frac{1}{2} \angle AOP$ and $\sqrt{I}$ Cos $\frac{1}{2} \angle AOP$ are rational, and for which $\mathcal{Q}$ is dense in the circle $\mathcal{D}$, is everywhere dense in the set of all triangles. Moreover*:

(i) *If $I = 1$, then any points in the set $\mathcal{Q} \cup \{A, O, P, A^*, B, C\}$ form a rational polygon with rational area.*

(ii) *If $I \neq 1$, then any points in the set $\mathcal{Q} \cup \{A, O, P, A^*\}$ form a rational polygon with rational area, but both $BQ$ and $CQ$ are irrational for $Q \in \mathcal{Q}$.*

Thus we have greatly strengthened Sheng's theorem 1 above, even though we were not able to prove our

CONJECTURE. *If $AOP$ is a Heron triangle then $\mathcal{Q}$ is dense in $\mathcal{D}$.*

We confess to having some reservations about the case when the Heron triangle is right-angled at $P$.

It is essential to work with the angle $AOP$ in the above theorem. The right angle triangle with sides 3, 4, 5 shows that the $I$ can change in a triangle from angle to angle. Also the two triangles 7, 65, 68 and 29, 65, 68 have 442 and 85 respectively as $I$ for the angle between the sides 65 and 68.

J. H. J. Almering called a point $E$ of the plane a convenient point if the distances $AE$, $OE$, $PE$ are all rational. Thus we have showed that convenient points are dense on the circle $\mathcal{D}$. Now Almering showed in [1] that convenient points are dense in the plane, but he did not require the distance between two convenient points to be rational, as we have done.

## 2. A basic property of point and circle polygons

Let $AOP$ be a given triangle and suppose that $Q_1$, $Q_2$, $Q_3$ are distinct points of $\mathcal{D}$. For $i = 1, 2, 3$ we write $2\theta_i$ for angle $AOQ_i$ and $Q'_i$ for the reflection of $Q_i$ in the line $OA$. We give a basic property of the polygon $AOQ_iQ'_i$ as

LEMMA 1. *If the 5-gon $AOQ_1Q_2Q_3$ is rational then there is a unique square free positive integer $I \in \mathscr{I}$ such that each of the six numbers* $\operatorname{Sin}\theta_i$ *and* $\operatorname{Cos}\theta_i$ *is a rational multiple of $\sqrt{I}$. Every triangle in the polygon*

$$A^*AOQ_1Q_2Q_3Q_1'Q_2'Q_3'$$

*is Heron. Also $BQ_i$, $CQ_i$ are rational iff $I = 1$.*

PROOF. For $i = 1, 2, 3$ we find, by the cosine rule, that $\operatorname{Cos} 2\theta_i$ is rational. Since $\operatorname{Cos} 2\theta_i = 1-2\operatorname{Sin}^2\theta_i = 2\operatorname{Cos}^2\theta_i-1$ it follows that there are positive square free integers $J_i$, $K_i$ and rational numbers $s_i$, $c_i$ such that

$$\operatorname{Sin}\theta_i = s_i\sqrt{J_i} \text{ and } \operatorname{Cos}\theta_i = c_i\sqrt{K_i} \text{ for } i = 1, 2, 3.$$

Now for $1 \leqq i < j \leqq 3$ the number

$$\pm Q_iQ_j/2a = \operatorname{Sin}(\theta_i-\theta_j) = s_ic_j\sqrt{(J_iK_j)}-s_jc_i\sqrt{(J_jK_i)}$$

is a non-zero rational. It follows from the identity

$$\{s_ic_j\sqrt{(J_iK_j)}-s_jc_i\sqrt{(J_jK_i)}\}\{s_ic_j\sqrt{(J_iK_j)}+s_jc_i\sqrt{(J_jK_i)}\} = s_i^2c_j^2J_iK_j-s_j^2c_i^2J_jK_i$$

