# ON THE PERMANENT OF A DOUBLY STOCHASTIC MATRIX

HERBERT S. WILF

If $A = (a_{ij})_{i,j=1}^{n}$ is an $n \times n$ matrix, the permanent of $A$, Per $A$, is defined by

$$\text{Per } A = \sum a_{1i_1} \ldots a_{ni_n} \tag{1}$$

where the sum is over all permutations. If $A$ is doubly stochastic (i.e., non-negative with row and column sums all equal to 1), then it has been conjectured that Per $A \geqslant n!/n^n$. When confronted with a vaguely similar problem about determinants, M. Kac **(1)** observed that the computation of minima can often be aided by knowledge of various averages. In this spirit we calculate here the average permanent of a class of doubly stochastic matrices and thereby obtain upper bounds for the minima. These turn out to be surprisingly sharp.

Let $n$ and $s$ be fixed positive integers. By the class $K_{ns}$ we mean the set of $n!^s$ matrices (not all different) that result from calculating

$$A = s^{-1}(P_1 + \ldots + P_s) \tag{2}$$

as $P_1, \ldots, P_s$ run independently over the $n \times n$ permutation matrices. Evidently the matrices of $K_{ns}$ are doubly stochastic. The average permanent of these matrices is

$$\begin{aligned}
\gamma_{ns} &= n!^{-s} \sum_{P_1,\ldots,P_s} \text{Per}\left(\frac{P_1 + \ldots + P_s}{s}\right) \\
&= s^{-n} n!^{-s} \sum_{P_1,\ldots,P_s} \text{Per } (P_1 + \ldots + P_s) \\
&= s^{-n} n!^{-s} \sum_{P_1,\ldots,P_s} \sum_{S_n} (P_1 + \ldots + P_s)_{1i_1} \ldots (P_1 + \ldots + P_s)_{ni_n} \\
&= s^{-n} n!^{-s} \sum_{S_n} \sum_{P_1,\ldots,P_s} (P_1 + \ldots + P_s)_{1i_1} \ldots (P_1 + \ldots + P_s)_{ni_n} \\
&= s^{-n} n!^{1-s} \sum_{P_1,\ldots,P_s} (P_1 + \ldots + P_s)_{11} \ldots (P_1 + \ldots + P_s)_{nn}.
\end{aligned}$$

In this last sum we select one of the $P$'s from each set of parentheses, and then do the sum for that particular selection. Suppose $P_1$ is selected $r_1$ times, $\ldots, P_s$ is selected $r_s$ times, where, of course, $r_1 + \ldots + r_s = n$. The number of non-zero contributions to the sum (each non-zero contribution is 1) would then be

$$(n - r_1) \ldots . (n - r_s)!$$

Received June 11, 1965.



https://doi.org/10.4153/CJM-1966-076-6 Published online by Cambridge University Press

The number of ways of selecting $P_1$ $r_1$ times, . . . , $P_s$ $r_s$ times is

$$\binom{n}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_1-\cdots-r_{s-1}}{r_s} = \frac{n!}{r_1!\ldots r_s!}.$$

Consequently the average permanent of the matrices in $K_{ns}$ is

$$(3) \qquad \gamma_{ns} = s^{-n}n!^{2-s} \sum_{r_1+\ldots+r_s=n} \frac{(n-r_1)!\ldots(n-r_s)!}{r_1!\ldots r_s!}.$$

We now establish two features of the asymptotic behaviour of $\gamma_{ns}$. First we claim that for each fixed $n$

$$(4) \qquad \lim_{s\to\infty} \gamma_{ns} = n!\,n^{-n}.$$

To see this, notice that the sum in (3) is the coefficient of $x^n$ in

$$\left(\sum_{j=0}^{n} \frac{(n-j)!}{j!}x^j\right)^s$$

and so

$$\begin{aligned}
\gamma_{ns} &= s^{-n}n!^{2-s}\frac{1}{2\pi i}\oint\left(\sum_{j=0}^{n}\frac{(n-j)!}{j!}z^j\right)^s z^{-n-1}\,dz\\
&= n!^2\frac{1}{2\pi i}\oint\left\{\sum_{j=0}^{n}\frac{(n-j)!}{j!\,n!}\left(\frac{w}{s}\right)^j\right\}^s w^{-n-1}\,dw\\
&= n!^2\frac{1}{2\pi i}\oint e^{w/n}\left\{1+\frac{w^2}{2n^2(n-1)s}+O(s^{-2})\right\}w^{-n-1}\,dw\\
&= \frac{n!}{n^n}\left[1+\frac{n}{2s}+O(s^{-2})\right] \qquad (s\to\infty).
\end{aligned}$$

To investigate $\gamma_{ns}$ for fixed $s$ as $n \to \infty$, we need a different integral representation. Since

$$(x_1+x_2+\ldots+x_s)^n = \sum_{r_1+\ldots+r_s=n}\frac{n!}{r_1!\ldots r_s!}x_1^{r_1}\ldots x_s^{r_s},$$

we have

$$(x_2\ldots x_s+\ldots+x_1\ldots x_{s-1})^n = \sum_{r_1+\ldots+r_s=n}\frac{n!}{r_1!\ldots r_s!}x_1^{n-r_1}x_2^{n-r_2}\ldots x_s^{n-r_s}.$$

