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NECESSARY CONDITIONS FOR UNIVERSAL INTERPOLATION IN $\hat{\mathscr{E}}'$

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1. Introduction. Let $\hat{\mathscr{E}}'$ be the space of Fourier transforms of distributions with compact support, or equivalently, the space of entire functions h satisfying the growth condition

$$(1) \qquad |g(z)| \leqq A \exp (Bp(z)) \text{ for all } z \in \mathbf{C}$$

where $p(z) = |\operatorname{Im} z| + \log (1 + |z|^2)$ and A, B are constants depending only on h. A sequence $\{z_k\}_{k=1}^{\infty} \subset \mathbf{C}$ with $|z_k| \uparrow \infty$ is said to be a *universal interpolation sequence for* $\hat{\mathscr{E}}'$ if for all $\{a_k\}_{k=1}^{\infty}$ such that

$$(2) \qquad |a_k| \leqq A \exp (Bp(z_k)) \quad k = 1, 2, \ldots$$

for constants A, B independent of k, there exists $f \in \hat{\mathscr{E}}'$ such that $f(z_k) = a_k$. In this note we will consider necessary conditions for universal interpolation in $\hat{\mathscr{E}}'$ and more general subspaces of the entire functions.

If $\{z_k\}_{k=1}^{\infty}$ is a universal interpolating sequence for $\hat{\mathscr{E}}'$ then for some $h \in \hat{\mathscr{E}}'$ we must have

$$\{z_k\}_{k=1}^{\infty} \subset Z(h) = \{z | \, h(z) = 0\}.$$

To see this note that $\{z_k\}_{k=1}^{\infty}$ a universal interpolation sequence implies there exists $f \in \hat{\mathscr{E}}'$ such that $f(z_1) = 1$, $f(z_k) = 0$, $k = 2, 3, \ldots$. Thus we have

$$\{z_k\}_{k=1}^{\infty} \subset Z((z - z_1) f (z))$$

where $f \not\equiv 0$.

If in (1) we let $p(z) = |z|$ the resulting space of entire functions is the space of functions of exponential type, denoted A_1. It is known (see [**2**]) that if $\{z_k\}_{k=1}^{\infty} = Z(h)$ for some $h \in A_1$ then $\{z_k\}_{k=1}^{\infty}$ is a universal interpolation sequence for A_1 if and only if

$$(3) \qquad |h'(z_k)| \geqq \epsilon \exp (-Cp(z_k)) \quad k = 1, 2, \ldots$$

with ϵ, C constants independent of k. This result is false for $\hat{\mathscr{E}}'$ as our example will show which answers the question posed in [**1**], page 34.

In a positive direction we have Theorem 1 which shows that if $\{z_k\}_{k=1}^{\infty}$ is a universal interpolation sequence for $\hat{\mathscr{E}}'$ then there exists

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$h \in \hat{\mathscr{E}}'$ such that $\{z_k\}_{k=1}^{\infty} \subset Z(h)$ and (3) holds for

$$p(z_k) = |\text{Im } z_k| + \log (1 + |z_k|^2), \quad k = 1, 2, \ldots .$$

Theorem 1, together with a result of Berenstein and Taylor gives us Theorem 2, namely, necessary and sufficient conditions for $\{z_k\}_{k=1}^{\infty}$ to be a universal interpolation sequence in $\hat{\mathscr{E}}'$. The condition for universal interpolation involves finding two defining functions $f_1, f_2 \in \hat{\mathscr{E}}'$ for $\{z_k\}_{k=1}^{\infty}$ such that

$$\{z_k\}_{k=1}^{\infty} = Z(f_1, f_2) = \{z | f_1(z) = f_2(z) = 0\}$$

and $|f_1'| + |f_2'|$ satisfies (3) for

$$p(z_k) = |\text{Im } z_k| + \log (1 + |z_k|^2).$$

However, this theorem does not give a practical way of determining whether or not $\{z_k\}_{k=1}^{\infty}$ is a universal interpolation sequence since we have no constructive way of finding f_1 and f_2.

