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Matchings in graphs

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Results of Tutte and of Anderson giving conditions for a simple graph G to have a perfect matching are generalized to give conditions for G to have a matching of defect d. A corollary to one of these results is a theorem of Berge on the size of a maximum matching in G.

Let G be a simple graph with vertex-set V and edge-set E. A *matching* in G is a set M of edges of G such that every vertex of G is incident to at most one edge in M. A matching M in G is *perfect* if every vertex of G is incident to exactly one edge in M.

If $S \subseteq V$, denote by G_S the subgraph of G obtained by removing all of the vertices in S and all of the edges to which vertices in S are incident. Denote by $p(S)$ the number of components of G_S having an odd number of vertices. In [4] Tutte obtained the following result.

TUTTE'S THEOREM. *The graph G has a perfect matching if and only if $p(S) \leq |S|$ for every subset S of V.*

The simplest proof of this theorem is the recent one by Anderson [1]. It makes use of Hall's Theorem on systems of distinct representatives. Earlier, Berge obtained Tutte's Theorem as a corollary to a result on the number of vertices of G not incident to the edges in a maximum matching in G [2; 3, Chapter 18]. We shall derive this result of Berge from Tutte's Theorem by using a technique popular in transversal theory.

A matching M in G has *defect* d if there are d vertices of G not incident to edges in M.

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THEOREM 1. *Let d be an integer with $0 \le d \le |V|$ and $|V| + d$ even. The graph G has a matching of defect d if and only if $p(S) \le |S| + d$ for every subset S of V.*

Proof. Let D be a set such that $|D| = d$ and $V \cap D$ is empty. Let G' be the graph with vertex-set $V \cup D$ and edge-set consisting of E and edges joining each element of D with each vertex in V. It is clear that G has a matching of defect d if and only if G' has a perfect matching. By Tutte's Theorem this is the case if and only if $p'(S) \le |S|$ for every subset S of $V \cup D$, where $p'(S)$ is the number of components of G'_S having an odd number of vertices.

If $V \subseteq S$, then $p'(S) = |D\backslash S| \le |D| = d \le |V| \le |S|$.

If $V \nsubseteq S$ and $D \nsubseteq S$, then $p'(S) = 0$ or 1 since for $v_1, v_2 \in V\backslash S$ and $x \in D\backslash S$ both $\{v_1, x\}$ and $\{v_2, x\}$ are edges of G'. If S is empty, then $p'(S) = 0$ since $|V| + d$ is even. Hence, we always have $p'(S) \le |S|$.

If $D \subseteq S$, let $S = T \cup D$, where $T \subseteq V$. Then $p'(S) = p(T)$, and so G' has a perfect matching if and only if for all such S we have $p(T) = p'(S) \le |S| = |T| + d$.

A matching M in G is a *maximum matching* if no matching in G has more edges than M.

COROLLARY (Berge). *The number of vertices of G not incident to any of the edges in a maximum matching in G is equal to* $\max\{p(S)-|S| : S \subseteq V\}$.

Note that if d is the stated maximum, then $|V| + d$ is even: in fact, for every $S \subseteq V$, the quantity $|V| + p(S) - |S|$ is even.

Let $|V| = 2n$. Anderson [1] obtained an interesting sufficient condition for there to exist a perfect matching in G, namely, that any $k \le \frac{3}{2}n$ vertices of G be adjacent to at least $\frac{4}{3}k$ vertices. We offer the following generalization.

THEOREM 2. *Let d be an integer such that $0 \le d \le \frac{3}{4}|V|$ and $|V| + d$ is even. If any $k \le \frac{3}{4}(|V|+d)$ vertices of G are adjacent to at least $\frac{4}{3}k - d$ vertices, then G has a matching of defect d.*

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Proof. Let d and G' be as in the proof of Theorem 1. By Anderson's result, G' has a perfect matching if any $k \le \frac{3}{4}(|V|+d)$ vertices of G' are adjacent to at least $\frac{4}{3}k$ vertices of G'. We show that this is the case when G satisfies the hypothesis of the theorem. Let $S \subseteq V \cup D$ be such that $|S| = k \le \frac{3}{4}(|V|+d)$.

If $S \subseteq D$, then the vertices in S are adjacent to the $|V|$ vertices in V, and $k \le d \le \frac{3}{4}|V|$: hence $\frac{4}{3}k \le |V|$.

If $S \nsubseteq V$ and $S \nsubseteq D$, then the vertices in S are adjacent to the $|V| + d$ vertices of G', and $\frac{4}{3}k \le |V| + d$.

If $S \subseteq V$, then the vertices in S are adjacent to the d vertices of G in D and to at least $\frac{4}{3}k - d$ vertices in V. Hence, the vertices in S are adjacent to at least $\frac{4}{3}k$ vertices of G'.

As a corollary we obtain a result for graphs with an odd number of vertices analogous to Anderson's result for graphs with an even number of vertices.

COROLLARY. *If* $|V| = 2n + 1$, *and if any* $k \le \frac{3}{2}(n+1)$ *vertices of* G *are adjacent to at least* $\frac{4}{3}k - 1$ *vertices, then* G *has a matching of defect one.*

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