that $s_ic_j\sqrt{(J_iK_j)}+s_jc_i\sqrt{(J_jK_i)}$ is rational, and hence the numbers $\sqrt{(J_iK_j)}$ and $\sqrt{(J_jK_i)}$ are rational. Since the $J$'s and the $K$'s are square free integers we therefore have $J_i = K_j$ and $J_j = K_i$. Varying $i$ and $j$ shows that all the $J$'s and $K$'s have the same value, $I$ say. Since $s_i^2I+c_i^2I = 1$ we have $I \in \mathscr{I}$. That the lengths $Q_iA^*$ are rational follows from the fact that triangles $AOQ_i$ and $OA^*Q_i$ are similar. The area of each of the triangles in the polygon is rational because $\operatorname{Sin}2\theta_i$ and $\operatorname{Cos}(\theta_i\pm\theta_j)$ are clearly rational for $1 \leqq i \leqq j \leqq 3$. Since $BQ_i = 2a\operatorname{Sin}\theta_i$ and $CQ_i = 2a\operatorname{Cos}\theta_i$ these lengths are rational iff $I = 1$.

If the point $P$ of the given triangle was one of the points $Q_i$ of lemma 1, then since

$$\operatorname{Cos}^2\theta = \frac{(a+b)^2-c^2}{4ab} \text{ and } \operatorname{Sin}^2\theta = \frac{c^2-(a-b)^2}{4ab}, \tag{1}$$

we could find $I$ by taking square roots.

## 3. To make $\mathscr{Q}$ void

We have just seen that the triangles given by $\mathscr{Q}$ must be Heron. So to show that $\mathscr{Q}$ can be void we take a triangle $AOP$ with $a = 1$, $b = 2$ and prove

LEMMA 2. *No Heron triangle has* 1 *and* 2 *as two of its sides.*

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PROOF. By the area and cosine formulae, a triangle is Heron iff $\operatorname{Sin} \psi$ and $\operatorname{Cos} \psi$ are rational for all its angles $\psi$. Assume that $AOP$ is a Heron triangle with $a = 1$, $b = 2$ and let $\psi$ and $\phi$ be the angles opposite sides 1 and 2 respectively. Since $\operatorname{Tan} \frac{1}{2}\psi = \operatorname{Sin} \psi/(1+\operatorname{Cos} \psi) = r$ say, we have $\operatorname{Sin} \psi = 2r/(1+r^2)$ and $\operatorname{Sin} \phi = 2s/(1+s^2)$ for rationals $r$, $s$. By the sine rule

$$s(1+r^2)-2r(1+s^2) = 0.$$

Solving this quadratic equation for $r$, we find that no non-zero rational solutions $r$, $s$ exist because it is known that

$$s^4+s^2+1 = t^2,$$

has no non-zero solutions (cf. [6] 636–8).

## 4. The general solution

Now lemma 2 has shown that we cannot start with any circle and point and get rational polygons. So in view of lemma 1, from now on we will start with $AOP$ as a Heron triangle. Also we may change the scale of $AOP$ so that $a$, $b$, $c$ are integers with greatest common divisor 1. If $a = b$ so that $A$ lies on $\mathscr{D}$, then $\mathscr{Q}$ is obtained simply taking $\operatorname{Tan} \frac{1}{2}\theta$ rational, so we assume that $a \neq b$. Then we have

LEMMA 3. *If $AOP$ is a Heron triangle and $a \neq b$ then the points of $\mathscr{Q}$ can be obtained from the rational solutions $X$, $Y$ of*

$$(2) \qquad X(X+1)(X+e^2) = Y^2, \text{ where } e^2 = (a+b)^2/(a-b)^2 > 1,$$

*and vice versa.*

PROOF. We wish to obtain the set $\mathscr{Q}$ from the triangle $AOP$. With $\mathscr{Q}$ there will be associated the number $\sqrt{I}$ of lemma 1. So points $Q$ of $\mathscr{Q}$ give triangles $OAQ$ such that $OA = b$, $OQ = a$, $AQ$ is rational, and if $2\omega$ is angle $AOQ$ then

$$\operatorname{Sin} \omega = s/\sqrt{I} \text{ and } \operatorname{Cos} \omega = r/\sqrt{I},$$

for some rationals $r$, $s$. Applying the cosine rule to $AOQ$ we find that

$$(3) \qquad (a-b)^2r^2+(a+b)^2s^2 = t^2I, \text{ where } t = AQ.$$

Putting $\theta = \omega$ in (1) shows that $r = r_0$, $s = s_0$, $t = t_0$ gives a solution to (3) when

$$r_0^2 = I\{(a+b)^2-c^2\}/4ab, \; s_0^2 = I\{c^2-(a-b)^2\}/4ab \text{ and } t_0 = c.$$