2. Notation and definitions. We shall always assume that $p(z)$ is a subharmonic function defined for all $z \in \mathbf{C}$, $p \not\equiv -\infty$, satisfying the following two conditions (see [**1**] for more details)

(4) $p(z) \geqq 0$ and $\log (1 + |z|^2) = O(p(z))$

(5) there exist constants C and D such that

$$|\zeta - z| \leqq \text{ implies } p(\zeta) \leqq Cp(z) + D.$$

Note that (5) says that p is approximately constant on discs of radius less than or equal to 1.

Definition. $A_p = \{ f \text{ entire} |\ |f(z)| \leqq A \exp (Bp(z))$ for some constants A, B depending on $f \}$.

It is easily seen that conditions (4) and (5) on $p(z)$ imply

(6) all polynomials belong to A_p

(7) A_p is closed under differentiation, that is, $f \in A_p$ implies $f' \in A_p$.

The two most important examples of such functions p are

$$p(z) = |z| \text{ and } p(z) = |\text{Im } z| + \log (1 + |z|^2)$$

corresponding to the spaces A_1 of entire functions of exponential type and $\hat{\mathscr{E}}'$.

We will now define what we mean by a universal interpolation sequence for the spaces A_p. Let $V = \{(z_k, m_k)\}_{k=1}^{\infty} \subset Z(h)$ for some $h \in A_p$ where (z_k, m_k) means a zero of multiplicity m_k at z_k.

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Definition. $A_p(V) = \{\gamma = \{\{\gamma_{kj}\}_{j=0}^{m_k-1}\}_{k=1}^{\infty} \mid |\gamma_{kj}| \leqq A \exp(Bp(z_k))$ for constants A and B, independent of k but depending on $\gamma\}$.

With the above definition define the restriction map $\rho: A_p \to A_p(V)$ by

$$\rho(f) = \gamma$$

where

$$\frac{f^{(j)}}{j!}(z_k) = \gamma_{kj} \quad j = 0, 1, \ldots, m_k - 1, k = 1, 2, \ldots .$$

Definition. A multiplicity sequence $V = \{(z_k, m_k)\}_{k=1}^{\infty}$ will be called a *universal interpolation sequence* if the restriction map ρ is onto.

3. Example. Now we will give an example of a variety $V = \{z_k\}_{k=1}^{\infty}$, each point having multiplicity one and $V = Z(h)$ for $h \in \hat{\mathscr{E}}'$. The variety will have the property that V is a universal interpolation sequence for $\hat{\mathscr{E}}'$ and h' is too small on V, that is, there exist no constants ϵ, C such that

$$|h'(z_k)| \geqq \epsilon \exp(-C[|\text{Im } z_k| + \log(1 + |z_k|^2)]).$$

Let

$$\phi(z) = \prod_{j=1}^{\infty}\left(1 - \left(\frac{z}{2^j}\right)^2\right)$$

and let

$$h(z) = \frac{\sin(\pi z)}{\phi(z) \cdot z}.$$

We will show that for each n there exists C_n such that

(8) $\quad |h(x)| \leqq C_n/(1 + |x|)^n$ for all $x \in \mathbf{R}$, $n = 0, 1, 2, \ldots .$

Since h is an even function and $Z(h) \subset Z(\sin(\pi z))$ it is clear that h is of exponential type. This fact and (8) imply $h \in \hat{\mathscr{D}}$, where $\hat{\mathscr{D}}$ is the space of Fourier transforms of C^∞ functions with compact support.

It is clear that $V = Z(h)$ is a universal interpolation sequence for $\hat{\mathscr{E}}'$ since $V \subset Z(\sin(\pi z))$ and $Z(\sin(\pi z))$ is certainly a universal interpolation sequence for $\hat{\mathscr{E}}'$ as is easily seen from Theorem 4 [**1**]. Since $h \in \hat{\mathscr{D}}$ we have $h' \in \hat{\mathscr{D}}$ and thus it is clear that h cannot satisfy inequality (3) for any constants ϵ, C.

