BASAL STRESS CONCENTRATIONS DUE TO ABRUPT CHANGES IN BOUNDARY CONDITIONS: A CAUSE FOR HIGH TILL CONCENTRATION AT THE BOTTOM OF A GLACIER

by

Kolumban Hutter

(Versuchsanstalt für Wasserbau, Hydrologie und Glaziologie, ETH-Zentrum, CH-8092 Zürich, Switzerland)

and Vincent O. S. Olunloyo

(Department of Mechanical Engineering, University of Ilorin, Ilorin, Nigeria)

ABSTRACT

The existence of cold patches at the base of a glacier suggests that the sliding law will depend on these patches, which will essentially affect the viscosity constant. In a polythermal glacier, such as a glacier which is cold in its lower part and temperate in its upper part, basal boundary conditions change from no-slip to viscous sliding. It is anticipated that the viscosity constant of this sliding law will depend on the distance from the transition line between cold and temperate ice.

The mixed boundary conditions, namely no-slip where the ice is cold and viscous sliding where it is temperate, induce large stresses and velocity changes close to the transition line. In fact, it is shown that, for a Newtonian fluid and all investigated discontinuities of boundary data, square-root singularities of the stresses will develop at the transition line. Asymptotic expressions for the basal stresses are derived. The explicit forms of these asymptotic expansions depend on the form of the spatial dependence of the sliding law and, furthermore, on the numerical values of the viscosity coefficient. It is, moreover, argued that the stress concentrations are sufficiently pronounced to account for the removal of basal rock especially in regions of high cleavage concentrations, the details again depending upon the sliding coefficients.

No mathematical details of the problem solved are presented as attention is focused on the physical processes.

1) INTRODUCTION

In seeking quantitative descriptions of glacier flow, attention is generally limited to glaciers which are either cold or temperate. With the exception of Fowler (1979[b]) and Hutter and Olunloyo (1980) there are no investigations known to us which deal with the state of stress and velocity in a *polythermal* environment. Such glaciers are partly temperate and partly cold, and, depending on which of these states applies at the base, the ice is thought to be either fully adhering or sliding over its bed. It does not seem to be necessary to defend the postulate of the existence of polythermal glaciers, as there is sufficient field evidence corroborating cold ice in glaciers that are believed to be temperate at their beds. Harrison (1972) reports that water in some bore holes slowly freezes; Theakstone (1967) mentions that air temperatures in a tunnel beneath 50 m of ice near the margin of Østerdalsisen in Norway remained almost constant at $-0.1^{\circ}C$, and Vivian and Bocquet (1973) state that the ice roof of natural subglacial caverns in the Glacier d'Argentière was often below the melting point. Robin (1976) has offered a "heat pump" argument to explain the existence of cold patches at the base of a temperate glacier. It appears that spatial, as well as temporal, variations of basal pressure cause freezing of ice to rock. Indirect evidence in support of this is the intermittent stick-slip motion of temperate glaciers reported by both Theakstone (1967) and Vivian and Bocquet (1973).

The cold patches at the base, described by Robin (1976), are assumed to be of the extent of the roughness amplitudes of the bed, or smaller, and must therefore affect the viscous sliding law supposed to apply at the smoothed-out surface of the global flow problem; in a first approximation, one may assume that they only affect the constant C in the sliding law

$$u_b = C\tau^m, \tag{1}$$

where u_b is the basal sliding velocity, τ the corresponding shear traction, and C and m are constants, m having the value $\frac{(n+1)}{2}$ (Lliboutry 1979, Weertman 1979) or n (Fowler 1979[a]), n being the exponent in Glen's flow law. A proper treatment would have to take into account possible stress concentrations at the separation lines of cold and temperate ice and would have to include the possibility of tearing off some rocks from below. A mathematical solution to this problem does not seem to be in sight as it would consist of a complex, mixed boundary-value problem.