Hence (cf. [7] p. 44) the general solution to (3) is

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$$\begin{aligned}\rho^{-1}r &= -(a-b)^2r_0u^2-2(a+b)^2s_0uv+(a+b)^2r_0v^2\\ \rho^{-1}s &= (a-b)^2s_0u^2-2(a-b)^2r_0uv-(a+b)^2s_0v^2\\ \rho^{-1}t &= t_0\{(a-b)^2u^2+(a+b)^2v^2\},\end{aligned}$$

and so

$$\frac{r}{s} = \frac{-r_0m^2-2e^2s_0m+e^2r_0}{s_0m^2-2r_0m-e^2s_0}, \tag{4}$$

where $e = (a+b)/(a-b)$ and $m = u/v$. Thus all ratios $r/s$ for solutions to (3) are obtained by giving $m$ all rational values in (4).

Now we must have $\mathrm{Sin}^2\,\omega+\mathrm{Cos}^2\,\omega = 1$, or in other words

$$s^2+r^2 = I. \tag{5}$$

Using the same method as we used for (3) we find that

$$\frac{r}{s} = \frac{-r_0n^2-2s_0n+r_0}{s_0n^2-2r_0n-s_0}, \tag{6}$$

gives all ratios $r/s$ for solutions to (5) when $n$ takes all rational values. Note also that because $r_0$, $s_0$ satisfy (5) and (3) we have

$$r_0^2+s_0^2 = I \text{ and } r_0^2+e^2s_0^2 = f^2I \tag{7}$$

where $f = c/(a-b)$. We evaluate the expression

$$(rs_0-sr_0)/(rs_0+sr_0)$$

using (4) and then using (6) and equate the two results. The resulting expression simplifies, by virtue of (7), to

$$n(r_0m+e^2s_0)(s_0m-r_0) = mf^2(r_0n+s_0)(s_0n-r_0). \tag{8}$$

We now put $m = -Xn$ in (8) and, again using (7), obtain

$$n^2s_0X(f^2+X)+2nXr_0(1-f^2)-s_0(Xf^2+e^2) = 0,$$

which is a quadratic equation in $n$. To obtain rational solutions of this equation the discriminant must be a square, so

$$s_0^2f^2X^3+gX^2+s_0^2e^2f^2X = z^2, \tag{9}$$

for some rational $z$, where

$$g = r_0^2-2r_0^2f^2+r_0^2f^4+s_0^2e^2+s_0^2f^4 = s_0^2f^2(1+e^2),$$

the last equation being obtained again by using (7). The lemma now follows by dividing (9) by $s_0^2f^2$, and putting $Y^2 = z^2/s_0^2f^2$.

We will next discuss the solutions of (2). Trivially $X = 0, -1, -e^2$ are solutions. Also from (8) we observe that

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(10) $\qquad X = -m/n = r_0^2/s_0^2 = \{(a+b)^2-c^2\}/\{c^2-(a-b)^2\} = \mathrm{Cot}^2\,\theta$

is a solution.

LEMMA 4. *If* (i) $X = \lambda$ *is any solution of* (2) *then* (ii) $X = e^2/\lambda$, (iii) $X = -(\lambda+e^2)/(\lambda+1)$, *and* (iv) $X = -e^2(\lambda+1)/(\lambda+e^2)$ *are also solutions.*

PROOF. The result can be verified by direct substitution. In fact we found (iii) and (iv) by determining the points of intersection of the curve (2) with straight lines through the known points $(\lambda, \xi)$, $(0, 0)$, $(-1, 0)$ and $(-e^2, 0)$ of (2).