Now we will prove (8). To prove (8) it suffices to prove that

(9) $\quad |h(x)| \leqq K_0/|x|^{n-8}$ for all x, $2^{n-1} \leqq x \leqq 2^n$

where K_0 is a constant independent of n.

First let $x \in I_n = [2^{n-1} + 1, 2^n - 1]$. Since we are assuming $x \leqq 2^n$ then for $j \geqq n$ we have

$$1 - \left(\frac{x}{2^j}\right)^2 \geqq 1 - \frac{1}{2^{2(j-n)}}$$

which implies

$$\prod_{j=n+1}^{\infty} \left(1 - \left(\frac{x}{2^j}\right)^2\right) \geqq \prod_{j=1}^{\infty} \left(1 - \frac{1}{2^j}\right) = C_0.$$

Thus we have

$$(10) \quad |h(x)| \leqq \prod_{j=1}^{n} \left|1 - \left(\frac{x}{2^j}\right)^2\right|^{-1} \cdot \frac{1}{C_0} \quad \text{for all } x \in I_n.$$

Next let us compute a lower bound for the product

$$\left|\prod_{j=1}^{n} \left(1 - \left(\frac{x}{2^j}\right)^2\right)\right|.$$

We have

$$\prod_{j=1}^{n} \left|1 - \left(\frac{x}{2^j}\right)^2\right| \geqq \prod_{j=1}^{n-2} \left|\frac{2^{2j} - x^2}{2^{2j}}\right| \cdot \frac{1}{2^{2n-2}} \cdot \frac{1}{2^{2n}}$$

$$\geqq \frac{1}{2^{4n-2}} \prod_{j=2}^{n-2} \frac{(2^{2n-2} - 2^{2j})}{2^{2j}} \geqq \frac{1}{2^{4n-2}} \cdot \frac{1}{2^{n-2}} \cdot \prod_{j=1}^{n-2} 2^{2n-2j-2}$$

$$\geqq \frac{1}{2^{5n-4}} \cdot 2^{(n-2)(n-1)} \geqq 2^6 (2^n)^{n-8}.$$

Hence

$$(11) \quad \prod_{j=1}^{n} \left|1 - \left(\frac{x}{2^j}\right)^2\right| \geqq |x|^{n-8}.$$

The last inequality along with (10) gives

(12) $\quad |h(x)| \leqq 1/C_0 |x|^{n-8}$ for all $x \in I_n$.

We will now consider x in the interval $J_n = [2^n - 1, 2^n + 1]$ and show that (12) holds for $x \in J_n$ with $1/C_0$ replaced by a larger constant. We will obtain a lower bound for ϕ on the circle $C_n = \{z \mid |z - 2^n| = 1\}$ which will give an upper bound for h on C_n and applying the maximum principle we get the desired upper bound for $x \in [2^n - 1, 2^n + 1]$.

For $z \in C_n$ it is easy to show

$$\prod_{j=n+1}^{\infty} \left|1 - \left(\frac{z}{2^j}\right)^2\right| \geqq \prod_{j=1}^{\infty} \left(1 - \frac{1}{2^j}\right) = C_0$$

and a calculation similar to (11) gives us

$$\prod_{j=1}^{n} \left|1 - \left(\frac{z}{2^j}\right)^2\right| \geqq |z|^{n-8}.$$

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Now $|\sin(\pi z)| \leqq (1 + (\sinh 1)^2)^{1/2}$ for $z \in C_n$ and thus

$$\left| \frac{\sin(\pi z)}{z \cdot \phi(z)} \right| \leqq \frac{(1 + (\sinh 1)^2)^{1/2}}{C_0 |z|^{n-8}} \quad \text{for all } z \in C_n.$$

An application of the maximum principle allows us to conclude that for all $x \in J_n$

$$(13) \quad |h(x)| = \left| \frac{\sin(\pi x)}{x \cdot \phi(x)} \right| \leqq \frac{K_0}{|x|^{n-8}}$$

where K_0 is a constant independent of n. From (12) and (13) we deduce $h \in \mathscr{D}$.

