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Rationality and Orbit Closures

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Abstract. Suppose we are given a finite-dimensional vector space V equipped with an F-rational action of a linearly algebraic group G, with F a characteristic zero field. We conjecture the following: to each vector $v \in V(F)$ there corresponds a canonical $G(F)$-orbit of semisimple vectors of V. In the case of the adjoint action, this orbit is the $G(F)$-orbit of the semisimple part of v, so this conjecture can be considered a generalization of the Jordan decomposition. We prove some cases of the conjecture.

0 Introduction

Let G be a linearly reductive algebraic group defined over a field F of characteristic 0, and write $\overline{F}$ for an algebraic closure of F. Consider the action of G on its Lie algebra $L(G)$. To any point $v \in L(G)$, we can assign a canonical point that has a Zariski-closed G-orbit: the semisimple part s of the Jordan decomposition of v. We also have the rationality result that if v lies in $V(F)$ then so does s.

Suppose now that we are given an action defined over F of G on a vector space V. There is no canonical map from V to V assigning to a point v a point s with a Zariski-closed G-orbit—this difficulty appears even with direct sums of the adjoint action. However one does have the following weaker result: given any vector $v \in V(\overline{F})$, the Zariski closure $\overline{G \cdot v}$ of $g \cdot v$ in $V(\overline{F})$ contains a unique closed $G(\overline{F})$-orbit. In the special case that the action is the adjoint action Ad of G on $L(G)$, this orbit is just $G(\overline{F}) \cdot s$.

In this paper our interest is a generalization of the rationality result above. One can hope to canonically assign to $v \in V(F)$ a single $G(F)$-orbit in the F-rational points of the unique closed $G(\overline{F})$-orbit in the closure of the orbit of v—recall that the F-rational points in a $G(\overline{F})$-orbit may consist of many $G(F)$-orbits. We formulate a conjecture (1.5) that allows just this. Conjecture 1.5 can be viewed as a weakening of the Jordan decomposition that is expected to hold for all representations. This conjecture is important in the development of extensions of the Arthur-Selberg trace formula, such as the relative trace formula.

In Section 1, we define a "rational closure", the 1PS closure, of $G(F) \cdot v$. The 1PS closure is $G(F)$-invariant, strictly contained in the Zariski closure, and it contains a $G(F)$-orbit. Conjecture 1.5 states that it contains a unique $G(F)$-orbit. The remainder of this paper is devoted to proving the conjecture for those classes of representations that are easiest to analyze. In Section 2 we deal with the case that G has rank 1. In Section 3 we deal with the case that the representation is a finite direct sum of copies of Ad.

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1 Definition of the Closure

Let G be a linearly reductive (not necessarily connected) algebraic group, defined over a field F of characteristic zero, and write $\overline{F}$ for an algebraic closure of F. Since $\operatorname{char} F = 0$, we know that the connected component G^0 of the identity in G is reductive. Given an algebraic representation, defined over F, of G on a vector space V, we will write $g \cdot v$ for the action of $g \in G$ on $v \in V$, and $S \cdot W$ for the set $\{g \cdot v \mid g \in S, v \in W\}$ for $S \subseteq G(\overline{F})$, $W \subseteq V(\overline{F})$. We will call the set $G(\overline{F}) \cdot v$ the *geometric orbit* of $v \in V(\overline{F})$, and $G(F) \cdot v$ the *rational orbit* or simply orbit of $v \in V(F)$. We say that an element $v \in V(\overline{F})$ is *semisimple* if its geometric orbit is Zariski closed in V. (This property has had other names in the literature—for example, the nonzero semisimple elements are called "nice semistable" in [7].) An element $v \in V(\overline{F})$ is said to *nilpotent* if the Zariski closure of its orbit contains $0 \in V(\overline{F})$. Notice that with this terminology, the vector 0 is both semisimple and nilpotent. We will also call the geometric or rational orbit of a semisimple (or, respectively, nilpotent) vector $v \in V(\overline{F})$ semisimple (or, respectively, nilpotent); every element of a semisimple (or, respectively, nilpotent) orbit is semisimple (or, respectively, nilpotent). If V is the Lie algebra of G, with the adjoint action, then this definition of semisimple and nilpotent agrees with the more usual one. It also agrees with Richardson's usage in [11].

The following lemma is well-known (*cf.* [7, Lemma 2.3]).

Lemma 1.1 *The (Zariski) closure of any orbit contains a unique semisimple geometric orbit.*

Luna proved in [6] a refinement, which he called property (A), of this lemma, in the case $F = \mathbb{R}$: the closure with respect to the usual topology on $V(\mathbb{R})$ of any rational orbit contains a unique rational semisimple orbit. This refinement has been more recently proven in [8] in the case that F is a p-adic field; in both these cases it follows from an application of the implicit function theorem.

For these fields, given any $\gamma \in V(F)$ we can assign a unique rational semisimple orbit in $\overline{G \cdot \gamma}$. In the case of the adjoint representation, we can of course do more—given γ we can assign a canonical point in this rational semisimple orbit, namely the semisimple part of the Jordan decomposition of γ. The following proposition implies that such a refinement cannot be expected for general representations.

Proposition 1.2 *If $G = \mathrm{SL}(2)$ acts on $V = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$, then there does not exist a canonical function $s \colon V(F) \to V(F)$ satisfying the following three properties for every $\gamma \in V(F)$:*

(i) *$s(\gamma)$ is semisimple.*
(ii) *$s(\gamma) \in \overline{G \cdot \gamma}$.*
(iii) *For any $g \in G(F)$, $s(g \cdot \gamma) = g \cdot s(\gamma)$.*

Note By a canonical function we mean a function $s \colon V(F) \to V(F)$ such that for

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every F-vector space automorphism h of $V(F)$ that preserves the $G(F)$-action, we have $hs = sh$.

