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ON A THEOREM OF NIVEN

BY

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In [4], Niven proved that the set A of integers $\mathfrak{D}_s(n)$ for all $s \geq 1$ and all $n \geq 1$ has density zero, $\mathfrak{D}_s(n)$ being the sum of the sth powers of all positive divisors of n. However his argument contains a mistake (see Remark 1). In this paper we give a proof of Niven's result and establish several related results, one of which generalizes a result of Dressler (See Theorem 3 and Remark 2).

THEOREM 1. *The set A of integers $\mathfrak{D}_s(n)$ for all $s \geq 1$ and all $n \geq 1$ has density zero. That is, if $A(n)$ is the number of positive integers not exceeding n that belong to A, then*

$$\lim_{n\to\infty} \frac{A(n)}{n} = 0.$$

Proof. We use the following result of Niven (cf. [4] corollary 2). (1) For any fixed positive integer k, if p_i is a set of primes for which $\sum p_i^{-1} = \infty$ and if A is any sequence whose members are divisible by atmost k of these primes only to the first degree, then $d(A) = 0$ (where $d(A)$ is the density of A).

Let B denote the set of all integers $\mathfrak{D}_s(n)$ for all $s \geq 2$ and for all $n \geq 1$. Since $\mathfrak{D}_s(m) \geq m^s$, for fixed s, the number of $\mathfrak{D}_s(m)$ counted by $B(n)$ is not more than $n^{1/s}$ and hence

$$B(n) \leq n^{1/2} + n^{1/3} + \cdots + n^{1/r}, \; pgr,$$

where $r \leq \log_2 n$ because for any larger value of r, $\mathfrak{D}_r(2) > n$. Thus $B(n) \leq n^{1/2} \log_2 n$ so that $d(B) = 0$. Let C denote the set of integers $\mathfrak{D}_1(n)$ for all $n \geq 1$. Given $\varepsilon > 0$, choose a positive integer k such that $1/2^k < \varepsilon/2$ and separate C into two disjoint sets C_1 and C_2 where C_1 consists of those elements of C that are divisible by 2^k. Hence for all n, $C_1(n) \leq n/2^k < \varepsilon n/2$. Also $C_2(n) =$ The number of $\mathfrak{D}_1(m) \leq n$ such that $2^k \nmid \mathfrak{D}_1(m)$ and this does not exceed the number of positive integers $m \leq n$ which are divisible by atmost k distinct primes to the first degree. Hence by (1), $d(C_2) = 0$ so that for all large n, $C(n) = C_1(n) + C_2(n) < \varepsilon n$. This proves that $d(C) = 0$ and since $A = B \cup C$, the theorem follows.

REMARK 1. Niven separates A into two possibly overlapping sets B and C, $\mathfrak{D}_s(n)$ being put in B if n has more than k distinct prime factors, otherwise,

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in C. In specifying the choice of k, he says "Any member of the set B satisfies the inequality

$$\mathfrak{D}_s(n) \geq n^s \prod_{j=1}^{k} (1+\rho_j^{-s}) = n^s c_s,$$

ρ_j being the jth prime. The last equality defines c_s, a function of s and k, and we choose k so that $\varepsilon c_1 > 4$". We note that the above inequality is true only if ρ_j is the jth prime dividing n but not the jth prime in the sequence of all rational primes. Thus c_s also depends on n and hence the choice of k mentioned above can not be done.

THEOREM 2. *The set B of all integers $J_s(n)$, for all $s \geq 1$ and all $n \geq 1$ has density zero, where $J_s(n)$ denotes the Jordan totient function of order s* (cf. [1], *Page* 147).

Proof. If B_1 is the set of all integers $J_1(n)$, for all $n \geq 1$, then it is well known that $d(B_1) = 0$ (cf. [5], Theorem 11.9, pp. 249). Let B_2 denote the set of all integers $J_s(n)$ for all $s \geq 2$ and $n \geq 1$. Since

$$J_s(m) = m^s \prod_{p|m} (1-p^{-s}) > m^s \prod_{p} (1-p^{-s}) \geq m^s \prod_{p} (1-p^{-2}),$$

the product ranging over all primes p, repeating the arguments in the first part of the proof of Theorem 1, we get $d(B_2) = 0$. Since $B = B_1 \cup B_2$, the result follows.

THEOREM 3. *Let r, s be fixed non-negative integers and t, u, k be fixed positive integers. Further let*

$$X_{s,t} = \{n \mid (\mathfrak{D}_s(n), J_t(n)) \leq k\},$$
$$Y_{r,s} = \{n \mid (\mathfrak{D}_r(n), \mathfrak{D}_s(n)) \leq k\},$$
$$Z_{t,u} = \{n \mid (J_t(n), J_u(n)) \leq k\}.$$

Then

$$d(X_{s,t}) = d(Y_{r,s}) = d(Z_{t,u}) = 0.$$

Proof. We prove $d(X_{s,t}) = 0$ and the rest are similar. If for an n, $(\mathfrak{D}_s(n), J_t(n)) \leq k$, then n is divisible by atmost k distinct primes only to the first degree. Hence by (1), $d(X_{s,t}) = 0$.

REMARK 2. Taking $t = 1$ in Theorem 3, we see that $d(X_{s,1}) = 0$, where s is a positive integer, which is a recent result due to Dressler (cf. [2], Theorem 2). We note that in addition to (1) mentioned in the proof of Theorem 1 above, Dressler uses the results of Hardy and Ramanujan on the normal order of $\mathfrak{W}(n)$ and $\Omega(n)$ (cf. [3], Theorem 431) which were avoided in our proof.

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