It is possible to define V by two "good" functions namely $h_1(z) = \sin(\pi z)$ and $h_2(z)$ which has the same zeros as $h_1(z)$ except for the zeros at $\pm 2^n$, $n = 1, 2, \ldots$ which are perturbed by some small amount. Then

$$|h_1'(n)| + |h_2'(n)| \geqq \epsilon \exp(-C \log(1 + n^2)) \quad \text{for all } n \neq 2^k, k = 1, 2, \ldots$$

where ϵ, C are constants independent of n.

4. Necessary conditions for interpolation. We see in the example that the universal interpolation sequence $Z(h)$ can be defined as a subset of the zero set of a function with large derivative, namely $\sin(\pi z)$. This leads us to the question as to whether or not every universal interpolation sequence $V = \{(z_k, m_k)\}_{k=1}^{\infty}$ in the space A_p is the zero set of a function F satisfying

$$\frac{|F^{(m_k)}(z_k)|}{m_k!} \geqq \epsilon \exp(-Cp(z_k)) \quad k = 1, 2, \ldots.$$

This problem was posed in [**4**] page 258 and the following theorem gives an affirmative answer to this question.

THEOREM 1. *If $V = \{(z_k, m_k)\}_{k=1}^{\infty} \subset Z(h)$ for $h \in A_p$ is a universal interpolation sequence for the space A_p then there exists a function $F \in A_p$ such that*

$$|F^{(m_k)}(z_k)|/m_k! \geqq \epsilon \exp(-Cp(z_k)) \text{ for all } k.$$

Proof. The basic idea of the proof was communicated to us by E. Kronstadt. The proof will follow easily from the following two lemmas (see [**3**]).

LEMMA 1. *If V is a universal interpolation sequence for A_p then for all $C > 0$, there exist constants A, B and functions $f_{kj} \in A_p$ such that $\rho(f_{kj}) = e_{kj}$ (e_{kj} is the sequence in $A_p(V)$ which is 0 except for a 1 at the (k, j) place) and*

$$|f_{kj}(z)| \leqq A \exp(Bp(z))/\exp(Cp(z_k)).$$

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LEMMA 2. *If* $V = \{(z_k, m_k)\}_{k=1}^{\infty}$ *is a universal interpolation sequence for* A_p *then there exists* $C > 0$ *such that*

$$\sum_{k=1}^{\infty} \exp(-Cp(z_k)) < \infty.$$

Assuming the two lemmas for the moment we will prove the theorem. Let

$$F(z) = \sum_{k=1}^{\infty} m_k(z - z_k) f_{km_k-1}(z) \cdot f_{k0}(z).$$

It suffices to show $|F(z)| \leqq A \exp(Bp(z))$ for some constants A and B since by differentiating the sum term by term we see

$$\frac{F^{(m_k)}(z_k)}{m_k!} = \frac{f_{km_k-1}^{(m_k-1)}(z_k)}{(m_k - 1)!} \cdot f_{k0}(z_k) = 1.$$

Now we have

$$|F(z)| \leqq \sum_{k=1}^{\infty} m_k(|z| + |z_k|)|f_{km_k-1}(z)|\,|f_{k0}(z)|$$

$$\leqq |z| \cdot A^2 \exp(2Bp(z)) \cdot \sum_{k=1}^{\infty} m_k \exp(-2Cp(z_k))$$

$$+ A^2 \exp(2Bp(z)) \cdot \sum_{k=1}^{\infty} m_k|z_k| \exp(-2Cp(z_k))$$

using the estimate obtained for $|f_{kj}(z)|$ in Lemma 1. The next estimates require the following two facts

(i) $m_k \leqq Ep(z_k) + F$ for some constants E, F independent of k (see [**1**] page 126).