However, when the cold and temperate portions of the base extend over several thicknesses of the glacier, the stress-concentration problem where no-slip goes over into viscous sliding becomes mathematically amenable to analysis, at least approximately. This situation corresponds to a glacier which is cold in the

lower and temperate in the upper parts. Our situation corresponds to that discussed by Müller (1976). The reverse also occurs, and is believed to arise more frequently; it can be obtained from our analysis by simply selecting the angle γ as negative (this angle is described in section 2 and Figure 1). The abrupt change in boundary condition from no-slip to viscous sliding will induce large stress and velocity variations. The maximal stresses are to be expected at the "discontinuity" of boundary data.
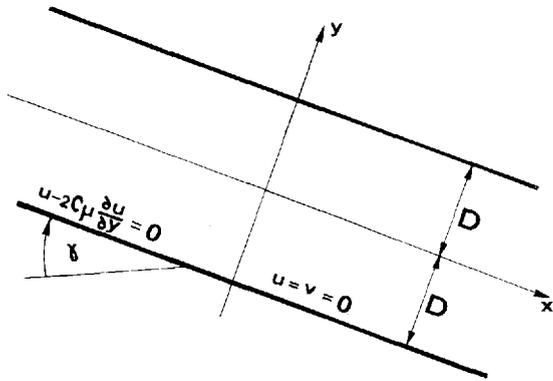
Fig. 1. Flow of ice down a strictly parallel-sided slab can be modelled approximately by treating a channel-flow problem of ice held between two plates which are twice as far apart as the ice strip is deep. Inclination of the slope is γ and the basal boundary conditions are no-slip ($u = v = 0$) and viscous sliding, respectively.

One question of geomorphological interest is whether these stresses are sufficiently high to tear off portions of the rock base. If they are, a partial explanation of the substantial amount of till observed at the bottom of an ice bore hole (Engelhardt and others 1978) could be explained. This, then, is the question to which we shall address ourselves, and indeed we shall demonstrate that the removal of rock from the base in the above-mentioned conditions is possible.

2) MATHEMATICAL PROBLEM

An infinite, parallel-sided ice slab under plane motion is studied, the upper half of the base of which adheres to the bed while the lower half of the bed is subject to viscous sliding. Of interest is the stress distribution induced by gravity and, in particular, its behaviour near the abrupt change of boundary data. The exact problem based on non-linear rheology (e.g. Glen's flow law) is hardly amenable to an analytical solution, but by restricting atten-tion to Newtonian behaviour approximate solutions can be obtained with reasonable effort. Although the Newtonian model suffers the disadvantage of not being materially realistic for ice, it is a useful one because it allows the solution of a well-defined mathematical problem free of *ad hoc* assumptions. The same applies to the many other linear models intro-duced into the ice literature (Nye 1969 and 1970, Kamb 1970, Lliboutry 1975, Morland 1976[a] and 1976[b], Morris 1979, Hutter and Olunloyo 1980).

Specifically, consider slow flow of a Newtonian viscous fluid held between two parallel planes whose inclination angle is γ and which are at a distance $2D$ apart (Fig.1). Let (x,y) be a Cartesian coordinate system; x is midway between the two planes and y perpendicular to it, so that

the two planes confine the fluid to the region $|y| \leq D$. We assume mixed boundary conditions, namely no-slip for $x>0$ and viscous sliding for $x \leq 0$. The sliding law will be taken in the form of Equation (1) with $m=1$ and a coefficient $C(x)$ that varies with x. This x-dependence will be further elaborated below. Let ρ be the density, g the gravity constant, μ the dynamic viscosity, (u,v) the longitudinal and transversal velocity components, and p the pressure. The governing equations describing steady flow through the above channel are given by the differential equations (Fig.1):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$\frac{\partial p}{\partial x} = \rho g \sin\gamma + \mu \nabla^2 u, \qquad (2)$$

$$\frac{\partial p}{\partial y} = -\rho g \cos\gamma + \mu \nabla^2 v,$$

subject to boundary conditions:

$$u = v = 0, \qquad |y| = D, \ x > 0,$$
$$u - C\mu\frac{\partial u}{\partial y} = v = 0, \quad |y| = D, \ x \leq 0. \qquad (3)$$

Equations (2) express mass balance for an incom-pressible body and momentum balance when convective acceleration terms are neglected. The first of Equations (3), on the other hand, expresses the no-slip boundary condition, whereas the second of (3) is the viscous sliding law. Because of its physical significance the latter warrants further attention.