Proof Suppose that such an s did exist. Then by (i) and (ii), we have that

$$\begin{aligned} s\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}\right) &\in G(\overline{F}) \cdot \left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right) \\ &= G(F) \cdot \left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right), \end{aligned}$$

so

$$s\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}\right) = \left((\operatorname{Ad} x)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, (\operatorname{Ad} x)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right)$$

for some $x \in G(F)$. Now, given any $a, a' \in F^\times$, and $n, n' \in F$, with $n/a \neq n'/a'$, consider the automorphism h on $V(F)$ given by

$$h(X_1, X_2) = \big((a-n)X_1 + nX_2, (a'-n')X_1 + n'X_2\big).$$

Since s is canonical,

$$\begin{aligned} s\left(\begin{pmatrix}a & n\\ 0 & -a\end{pmatrix}, \begin{pmatrix}a' & n'\\ 0 & -a'\end{pmatrix}\right) &= s\left(h\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}\right)\right) \\ &= h\left(s\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}\right)\right) \\ &= h\left((\operatorname{Ad} x)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, (\operatorname{Ad} x)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right) \\ &= \left((\operatorname{Ad} x)\begin{pmatrix}a & 0\\ 0 & -a\end{pmatrix}, (\operatorname{Ad} x)\begin{pmatrix}a' & 0\\ 0 & -a'\end{pmatrix}\right). \end{aligned}$$

But this contradicts (iii), if we take $g = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ for example. ∎

Remark Luna's property (A) is stronger than the property (A) mentioned in [1] and [9]. The latter was proven by Kempf in [4], and is an essential tool in this paper.

We want to write down and test a version of Luna's property (A) that does not require a topology on the field F. To do this, we need a notion of the closure of an orbit that could conceivably distinguish between rational semisimple orbits.

Before we can introduce this notion, we must first review some definitions and theorems in geometric invariant theory. Let $X_*(G)$ denote the set of one-parameter subgroups $\lambda\colon \mathrm{GL}(1) \to G$, and let $X_*(G)_F$ denote the set of $\lambda \in X_*(G)$ defined over F. Given $\lambda \in X_*(G)_F$, $\gamma \in V(\overline{F})$, we say that the *limit*

$$\lim_{t \to 0} \lambda(t) \cdot \gamma$$

exists and equals v if there is a (necessarily unique) morphism $\ell\colon \mathbb{A}^1 \to V$ with $\ell(t) = \lambda(t)\cdot\gamma$ for $t \neq 0$ and $\ell(0) = v$. Notice that for $\gamma \in V(F)$, $\lambda \in X_*(G)_F$, if the above limit exists, it must lie in $V(F)$. We will apply this definition not just for representations, but for any variety with a G-action, such as G itself, under conjugation.

Let γ be in $V(F)$, and let $G(\overline{F})\cdot v$ be the unique semisimple geometric orbit given by Lemma 1.1, with $v \in V(\overline{F})$. Then the Hilbert-Mumford theorem, as refined by Kempf and Rousseau (*cf.* Corollary 4.3 of [4]) says the following.

Lemma 1.3 *$G(\overline{F})\cdot v \cap V(F)$ contains points of the form*

$$\lim_{t\to 0} \lambda(t)\cdot\gamma,$$

with $\lambda \in X_*(G)_F$.

This lemma leads to a new notion of closure:

Definition 1.4 The *one-parameter subgroup closure*, or 1PS-*closure* of the rational orbit $G(F)\cdot\gamma$, $\gamma \in V(F)$, is the set

$$\begin{aligned} G(F)\cdot\{&\lim_{t\to 0}\lambda(t)\cdot\gamma \mid \lambda \in X_*(G)_F \text{ and the limit exists}\} \\ &= \{\lim_{t\to 0}\lambda(t)g\cdot\gamma \mid \lambda \in X_*(G)_F, g \in G(F), \text{ and the limit exists}\}. \end{aligned}$$

It is clear that the 1PS-closure of on orbit $G(F)\cdot\gamma$ does not depend on the point γ in the orbit, and that if F is a topological field, then the 1PS-closure of $G(F)\cdot\gamma$ is contained in its topological closure. Notice though that the inclusion may be strict: if the representation is $\mathrm{SL}(2)$ acting on the space of binary cubic forms, the 1PS-closure of the orbit of x^2y does not contain x^3, while the topological closure does, *cf.* [3].

If F is real or p-adic, then Luna's property (A) holds, and combined with Lemma 1.3 it implies that the 1PS-closure of a rational orbit contains a unique semisimple rational orbit. This latter property does not require a topology on the field, and so it can be seen as an extension of Luna's property (A). It seems natural to conjecture that it also always holds.

Conjecture 1.5 Suppose we are given an algebraic representation of a linearly reductive algebraic group G on a finite-dimensional vector space V, all defined over a field F of characteristic zero. Then for any $\gamma \in V(F)$, the 1PS-closure of $G(F)\cdot\gamma$ contains a unique semisimple rational orbit.

For the remainder of this paper we will prove some special cases of this conjecture in a way that does not depend upon the field.

Remarks (1) The results of Kempf and Rousseau on the Hilbert-Mumford theorem do not address this conjecture, because they only identify "optimal" classes of one-parameter subgroups λ such that the limit $\lim_{t\to 0}\lambda(t)\cdot\gamma$ exists, but do not describe the possible limit points under the action of other one-parameter subgroups. On the other hand, their result is essential to all the results in this paper.