(ii) $|z| \leqq \exp(Kp(z))$ (a consequence of (4)).

Thus

$$|F(z)| \leqq A' \exp[(2B + K)p(z)] \sum_{k=1}^{\infty} \{\exp[(-2C + E)p(z_k)]$$

$$+ \exp[(-2C + E + K)p(z_k)]\}$$

$$\leqq A'' \exp[(2B + K)p(z)]$$

if C is chosen sufficiently large so that the sum converges. This completes the proof of the theorem.

Proof of Lemma 1. Let

$$D = \{\gamma \in A_p(V) |\, |\gamma_{kj}| \leqq \exp(Cp(z_k)),$$
$$j = 0, 1, \ldots, m_k - 1, k = 1, 2, \ldots\}$$

and let D have the topology induced by the norm

$$\|\gamma\| = \sup_{k,j} |\gamma_{kj}| \exp(-Cp(z_k)).$$

With this norm we see that $D = \{\gamma | \|\gamma\| \leqq 1\}$ and D is closed in this topology.

Now let $U_n = \{ f \text{ entire} \mid |f(z)| \leqq n \exp (np(z))\}$.

Claim. $\rho(U_n) \cap D$ is closed in D, in the topology induced by $\| \ \|$.

Let $\{\rho(f_j)\}_{j=1}^{\infty} \subset \rho(U_n \cap D)$ such that $\rho(f_j) \to \gamma \in D$. Now all the f satisfy the uniform bound $|f_j(z)| \leqq n \exp (np(z))$ and thus, because it is a normal family, there exists a subsequence $\{f_{jl}\}_{l=1}^{\infty}$ such that $f_{jl} \to f \in U_n$ and $\rho(f) = \gamma$. From this we conclude $\rho(U_n) \cap D$ is closed and the claim is proved.

The hypothesis that V is a universal interpolation sequence is equivalent to ρ being onto and thus we have

$$\bigcup_{n=1}^{\infty} [\rho(U_n) \cap D] = D.$$

Now we apply the Baire Category theorem to conclude for some n, $\rho(U_n) \cap D$ contains an open set. Without loss of generality we can assume

$$\rho(U_n) \cap D \supset \{\gamma | \|\gamma\| \leqq \epsilon\}$$

and it easily follows that $\rho(1/\epsilon U_n) \cap D = D$.

Thus we have shown there exist $\hat{f}_{kj}(z)$ such that

$$\rho(\hat{f}_{kj}) = \exp (Cp(z_k))e_{kj} \text{ and } |\hat{f}_{kj}(z)| \leqq A \exp (Bp(z)).$$

If we let $f_{kj}(z) = \hat{f}_{kj}(z)/\exp (Cp(z_k))$ then

$$\rho(f_{kj}) = e_{kj} \text{ and } |f_{kj}(z)| \leqq A \exp (Bp(z))/\exp (Cp(z_k)).$$

This completes the proof of Lemma 1.

Proof of Lemma 2. We know that since p satisfies (1) there exists $C_1 > 0$ such that

$$\int_{\mathbf{C}} \exp (-C_1 p(z))dxdy < \infty.$$

Let $d_j = \min_{k \neq j} |z_k - z_j|$, the distance of the closest zero to z_j, and let B_j be the disc of radius $r_j = \min \{d_j/2, 1\}$ about z_j. Then we have

$$\sum_{j=1}^{\infty} \int_{B_j} \exp (-C_1 p(z))dxdy \leqq \int_{\mathbf{C}} \exp (-C_1 p(z))dxdy < \infty.$$

It can be shown that $d_j/2 \geqq \epsilon \exp (-C_2 p(z_j))$ (see [**1**], page 126) which implies

$$\epsilon \exp (-C_2 p(z_j)) \leqq r_j \leqq 1.$$

From the hypothesis that p satisfies (2) we have $p(z) \leqq Ap(z_j) + B$ for

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all $z_j \in B_j$. Thus it follows that

$$\int_{B_j} \exp(-C_1 p(z))dxdy \geqq \int_{B_j} \epsilon \exp(-C_3 p(z_j))dxdy$$
$$\geqq \epsilon \exp(-C_3 p(z_j)) \int_{B_j} 1\, dxdy \geqq \epsilon \exp(-Cp(z_j))$$

for some constant C larger than C_2 and C_3. From the above inequality we can conclude

$$\sum_{j=1}^{\infty} \exp(-Cp(z_j)) < \infty .$$

This completes the proof of Lemma 2.