## 5. Reduction to canonical form

To reduce (2) to canonical form we put $X = x'-3^{-1}(1+e^2)$ in (2) to obtain

$$x'^3-3^{-1}(e^4-e^2+1)x'+3^{-3}(e^2+1)(2e^2-1)(e^2-2) = Y^2.$$

Then we make the change of variables

$x = 9x'(a-b)^2/h^2$ and $y = 27(a-b)^3Y/h^3$ where $h = h.c.f.\ (a+b, a-b)$,

and find that the canonical form of (2) is

(11) $$x^3-Kx-L = y^2,$$

where $K$, $L$ are the integers

$$K = 27\{(a+b)^4-(a+b)^2(a-b)^2+(a-b)^4\}/h^4,$$

and

$$L = -27\{(a+b)^2+(a-b)^2\}\{(a+b)^2-2(a-b)^2\}\{2(a+b)^2-(a-b)^2\}/h^6.$$

We are now able to use a result proved earlier by one of us ([4], Lemma 1).

LEMMA 5. *Let $K$ and $L$ be integers. Let the equation*

(12) $$x^3-Kx-L = 0$$

*have real roots $\alpha, \beta, \gamma$ with $\alpha < \beta < \gamma$. Suppose that on the cubic curve*

$$x^3-Kx-L = y^2$$

*there are rational points $(\xi_1, \eta_1)$ and $(\xi_2, \eta_2)$ with $\alpha \leqq \xi_1 \leqq \beta$ and $\xi_2$ not an integer. Then the set of rational points of the curve is everywhere dense on it.*

The roots of (12) obtained from the roots $X = 0, -1, -e^2$ of (2) are the rationals

$$\alpha = -3(a^2+b^2+3ab)h^{-2} < \beta = -3(a^2+b^2-6ab)h^{-2} < \gamma = 6(a^2+b^2)h^{-2},$$

so in lemma 5 we take $\xi_1 = \alpha$. Hence the rational solutions to (11) and hence to (2) will be everywhere dense if we can find one solution $(\xi_2, \eta_2)$ with $\xi_2$ not an integer. If $X$ is a solution of (2) then by the transformations which reduced (2) to (11) we find that

$$(13) \qquad \xi = \frac{9(a-b)^2}{h^2}\left[X+\frac{(a+b)^2+(a-b)^2}{3(a-b)^2}\right] \doteq \frac{9(a-b)^2}{h^2}X + \text{integer}$$

is a solution of (11). We recall that (10) is a solution of (2). Using this solution, lemma 4, and the transformation (13) yields

LEMMA 6. *The set of rational points of* (2) *is everywhere dense on* (2) *if any one of the numbers*

(i) $9(a-b)^2\{(a+b)^2-c^2\}/h^2\{c^2-(a-b)^2\}$,
(ii) $9(a+b)^2\{c^2-(a-b)^2\}/h^2\{(a+b)^2-c^2\}$,
(iii) $3c/h$, *and*
(iv) $3(a^2-b^2)/hc$,

*is not an integer.*

## 6. Particular cases

Our first two cases show that if the given Heron triangle $AOP$ is right angled at $O$, or isosceles with $b = c$, then $\mathscr{Q}$ is dense in $\mathscr{D}$. In both these cases we show that the number $3(a^2-b^2)/hc$ of lemma 6 is not an integer.

CASE 1. $a^2+b^2 = c^2$. If $\pi$ divides $c^2 = a^2+b^2$ and $a^2-b^2$ then $\pi$ divides $2a^2$ and $2b^2$. Clearly h.c.f. $(a, b) = 1$ so $\pi$ is 1 or 2. Reducing the equation $a^2+b^2 = c^2$ modulo 4 shows that of $a$ and $b$ one is odd and one is even, so $\pi = 1$. Hence if $3(a^2-b^2)/hc$ is an integer then $c$ is 1 or 3, neither of which is possible.

CASE 2. $b = c$. In this case h.c.f. $(a, b) = 1$ so if $3(a^2-b^2)/hc$ is an integer then $c$ is 1 or 3, but none of the 5 triangles which need be considered is Heron.

It is worth remarking that if $R$ is the point of intersection of $AP$ produced and $\mathscr{D}$, then $R \in \mathscr{Q}$ if $AOP \in \mathscr{Q}$. Thus the Heron triangles occur in pairs. However when $AOP$ is right-angled at $P$, then $R = P$ and we have no non-trivial solution of (2). Not surprisingly all the numbers in lemma 6 are integers under these circumstances. Thus lemma 6 is not strong enough to prove our conjecture. However of all the 26 integer Heron triangles with all sides $\leqq 30$ the only ones which are not covered by the cases mentioned in this section, and for which lemma 6 does not prove $\mathscr{Q}$ dense are the two pairs (13, 20, 11), (13, 20, 21), (17, 25, 12), (17, 25, 28).