(2) Since $X_*(G) = X_*(G^0)$, it is sufficient to prove Conjecture 1.5 for connected groups.

(3) The conjecture is trivial if γ is nilpotent because then the semisimple geometric orbit given by Lemma 1.1 is the set $\{0\}$.

(4) The conjecture is also trivial if the F-rank of G is zero, because then the only one-parameter subgroup λ in $X_*(G)_F$ is the constant function e.

(5) If the conjecture holds for any given representation then it holds for any subrepresentation.

The following basic rationality lemma is important for our proofs.

Lemma 1.6 *Let N be a unipotent algebraic group defined over F, acting on a vector space V by a representation defined over F. The given a point $\gamma \in V(F)$, the set $\big(N(\overline{F}) \cdot \gamma\big) \cap V(F)$ of rational points in the geometric N-orbit of γ consists of a single $N(F)$-orbit.*

Proof This is an elementary application of Galois cohomology. Write N_γ for the stabilizer in N of γ, so that N_γ is a unipotent group defined over F. Since char F is zero, N_γ is connected, so $H^1(F, N_\gamma) = 0$ (*cf.* [12, 4.1]). Since N/N_γ is isomorphic, as a set with both $\mathrm{Gal}(\overline{F}/F)$- and N-actions, to the orbit $N \cdot \gamma$, Proposition 2.1(2) of [12] implies the desired result. ∎

We will require a little more notation. Suppose that G is connected. Write $X^*(G)$ for the set of characters $\pi\colon G \to \mathrm{GL}(1)$, and $X^*(G)_F$ for the set of those $\pi \in X^*(G)$ defined over F; these are both abelian groups. There is a natural non-degenerate bilinear pairing $(\chi, \lambda) \mapsto \chi(\lambda) \in \mathbb{Z}$ between $X^*(G)_F$ and $X_*(G)_F$, given by $\chi\big(\lambda(t)\big) = t^{\chi(\lambda)}$, $\chi \in X^*(G)_F$, $\lambda \in X_*(G)_F$. Write G^1 for the subgroup

$$\bigcap_{\pi \in X^*(G)_F} \ker \pi$$

of G; it is a semisimple group defined over F, whose rank is the semisimple rank of G. Given $H \leq G$ and $\pi \in X^*(H)$, write V^π for $\{v \in V \mid h \cdot v = \pi(h)v, \text{ for all } h \in H\}$, the π-weight space of V.

2 Rank One Groups

In this section we will prove Conjecture 1.5 given a strong restriction on G.

Theorem 2.1 *If the semisimple F-rank of G^0 is at most one, then Conjecture 1.5 holds.*

If G is semisimple, then the proof goes roughly as follows: if v is a semisimple vector in the 1PS-closure of a vector $\gamma \in V(F)$, then either $v = \gamma$ or for some minimal parabolic subgroup P with maximal split torus A, the vector v lies in the 0-weight space with respect to A, and γ lies in the sum of v and the direct sum of the positive weight spaces. Given two such vectors v and v', there are only two non-trivial cases to consider: when neither equals γ or when exactly one (say v') does. In the first

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case, one obtains two maximal split tori, which are conjugate by an element g of $G(F)$, and we show that with an appropriate choice of g, v' is $g^{-1}\cdot v$. In the second case, we know that $v = g\cdot\gamma$ for some $g \in G(\overline{F})$. Using the Bruhat decomposition with respect to P and basic knowledge of the action of each term of the Bruhat decomposition ($M = Z_G(A)$, $N = R_u(P)$, and W) on the weight spaces in V, we conclude that for some $n_2 \in N(\overline{F})$ we have $n_2 \cdot \gamma = v$, at which point Lemma 1.6 lets us conclude that $v \in G(F)\cdot\gamma$.

For general reductive groups of semisimple rank 1, the proof is somewhat more involved and ends with some geometry on $X^*(G)_F \otimes \mathbb{R}$.

Proof Assume that G is connected; this can be done without loss of generality by Remark 1.5 (2). Suppose we are given $\gamma \in V(F)$ and $\lambda, \lambda' \in X_*(G)_F$, $v, v' \in V(F)$, with

$$\lim_{t\to 0}\lambda(t)\cdot\gamma = v, \quad \lim_{t\to 0}\lambda'(t)\cdot\gamma = v',$$

and v, v' semisimple. We must show that $v' \in G(F)\cdot v$.

Write Z for the maximal split torus contained in the centre of G. Decompose V into weight spaces according to the action of Z,

$$V = \bigoplus_{\nu\in X^*(Z)_F} V^{\nu},$$

and for each vector $x \in V$, write $\operatorname{supp}_Z(x)$ for the set of weights $\nu \in X^*(Z)_F$ such that the component of x in V^ν is non-zero. Since Z is central in G, the action of G preserves each weight space V^ν.