It follows that

$$\gamma_{ns} = s^{-n}n!^{1-s}\int_0^\infty\ldots\int_0^\infty\left[(x_1\ldots x_s)\left(\frac{1}{x_1}+\ldots+\frac{1}{x_s}\right)\right]^n e^{-(x_1+\ldots+x_s)} \times dx_1\ldots dx_s\,.$$

The unique maximum of the integrand occurs at

$$x_1^0 = \ldots = x_s^0 = n(1-1/s).$$

Hence we put

$$x_i = nu_i + n(1-1/s) \qquad (i = 1,\ldots,s)$$

https://doi.org/10.4153/CJM-1966-076-6 Published online by Cambridge University Press

in the integral, and obtain

$$(5)\quad \gamma_{ns} \sim s^{-n}\sqrt{2n\pi}\left(\frac{n}{2\pi}\right)^{\frac{1}{2}s}\int_{-\alpha}^{\infty}\dots\int_{-\alpha}^{\infty}\Big\{(u_1+\alpha)\dots(u_s+\alpha)$$
$$\times\left(\frac{1}{u_1+\alpha}+\dots+\frac{1}{u_s+\alpha}\right)e^{-(u_1+\dots+u_s)}\Big\}^n du_1\dots du_s \qquad (n\to\infty),$$

where we have written $\alpha = (1 - s^{-1})$. The maximum of the integrand now occurs at the origin and so by standard asymptotic methods

$$\gamma_{ns} \sim \sqrt{2n\pi}\left(\frac{n}{2\pi}\right)^{\frac{1}{2}s}(2\pi)^{\frac{1}{2}s}|D|^{-\frac{1}{2}}n^{-\frac{1}{2}s}(1-1/s)^{ns-n}$$
$$= \sqrt{\frac{2\pi n}{|D|}}\,(1-1/s)^{ns-n},$$

where $D$ is the determinant of the $s \times s$ matrix of second partial derivatives of the logarithm of the braces in (5), evaluated at the origin. This matrix is

$$f_{\mu\nu} = \begin{cases} -1, & \mu = \nu, \\ -1/(s-1)^2, & \mu \neq \nu, \end{cases}$$

whence $|D| = (1 - 2s^{-1})^{s-1}(1 - s^{-1})^{1-2s}$. We have finally

$$(6)\qquad \gamma_{ns} \sim \sqrt{2\pi n}\,(1-2/s)^{-\frac{1}{2}(s-1)}(1-1/s)^{ns-n+s-\frac{1}{2}} \qquad (n\to\infty).$$

In particular

$$\lim_{n\to\infty}\gamma_{ns}^{1/n} = (1-1/s)^{s-1}$$
$$= e^{-1}(1+1/2s+O(s^{-2})) \qquad (s\to\infty).$$

We conclude by showing that

$$(7)\qquad \gamma_{ns} \geqslant n!\,n^{-n} \qquad (n, s = 1, 2, \dots).$$

From (3), we have

$$\gamma_{ns} = s^{-n}n!^2\sum_{r_1+\dots+r_s=n}\left[\frac{(n-r_1)!}{r_1!\,n!}\right]\dots\left[\frac{(n-r_s)!}{n!\,r_s!}\right]$$
$$= s^{-n}n!^2\sum_{r_1+\dots+r_s=n}\left[\frac{1}{r_1!\,n(n-1)\dots(n-r_1+1)}\right]$$
$$\dots\left[\frac{1}{r_s!\,n(n-1)\dots(n-r_s+1)}\right]$$
$$\geqslant s^{-n}n!^2\sum_{r_1+\dots+r_s=n}\left[\frac{1}{r_1!\,n^{r_1}}\right]\dots\left[\frac{1}{r_s!\,n^{r_s}}\right]$$
$$= n!\,n^{-n}$$

as required.

https://doi.org/10.4153/CJM-1966-076-6 Published online by Cambridge University Press

As a numerical example, consider the set of all $k$-stochastic $n \times n$ matrices with non-negative integer elements. This set, aside from a factor of $k$, is our set $K_{nk}$. Hence the minimum permanent of matrices in this set is $\leqslant k^n \gamma_{nk}$. When $n = 19$, $k = 10$, for instance, $k^n \gamma_{nk}$, from (6), is about $16.6 \times 10^{11}$. The incidence matrix of a 19–10–5 configuration **(3)** has a permanent of $9.53 \times 10^{11}$ **(2)**, and it has been suggested **(3)** that this may be minimal in the class.

For future research I would raise the question of computing the mean square permanent of matrices in the class (2). If the arithmetic mean and the mean square are both available, improved estimates of the minimum can be made. Another interesting question concerns the average permanent of (2) when the $P_i$ are restricted to the cyclic permutations.

## References

**1.** M. Kac, *Probability and related topics in physical sciences*, Seminar at Boulder, Colorado, Summer, 1957 (lecture notes).

**2.** P. J. Nikolai, *Permanents of incidence matrices*, MTAC(1960), 262–266.

**3.** H. J. Ryser, *Matrices of zeros and ones*, Bull. Amer. Math. Soc., *66* (1960), 442–464.

*University of Pennsylvania*

https://doi.org/10.4153/CJM-1966-076-6 Published online by Cambridge University Press