## 7. Proof of the main result

We need an elementary lemma.

LEMMA 7. *Given $I \in \mathscr{I}$, the set of Heron isosceles triangles $AOP$ with $AO = AP$, and such that $\sqrt{I}$ Sin $\frac{1}{2} \angle AOP$ and $\sqrt{I}$ Cos $\frac{1}{2} \angle AOP$ are both rational, is everywhere dense in the set of all isosceles triangles.*

PROOF. If we let $2\theta = \angle AOP = \angle APO$, then the triangle $AOP$ is uniquely defined when the angle $\theta$ and the side $a = OP$ are given. Now if we choose $a$ rational and $\mathrm{Tan}\, \theta = s/r$ where $r, s$ satisfy equation (5), then $\sqrt{I}\, \mathrm{Sin}\, \theta = s$ and $\sqrt{I}\, \mathrm{Cos}\, \theta = r$ whilst $\mathrm{Sin}\, 2\theta = 2rs/I$ and $\mathrm{Cos}\, 2\theta = (r^2 - s^2)/I$, so that triangle $AOP$ is one of the Heron isosceles triangles described in the lemma.

Varying $r$ and $s$ varies $\theta$. Thus the lemma will be proved if we can show that the solutions $s/r = \mathrm{Tan}\, \theta$ to (5) are everywhere dense in the reals. By definition of $\mathscr{I}$, that $I \in \mathscr{I}$ implies that there exist rationals $r_0 = R/W$ and $s_0 = S/W$ such that $(r_0, s_0)$ is a solution to (5). As mentioned earlier, the general solution to (5) is then (6). The right-hand side of (6) is continuous in $n$, with the real axis as its range, when $n$ is considered as a real variable. Hence the solutions $r/s$ to (5) are everywhere dense in the reals and the lemma is proved.

We can now prove theorem 2. Let $A'OP'$ be any triangle, not necessarily rational, with associated circle $\mathscr{D}'$. If $A'$ is inside $\mathscr{D}'$ let $A''$ be the inverse image of $A'$ in $\mathscr{D}'$, otherwise let $A''$ be $A'$. Let $P''$ be the point of $\mathscr{D}'$ such that the triangle $A''OP''$ is an isosceles triangle with $A''O = AP''$. By lemma 7, we can approximate $A''OP''$ as closely as we like by a Heron rational isosceles triangle $AOP$ with $AO = AP$ and $\sqrt{I}\, \mathrm{Sin}\, \frac{1}{2} \angle AOP$, $\sqrt{I}\, \mathrm{Cos}\, \frac{1}{2} \angle AOP$ are both rational. By case 2 of section 6 the set $\mathscr{Q}$ for $AOP$ is dense in its associated circle $\mathscr{D}$ and theorem 2 follows.

## References


[1] J. H. J. Almering, Rational quadrilaterals, *Proc. K. Nederl. Akad. Wetensch.* (A)66 *and Indag Math.* 25 (1963), 192–199.

[2] J. H. J. Almering, Rational quadrilaterals, *Proc. K. Nederl. Akad. Wetensch.* (A)68 *and Indag Math.* 27 (1965), 290–304.

[3] A. S. Besicovitch, Rational polygons, *Mathematika*, 6 (1959), 98.

[4] D. E. Daykin, Rational polygons, *Mathematika*, 10 (1963), 125–131.

[5] D. E. Daykin, Rational triangles and parallelograms, *Mathematics Magazine*, 38 (1965), 46–47.

[6] L. E. Dickson, *History of the Theory of Numbers, Vol. II*, Chicago, 1920.

[7] L. E. Dickson, *Introduction to the Theory of Numbers*, New York, 1957.

[8] L. J. Mordell, Rational quadrilaterals, *J. London Math. Soc.* 35 (1960), 277–282.

[9] T. K. Sheng and D. E. Daykin, On approximating polygons by rational polygons, *Mathematics Magazine*, 38 (1966), 299–300.

[10] T. K. Sheng, Rational polygons, *J. Aust. Math. Soc.* 6 (1966), 452–459.


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