By our assumption of the limit points v, v', they lie in the same $G(\overline{F})$-orbit, so that $\operatorname{supp}_Z v = \operatorname{supp}_Z v' \subseteq \operatorname{supp}_Z \gamma$, and so replacing γ with its component in $\bigoplus_{\nu\in\operatorname{supp}_Z(v)} V^\nu$ will not change v or v'. Hence we may assume, without loss of generality, that $\operatorname{supp}_Z v = \operatorname{supp}_Z \gamma$. If we write γ^ν for the projection of γ to V^ν, for $\nu \in \operatorname{supp}_Z\gamma$, and similarly with v, v', we must have

$$v^\nu = \lim_{t\to 0}\lambda(t)\cdot\gamma^\nu \quad\text{and}\quad (v')^\nu = \lim_{t\to 0}\lambda'(t)\cdot\gamma^\nu, \quad \text{for all } \nu \in \operatorname{supp}_Z\gamma.$$

If both one-parameter subgroups λ, λ' lie in $X_*(Z)_F$, then the existence of the limits v and v' implies that both v and v' equal γ, hence are equal. We may therefore assume, writing $\operatorname{im}(\lambda)$ for the image in G of λ, that G has semisimple rank 1 and that the product $\operatorname{im}(\lambda)Z$ is a maximal split torus T of G. Then the torus $A = (T\cap G^1)^0$ is a maximal F-split torus of G^1, and $T = AZ$. Let ϖ be the generator of $X^*(A)_F \cong \mathbb{Z}$ such that $\varpi(\lambda) > 0$. The map $\iota\colon X^*(T) \to X^*(Z)\oplus X^*(A)$ sending $\chi \in X^*(T)$ to $(\chi|_Z, \chi|_A)$ is clearly injective and $\operatorname{Gal}(\overline{F}/F)$-invariant. Given $(\nu, i\varpi) \in X^*(Z)\oplus X^*(A)$ (so that $i \in \mathbb{Z}$), write $V^{\nu,i}$ for the weight space of V under the action of T corresponding to $\chi \in X^*(T)_F$ if $\iota(\chi) = (\nu, i\varpi)$, and the trivial subspace if $(\nu, i\varpi)$ is not in the image of $X^*(T)_F$ under ι. Notice that for $\nu \in \operatorname{supp}_Z\gamma$, V^ν is the direct sum of the $V^{\nu,i}$, $i \in \mathbb{Z}$.

The group $(A \cap Z)(F)$ is finite; write f for its order. The one-parameter subgroup λ^f, defined by $\lambda^f(t) = \lambda(t^f)$, can be written uniquely as a product of two one-parameter subgroups, $\lambda^f(t) = \lambda_1(t)\lambda_2(t)$ with $\lambda_1 \in X_*(A)_F$ and $\lambda_2 \in X_*(Z)_F$. Notice that the limit point v^ν also equals the limits

$$v^\nu = \lim_{t \to 0} \lambda(t) \cdot \gamma^\nu = \lim_{t \to 0} \lambda^f(t) \cdot \gamma^\nu = \lim_{t \to 0} t^{\nu(\lambda_2)} \lambda_1(t) \cdot \gamma^\nu.$$

We similarly decompose a power of λ' as $(\lambda')^{f'} = \lambda'_1 \lambda'_2$ with $\lambda'_2 \in X_*(Z)$ and λ'_1 a (now possibly trivial) one-parameter subgroup in a maximal split torus A' of G^1.

For each $\nu \in \operatorname{supp}_Z \gamma$, the existence and non-vanishing of the limit $v^\nu = \lim_{t \to 0} \lambda(t) \cdot \gamma^\nu$ implies that the number $i_\nu = -\nu(\lambda_2)/\varpi(\lambda_1)$ is an integer, that γ^ν lies in the subspace

$$\bigoplus_{j \geq i_\nu} V^{\nu,j} \tag{2.1}$$

of V, and that v^ν equals γ^{ν,i_ν}, the component of γ in the weight space V^{ν,i_ν}.

Let P be the minimal parabolic subgroup of G^1 defined over F that has $M = C_{G^1}(A) \supseteq A$ as a Levi component, and whose positive roots are positive multiples of ϖ. Write N for the unipotent radical of P. Both M and N are defined over F. Pick a representative $w \in G^1(F)$ for the non-trivial element of the Weyl group of the rank one group G^1, so that $\{1, w\}$ is a set of representatives of the Weyl group. Then the Bruhat decomposition says that for any field F' containing F, $G^1(F')$ is the disjoint union of $P(F') = M(F')N(F')$ and $N(F')wM(F')N(F') = M(F')N(F')wN(F')$.

Remark 2.1 The Bruhat decomposition is important for our purposes because the behaviour of M, N, and w on weight spaces is relatively simple: M preserves every weight space, w is an isomorphism from $V^{\nu,j}$ to $V^{\nu,-j}$, $\nu \in \operatorname{supp}_Z \gamma$, $j \in \mathbb{Z}$. The image of the action of N on a weight space $V^{\nu,j}$ is more complicated, but the following fact is elementary and sufficient for our purposes: given $n \in N$ and $x \in V^{\nu,j}$, the point $n \cdot x$ lies in $x + \bigoplus_{l > j} V^{\nu,l}$.

Let us first suppose that λ' lies in $X_*(Z)$, so that λ'_1 is trivial. The condition $\operatorname{supp}_Z \gamma = \operatorname{supp}_Z v'$ implies that $v' = \gamma$. By assumption, both γ and $v = \sum_{\nu \in \operatorname{supp}_Z \gamma} \gamma^{\nu,i_\nu}$ are semisimple, so that for some element $g \in G(\overline{F})$, we have $g \cdot \gamma^\nu = \gamma^{\nu,i_\nu}$ for each $\nu \in \operatorname{supp}_Z \gamma$. We will show that if the Bruhat decomposition of wg is of the form $n_1 x m_1 n_2$, with $n_1, n_2 \in N(\overline{F})$, $m_1 \in M(\overline{F})$, $x \in \{1, w\}$ (with the proviso that $n_1 = 1$ if $x = 1$), then

$$n_2 \cdot \gamma^\nu \in V^{\nu,i_\nu}, \quad \text{for all } \nu \in \operatorname{supp}_Z \gamma. \tag{2.2}$$

By Remark 2.1, this implies that $n_2 \cdot \gamma = \sum_{\nu \in \operatorname{supp}_Z \gamma} \gamma^{\nu,i_\nu} = v$, and Lemma 1.6 will then imply that v and $v' = \gamma$ are in the same $N(F)$- and hence $G(F)$-orbit, proving the result in this case.

