THESIS ABSTRACT

R.M. Raphael, Algebraic closures of commutative regular rings, McGill University. April 1969. (Supervisor: J. Lambek)

All rings discussed are commutative with 1. Let S be an over-ring of R. S is essential over R, if every non-zero ideal of S has non-zero meet with R. S is algebraic over R if is essential over R, and if it is integral over R in the sense of Zariski-Samuel. S is weakly algebraic over R if all between rings of R and S are essential over R. The definitions 'algebraic' and 'weakly algebraic' coincide with the usual definition of algebraic in the case of fields. One shows that algebraic extensions of semiprime rings are weakly algebraic, but that the converse is not true in general. As well, an algebraic extension of a regular ring is regular, whereas examples show that a weakly algebraic extension of a regular ring need not be regular.

One calls a regular ring R algebraically closed if it satisfies any of the following conditions, which can be shown to be equivalent:

1. every algebraic embedding with domain R is an isomorphism;
2. R is Baer (i.e. annihilators are direct summands) and every monic polynomial over R has a root in R;
3. R is Baer, and all of its quotient fields are algebraically closed.

One calls a regular ring R weakly algebraically closed if it satisfies one of the two following equivalent conditions:

1. every weak algebraic embedding with domain R is an isomorphism;
2. R is algebraically closed and self-injective.

Main result: Let R be regular. Then R can be embedded algebraically into an algebraically closed regular ring S, which is unique up to isomorphism over R, and which contains a copy (over R) of any algebraic extension of R. One shows non-trivially that the same theorem is valid for weak algebraic extensions, i.e., there is a weak algebraic closure which is unique up to isomorphism and has the universal property. Furthermore, the weak algebraic closure is the complete ring of quotients of the algebraic closure.

The simplest example of a ring, for which the two closures differ is as follows: let R be the product of countably infinitely many copies of the two element field. Let S be the product over the same index set of copies of the algebraic closure of the two element field. Then S is the weak algebraic closure of R. However, the algebraic closure of R is the integral closure of R in S, and this is shown to be strictly proper in S.

The preservation of the properties 'algebraically closed' and 'weakly algebraically closed' under the formation of products and quotient objects is discussed.

The algebraic closure is given for Boolean rings, for rings of continuous functions into finite fields, and for Q(X), the complete ring of quotients of the ring of real valued continuous functions defined on a space.

A question posed by Enochs (page 706, Proceedings AMS, June 1968) concerning 'totally integrally closed' rings is answered in the negative.

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