Theorem 1 and a result of Berenstein and Taylor (see [**1**]) gives us necessary and sufficient conditions for universal interpolation in the spaces A_p. This is the content of Theorem 2.

THEOREM 2. *Let* $V = \{(z_k, m_k)\}_{k=1}^{\infty} \subset Z(h)$ *for some* $h \in A_p$. *Then* V *is a universal interpolation sequence in* A_p *if and only if there exist* F_1, $F_2 \in A_p$ *such that* $V = Z(F_1, F_2)$ *and*

$$(14)\quad \frac{|F_1{}^{(m_k)}(z_k)|}{m_k!} + \frac{|F_2{}^{(m_k)}(z_k)|}{m_k!} \geqq \epsilon \exp(-Cp(z_k)) \quad k = 1, 2, \ldots$$

for some constants ϵ, C *independent of* k.

Proof. ($\Rightarrow$) Theorem 1 shows that there exists one function F_1 which satisfies (14) at every point $z_k \in V$. We will show, by perturbing those zeros of F_1 which are not in V, there exists a function $F_2 \in A_p$ such that $V = Z(F_1, F_2)$.

Define

$$F_2(z) = F_1(z) \prod_{k=1}^{\infty} \left(\frac{z - w_k + \epsilon_k}{z - w_k}\right)^{n_k}$$

where $\{(w_k, n_k)\}_{k=1}^{\infty} = Z(F_1) \sim V$ and $\{\epsilon_k\}_{k=1}^{\infty}$ is a sequence of small constants to be chosen later. Now let $\{a_k\}_{k=1}^{\infty}$, $\{b_k\}_{k=1}^{\infty}$ be two sequences of positive numbers satisfying the following conditions:

(i) The discs $D(w_k, b_k) = \{z|\ |z - w_k| \leqq b_k\}$ are pairwise disjoint.

(ii) $b_k \leqq 1$ for all k so that $p(z) \leqq Cp(w_k) + D$ for $z \in D(w_k, b_k)$.

(iii) $\displaystyle\sum_{k=1}^{\infty} \frac{n_k}{a_k} = K < \infty .$

Define $\epsilon_k = b_k/a_k$. It remains to prove $F_2 \in A_p$ and it will suffice to show $|F_2(z)| \leqq K_1|F_1(z)|$ for some constant K_1. First assume $|z - w_k| \geqq$

b_k for all k. Then we have

$$|F_2(z)| \leqq |F_1(z)| \prod_{k=1}^{\infty} \left|1 + \frac{\epsilon_k}{z - w_k}\right|^{n_k} \leqq |F_1(z)| \prod_{k=1}^{\infty} \left|1 + \frac{1}{a_k}\right|^{n_k}$$
$$\leqq \exp (K) |F_1(z)|.$$

Now suppose $|z - w_k| \leqq b_k$ for some k (exactly one k by (i)). By applying the maximum principle and noting that (ii) holds it suffices to consider $|z - w_k| = b_k$. The above estimate still holds in this case, namely

$$|F_2(z)| \leqq \exp (K)|F_1(z)| \text{ for } |z - w_k| = b_k.$$

Thus we conclude $F_2 \in A_p$ as desired.

($\Rightarrow$) This follows from Theorem 4 page 126 of [**1**].

Remark. Theorem 2 actually gives no more information than Theorem 1 since given an arbitrary sequence $V \subset Z(h)$ for some $h \in A_p$ we have no constructive procedure for finding F_1 and F_2.

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