While proving (2.2), we fix $\nu \in \operatorname{supp}_Z \gamma$, and write i for i_ν. If g lies in $M(\overline{F})N(\overline{F})$, say $g = mn$, then $n_2 = n$ and $n \cdot \gamma^\nu = m^{-1} \cdot \gamma^{\nu,i} \in V^{\nu,i}$, so (2.2) holds.

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If instead g lies in $N(\overline{F})wM(\overline{F})N(\overline{F})$, say $g = n'wmn$, then the first form of $mn \cdot \gamma^\nu = w^{-1}(n')^{-1} \cdot \gamma^{\nu,i}$ clearly has a non-zero component in $V^{\nu,i}$, while the second form clearly has a non-zero component in $V^{\nu,-i}$. Since $mn \cdot \gamma$ lies in the space (2.1), we see that $i \le 0$.

Suppose first that $i = 0$. Then

$$wmn \cdot \gamma^\nu \in \bigoplus_{j \le 0} V^{\nu,j} \quad \text{and} \quad (n')^{-1} \cdot \gamma^{\nu,i} \in \gamma^{\nu,0} + \bigoplus_{j>0} V^{\nu,j}. \tag{2.3}$$

Since these two vectors are equal, $wmn \cdot \gamma^\nu$ must equal $\gamma^{\nu,0}$. If $n' = 1$, then $n_2 = n$ and $n \cdot \gamma^\nu \in (wm)^{-1} \cdot V^{\nu,0} = V^{\nu,0}$, so (2.2) holds. If $n' \ne 1$, then $wn'w \notin M(\overline{F})N(\overline{F})$ so $x = w$ and the Bruhat decomposition of wg is $wg = n_1 w m_1 n_2$, for some $n_1, n_2 \in N(\overline{F})$, $m_1 \in M(\overline{F})$. Then

$$n_1 w m_1 n_2 \cdot \gamma^\nu = wg \cdot \gamma^\nu \in w \cdot V^{\nu,0} = V^{\nu,0},$$

and we obtain as in (2.3) that $wm_1 n_2 \cdot \gamma^\nu$ and hence $n_2 \cdot \gamma^\nu$ lies in $V^{\nu,0}$ and (2.2) holds.

If $i < 0$, then since $wmn \cdot \gamma$ has a non-trivial component in $V^{\nu,-i}$, while $n'wmn \cdot \gamma^\nu = \gamma^{\nu,i}$ does not, the element n' is not 1. Therefore we again have that $x = w$, so $wn'wmn = n_1 w m_1 n_2$, for some $n_1, n_2 \in N(\overline{F})$, $m_1 \in M(\overline{F})$. Notice that $wn'wmn \cdot \gamma^\nu = w \cdot \gamma^{\nu,i} \in V^{\nu,-i}$, so that $n_2 \cdot \gamma^\nu = m_1^{-1} w^{-1} n_1^{-1} \cdot (w \cdot \gamma^{\nu,i}) \in \bigoplus_{j \le i} V^{\nu,j}$. On the other hand, we know that $n_2 \cdot \gamma^\nu \in \bigoplus_{j \ge i} V^{\nu,j}$, so that $n_2 \cdot \gamma \in V^i$, so (2.2) holds.

We have proven the theorem in the case that λ_1' is trivial. We now suppose that λ_1' lies in $X_*(A')_F \setminus \{1\}$. The maximal F-split tori A and A' of G^1 are conjugate over $G^1(F)$, so for some $g \in G(F)$, we have $A' = g^{-1}Ag$, and $\lambda_A'(t) = g\lambda_1'(t)g^{-1}$ defines a one-parameter subgroup in $X_*(A)_F \setminus \{1\} \subseteq X_*(G)_F$. In fact, by multiplying g on the left by w if necessary, we can arrange to have $\varpi(\lambda_A') > 0$. Since any $m \in M(F)$ commutes with A, we can take g to have Bruhat decomposition $g = n'xn$, $n, n' \in N(F)$, $x \in \{1, w\}$. Notice that

$$v' = \lim_{t \to 0} \lambda'(t) \cdot \gamma = \lim_{t \to 0} t^{\nu(\lambda_2')} \lambda_1'(t) \cdot \gamma = g^{-1} \cdot \Big(\sum_{\nu \in \operatorname{supp}_Z \gamma} \lim_{t \to 0} t^{\nu(\lambda_2')} \big(\lambda_A'(t) g \cdot \gamma^\nu \big) \Big),$$

where the existence of each of the limits on the right-hand side is forced, and that it suffices to show that this latter sum lies in $G(F) \cdot v$.

If $x = 1$, then g lies in $N(F)$, so that for each $\nu \in \operatorname{supp}_Z \gamma$, $g \cdot \gamma^\nu - \gamma^\nu$, and hence $g \cdot \gamma^\nu - \gamma^{\nu,i_\nu}$, lies in $\bigoplus_{j > i_\nu} V^{\nu,j}$—see (2.1). The existence and non-vanishing of the limit $\lim_{t \to 0} t^{\nu(\lambda_2')} \big(\lambda_A'(t) g \cdot \gamma^\nu \big)$ with $\varpi(\lambda_A') > 0$ implies that this limit equals $\gamma^{\nu,i_\nu} = v^\nu$.

We are left with the case that $x = w$. Since $\lim_{t \to 0} \lambda_A'(t) n' \lambda_A'(t^{-1}) = 1$, the existence of the limit

$$\lim_{t \to 0} t^{\nu(\lambda_2')} \lambda_A'(t) g \cdot \gamma^\nu = \lim_{t \to 0} t^{\nu(\lambda_2')} \Big(\big(\lambda_A'(t) n' \lambda_A'(t^{-1}) \big) w \lambda_A'(t^{-1}) n \cdot \gamma^\nu \Big),$$

for each $\nu \in \operatorname{supp}_Z \gamma$ implies the existence of the limit

$$\lim_{t \to 0} t^{\nu(\lambda_2')} \lambda_A'(t^{-1}) n \cdot \gamma^\nu. \tag{2.4}$$

We now show that for each $\nu \in \operatorname{supp}_Z \gamma$, (2.4) equals v^ν. This will complete the proof of the theorem, as we then have that $v' = g^{-1} w \cdot v$.

Let us replace γ with $n \cdot \gamma$; this does not change the limits $v^\nu = \lim_{t \to 0} \lambda(t) \cdot \gamma^\nu$, and simplifies (2.4) to

$$\lim_{t \to 0} t^{\nu(\lambda_2')} \lambda_A'(t^{-1}) \cdot \gamma^\nu. \tag{2.5}$$

Recall that v^ν has a very similar expression,

$$v^\nu = \lim_{t \to 0} t^{\nu(\lambda_2)} \lambda_1(t) \cdot \gamma^\nu. \tag{2.6}$$

Let us write i_ν' for $-\nu(\lambda_2')/\varpi(\lambda_A')$. Then the convergence of (2.5) and (2.6) implies that for each $\nu \in \operatorname{supp}_Z \gamma$,

$$\gamma^\nu \in \bigoplus_{j \le i_\nu'} V^{\nu,j}, \quad \gamma^\nu \in \bigoplus_{j \ge i_\nu} V^{\nu,j},$$

hence that $i_\nu' \ge i_\nu$.

Since v is semisimple, there does not exist a $\Lambda \in X_*(Z)_F$ such that the limit $\lim_{t \to 0} \Lambda(t) \cdot v$ exists and has support strictly contained in $\operatorname{supp}_Z v$. This implies that 0 lies in the interior of the convex hull of $\operatorname{supp}_Z v = \operatorname{supp}_Z \gamma$, so that there exist constants $c_\nu \in (0, \infty)$, $\nu \in \operatorname{supp}_Z \gamma$, such that $\sum_{\nu \in \operatorname{supp}_Z \gamma} c_\nu \nu = 0$. Therefore

$$0 = \sum_{\nu \in \operatorname{supp}_Z \gamma} c_\nu \frac{-\nu(\lambda_2)}{\varpi(\lambda_1)} = \sum_{\nu \in \operatorname{supp}_Z \gamma} c_\nu i_\nu \le \sum_{\nu \in \operatorname{supp}_Z \gamma} c_\nu i_\nu' = 0,$$

since $i_\nu \le i_\nu'$ for all ν. This implies that $i_\nu = i_\nu'$, so that (2.5) equals $\gamma^{\nu, i_\nu'} = \gamma^{\nu, i_\nu} = v^\nu$. ∎

3 m-Tuples in Lie Algebras

In this section we will prove Conjecture 1.5 given a strong restriction on the representation of G. Write $L(G)$ for the Lie algebra of G, and Ad for the adjoint action of G on $L(G)$. Given a positive integer m, write $L(G)^m$ for the space of m-tuples $\{(x_1, \dots, x_m) \mid x_i \in L(G)\}$.

Theorem 3.1 *If G is a linearly reductive algebraic group defined over F, and m is a positive integer, set $V = L(G)^m$, with G-action given by $g \cdot (x_1, \dots, x_m) = \big(\operatorname{Ad} g(x_1), \dots, \operatorname{Ad} g(x_m)\big)$. Then Conjecture 1.5 holds.*

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The main tool in our proof of this is the Levi decomposition in $L(G)^m$, an analogue of the Jordan decomposition in $L(G)$, introduced by Richardson in [11]. If $m = 1$, the Levi decomposition reduces to the Jordan decomposition, but for higher m, it need not be unique.

Let us briefly review the definitions and results of [11] that we will use.

Call a Lie subalgebra $\mathfrak{a}$ of $L(G)$ an *algebraic Lie subalgebra* of $L(G)$ if there exists a (necessarily unique) connected closed subgroup A of G such that $L(A) = \mathfrak{a}$. Given $\gamma = (\gamma_1, \dots, \gamma_m) \in L(G)^m$, write $\mathfrak{a}(\gamma)$ for the *algebraic hull* of $\gamma_1, \dots, \gamma_m$, that is, the smallest algebraic Lie subalgebra of $L(G)$ containing $\gamma_1, \dots, \gamma_m$, and write $A(\gamma)$ for the connected closed subgroup of G such that $L\big(A(\gamma)\big) = \mathfrak{a}(\gamma)$.

We say that a decomposition $\gamma = s + n$ is a *Levi F-decomposition* if $\gamma, s, n \in L(G)^m(F)$, $A(s)$ is a Levi F-subgroup of $A(\gamma)$, and $A(n) \subseteq R_u\big(A(\gamma)\big)$. The following results about Levi F-decompositions follow from [11]: see 1.2.4, 3.6, 3.7, and Section 5 there.

Proposition 3.2

(a) *s is semisimple if and only if $A(s)$ is reductive. n is nilpotent if and only if $A(n)$ is a unipotent group.*
(b) *Every $\gamma \in L(G)^m(F)$ has a Levi F-decomposition, and given any two decompositions $\gamma = s + n = s' + n'$, the elements s, s' are in the same $R_u\big(A(\gamma)\big)(F)$-orbit.*
(c) *If $\gamma = s + n$ is a Levi F-decomposition, then there exists $\lambda \in X_*(G)_F$ such that $\lim_{t\to 0} \lambda(t) \cdot \gamma = s$.*

By Proposition 3.2(c), Theorem 3.1 is equivalent to the following statement.

Theorem 3.1′ *If $\gamma \in L(G)^m(F)$ has Levi decomposition $\gamma = s + n$, then the unique semisimple rational orbit in the* 1PS*-closure of $G(F) \cdot \gamma$ is the rational orbit of s.*

Notice that the Levi decomposition gives a set of rational semisimple vectors, all belonging to a single rational $G(F)$-orbit, while Conjecture 1.5 assigns the entire $G(F)$-orbit.

We begin with a Lemma.

Lemma 3.3 *Suppose that $\gamma = s + n$ is a Levi F-decomposition in $L(G)^m$, and that $\gamma_0 = \lim_{t\to 0} \lambda(t) \cdot \gamma$ exists, with $\lambda \in X_*(G)_F$. Then $s_0 = \lim_{t\to 0} \lambda(t) \cdot s$ and $n_0 = \lim_{t\to 0} \lambda(t) \cdot n$ also exist, and $\gamma_0 = s_0 + n_0$ is a Levi F-decomposition.*

Proof Since the limit $\lim_{t\to 0} \lambda(t) \cdot \gamma$ exists, γ and hence s and n lies in $L\big(P(\lambda)\big)^m$, so that the limit points s_0 and n_0 also exist and are in $L(G)^m(F)$. It remains to show that $A(s_0)$ is a Levi F-subgroup of $A(\gamma_0)$ and that $A(n_0) \subseteq R_u\big(A(\gamma_0)\big)$.

Define the homomorphism $h_\lambda \colon P(\lambda) \to G^\lambda$ by $h_\lambda(g) = \lim_{t\to 0} \lambda(t) \cdot g$, as in [11, 2.2]. Then [11, 3.1] says that $A(\gamma_0) = h_\lambda\big(A(\gamma)\big)$, $A(s_0) = h_\lambda\big(A(s)\big)$, $A(n_0) = h_\lambda\big(A(n)\big)$, and [2, 14.11] says that $R_u\Big(h_\lambda\big(A(\gamma)\big)\Big) = h_\lambda\Big(R_u\big(A(\gamma)\big)\Big)$. Putting these facts together, we see that $A(n_0) = h_\lambda\big(A(n)\big) \subset h_\lambda\Big(R_u\big(A(\gamma)\big)\Big) = R_u\big(A(\gamma_0)\big)$. Also, $A(s_0)$ is reductive, hence contained in a Levi F-subgroup of $A(\gamma_0)$; since on the other hand $A(\gamma_0) = h_\lambda\big(A(\gamma)\big) = h_\lambda\big(A(s)\big)\, h_\lambda\Big(R_u\big(A(\gamma)\big)\Big) = A(s_0) R_u\big(A(\gamma_0)\big)$, we conclude that $A(s_0)$ is a Levi subgroup of $A(\gamma_0)$. ■

Proof of Theorem 3.1$'$ Suppose that $\lim_{t\to 0} \lambda(t) \cdot \gamma$ exists and is a semisimple element s_0. Then $R_u\bigl(A(s_0)\bigr) = \{1\}$, so by Lemma 3.3, $\lim_{t\to 0} \lambda(t) \cdot n = 0$ and $\lim_{t\to 0} \lambda(t) \cdot s = s_0$. Now the reductive F-group $A(s) \subseteq P(\lambda)$ is contained in a Levi F-subgroup M of $P(\lambda)$. Since G^λ is also a Levi F-subgroup of $P(\lambda)$, there exists $u \in R_u\bigl(P(\lambda)\bigr)(F)$ such that $uMu^{-1} = G^\lambda$. Then $u \cdot s \in L(G^\lambda)^m$ and $s - u \cdot s \in L\Bigl(R_u\bigl(P(\lambda)\bigr)\Bigr)^m$, so that

$$s_0 = \lim_{t\to 0} \lambda(t) \cdot s = \lim_{t\to 0} \lambda(t) \cdot (u \cdot s) + \lim_{t\to 0} \lambda(t) \cdot (s - u \cdot s) = u \cdot s.$$

This completes the proof. ■

Richardson in [11] defined a Levi decomposition in a more general context, and the analogue of Theorem 3.1$'$ is valid in this context as well. We begin with notation, and some assumptions that will hold for the remainder of this section.

Let G and Θ be linearly reductive algebraic groups defined over F. We say that G is a Θ-group defined over F if Θ acts F-morphically on G in such a way that for every $\theta \in \Theta$, the morphism $G \to G$ given by $g \mapsto \Theta \cdot g$ is an automorphism of algebraic groups. In this case we write $K = G^\Theta$; it is a linearly reductive subgroup of G defined over F [10, Prop. 10.1.5]. An important example of this is when Θ is a 2-element group.

Given $\gamma = (\gamma_1, \dots, \gamma_m) \in L(G)^m$, $m \geq 1$, we let $\mathfrak{a}_\Theta(\gamma)$ be the algebraic hull of $\{\theta \cdot \gamma_i \mid \theta \in \Theta, i = 1, \dots, m\}$, and $A_\Theta(g)$ be the unique closed connected subgroup of G such that $L\bigl(A_\Theta(\gamma)\bigr) = \mathfrak{a}_\Theta(\gamma)$. We say that a decomposition $\gamma = s + n$ is a Levi Θ-decomposition of γ defined over F if $\gamma, s, n \in L(G)^m(F)$, $A_\Theta(s)$ is a Θ-stable Levi F-subgroup of $A_\Theta(\gamma)$ and $A_\Theta(n) \subseteq R_u\bigl(A_\Theta(\gamma)\bigr)$. Consider the action of K on $L(G)^m$. Richardson proves [11], 12.1, 13.2–13.5] the following statements.

Proposition 3.4

(a) *s is semisimple under the action of K if and only if $A_\Theta(s)$ is reductive. n is nilpotent under the action of k if and only if $A_\Theta(n)$ is a unipotent group.*
(b) *Every $\gamma \in L(G)^m(F)$ has a Levi Θ-decomposition defined over F.*
(c) *If $\gamma = s+n$ is a Levi Θ-decomposition defined over F, then there exists $\lambda \in X_*(K)_F$ such that $\lim_{t\to 0} \lambda(t) \cdot \gamma = s$.*

We also have an analogue of Lemma 3.3.

Lemma 3.5 *Suppose that $\gamma = s+n$ is a Levi Θ-decomposition defined over F in $L(G)^m$, and that $\gamma_0 = \lim_{t\to 0} \lambda(t) \cdot \gamma$ exists, with $\lambda \in X_*(K)_F$. Then $s_0 = \lim_{t\to 0} \lambda(t) \cdot s$ and $n_0 = \lim_{t\to 0} \lambda(t) \cdot n$ also exist, and $\gamma_0 = s_0 + n_0$ is a Levi Θ-decomposition defined over F.*

Proof This is almost exactly the same as for Lemma 3.3. Notice that since $\lambda \in X_*(K)_F$, the parabolic subgroup $P(\lambda)$ is Θ-stable, and h_λ commutes with Θ. ■

Theorem 3.6 *If $\gamma \in L(G)^m(F)$ has Levi Θ-decomposition $\gamma = s + n$ defined over F, then the unique semisimple $K(F)$-orbit in the* 1PS-*closure of $K(F) \cdot \gamma$ is $K(F) \cdot s$.*

Proof This is almost exactly the same as for Theorem 3.1$'$. The only addition we need to make is to prove that $u \in K$. Since $\lambda \in X_*(K)_F$, $P(\lambda)$ and hence $R_u\big(P(\lambda)\big)$ is Θ-stable. Therefore for any $\theta \in \Theta$, $\theta(u)$ lies in $R_u\big(P(\lambda)\big)$ and also satisfies $\theta(u)M\theta(u)^{-1} = G^\lambda$. Therefore $u^{-1}\theta(u)$ both normalizes the Levi subgroup M of $P(\lambda)$ and lies in $R_u\big(P(\lambda)\big)$, hence is the identity, so $u = \theta(u)$. Since θ was arbitrary, $u \in K$. ■

Corollary 3.7 *Suppose that G and Θ are linearly reductive algebraic groups defined over F, and that m is a positive integer. Set V to be the vector space $L(G)^m$, with K-action given by $k \cdot (x_1, \dots, x_m) = \big(\operatorname{Ad} k(x_1), \dots, \operatorname{Ad} k(x_m)\big)$, or any K-invariant subspace. Then Conjecture 1.5 holds.*

Remark If $\Theta = \{1, \theta\}$ with θ an involution of G defined over F, then the ± 1-eigenspaces $\mathfrak{k}$ and $\mathfrak{p}$ under the action of θ are invariant under the action of K, and hence Conjecture 1.5 holds for representations of the form $(\mathfrak{k})^{m_1} \oplus (\mathfrak{p})^{m_2}$.

References

[1] David Birkes, *Orbits of linear algebraic groups.* Ann. of Math. (2) **93**(1971), 459–475.
[2] Armand Borel, *Linear algebraic groups.* 2nd edition, Graduate Texts in Math. **126**, Springer-Verlag, New York, 1991.
[3] W. H. Hesselink, *Concentration under actions of algebraic groups.* Lecture Notes in Math. **867**, Springer, Berlin, 1981, 55–89.
[4] G. Kempf, *Instability in invariant theory.* Ann. of Math. (2) **108**(1978), 299–316.
[5] J. Levy, *A truncated integral of the Poisson summation formula.* Canad. J. Math. **53**(2001), 122–160.
[6] D. Luna, *Sur certaines opérations différentiables des groupes de Lie.* Amer. J. Math. **97**(1975), 172–181.
[7] L. Ness, *A stratification of the null cone via the moment map.* Amer. J. Math. **106**(1984), 1281–1329.
[8] Cary Rader and Steve Rallis, *Spherical characters on p-adic symmetric spaces.* Amer. J. Math. **118**(1996), 91–178.
[9] M. S. Raghunathan, *A note on orbits of reductive groups.* J. Indian Math. Soc. (N.S.) **38**(1974), 65–70.
[10] R. W. Richardson, *On orbits of algebraic groups and Lie groups.* Bull. Austral. Math. Soc. **25**(1982), 1–29.
[11] ———, *Conjugacy classes of n-tuples in Lie algebras and algebraic groups.* Duke Math. J. **57**(1988), 1–35.
[12] T. A. Springer, *Galois cohomology of linear algebraic groups.* Proc. Sympos. Pure Math. **9**, Amer. Math. Soc., Providence, RI, 1966, 149–158.

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