We assume the sliding coefficient to vary along the temperate part of the boundary, typically as shown in Figure 2. At large values
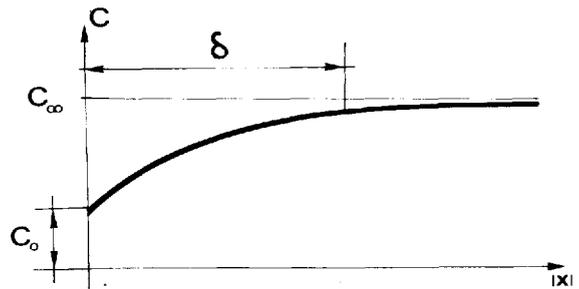
Fig. 2. A typical assumed variation of the viscosity coefficient in the basal sliding law. At the transition point where temperate ice becomes cold $C = C_0$ is minimal and may even have the value $C_0 = 0$. This would then corres-pond to a continuous transition from no-slip to viscous sliding.

of $|x|$, effectively for $|x| > \delta$, where δ is a *transition length*, the sliding coefficient is nearly constant with a value approaching C_∞ as $|x| \to \infty$. For $0 \leq |x| \leq \delta$, C depends on x, starting at $x = 0^-$ with a value C_0 and then monotonically increasing, approaching the value C_∞ effectively at $x = -\delta$. We regard this variation as a reasonable model for the description of the transition from viscous sliding to no-slip. Close to $x = 0$ a decrease of C with decreasing $|x|$ is most likely to occur since the number of cold patches will increase as the cold portion of the base is approached. We assume that the transition depth δ must be small compared with the glacier thickness D; it is, however, physically not clear whether C_0 differs from zero or must vanish. Fowler's work (1979[a])

suggests that C_0 should vanish, and he claims that in this case both stresses and velocities remain bounded at the point of discontinuity of boundary data. Hutter and Olunloyo (1980), on the other hand, have shown that for $C_0 = C_\infty \to \infty$ (corresponding to perfect slip), the stresses show half-order singularities at the point $x = 0$. It is thus expected that the details of the stress and velocity distributions will depend on the transition properties of the sliding law in the neighbourhood of the transition point "temperate-cold". As a consequence, the failure criterion for the rock beneath will also depend on the pertinent details of the sliding law.

Mathematically, the boundary value problem set forth by Equations (2) and (3) can be shown to be reducible to a problem of the Wiener-Hopf type (Hutter and Olunloyo 1980), and it is in the form of the latter that an approximate solution can be found. Symmetry of the mathematical formulation about $y = 0$ is obtained by having identical boundary conditions on $y = \pm D$, by redefining pressure to incorporate transverse components of the gravity forces, and by imposing zero transverse velocity and zero shear traction, but not zero normal traction, on the surface $y = 0$. This is not entirely realistic as shear and normal tractions should vanish on the true free surface but, for perfect slip on the temperate side, calculations indicate that p on $y = 0$ is small and does not affect the singularity. The details of the construction of the solution of the Wiener-Hopf problem are very elaborate (as shown by Olunloyo and Hutter, unpublished). In the following section the most important results pertinent to the problem at hand will be discussed.

3) BASAL STRESSES NEAR THE POINT OF ABRUPT CHANGES OF BOUNDARY DATA

A possible mathematical representation for the dimensionless viscosity coefficient in the sliding law as depicted in Figure 2 is

$$C(x) = \begin{cases} C_\infty - (C_\infty - C_0)\ \exp(-|x|/\delta), & x < 0, \\ 0 & , \ x \geq 0. \end{cases} \quad (4)$$

In this equation the various forms of C are made dimensionless with $D/(2\mu)$ so that $C_{dim} = CD/(2\mu)$. Similarly, x and δ are dimensionless lengths based on D, the glacier thickness. With $C_{dim} = u_b/\tau_b$ and with $u_b = 100$ m a^{-1}, $\tau_b = 10^5$Pa, $D = 10^2$m and $\mu = 10^5$Pa, a reasonable value for C is about 2.

The Wiener-Hopf problem based on Equations (2), (3) and (4) was solved by approximating the kernel of the Wiener-Hopf equation. Two approximations have been found. The first is valid, provided that $C_0 = C_\infty \to \infty$ and when the boundary condition at the temperate side is that of perfect slip (Hutter and Olunloyo 1980). The other applies for viscous sliding according to the functional representation of Equation (4) and holds for any finite value of C_∞ and any value of C_0/C_∞ in the interval $[0, 1]$.

Let Σ_x, Σ_y, Θ be dimensionless normal and shear stresses at the base, respectively, which have no dimensions with respect to the basal shear stress $\rho g D \sin \gamma$ of the strictly parallel-sided slab. Hence, orders of magnitudes to be expected for Σ_x, Σ_y and Θ in a strictly parallel-sided slab are 10 and 1, respectively. For the special case that $C_0 = C_\infty \to \infty$ the functions $\Sigma_x(x)$, $\Sigma_y(x)$, and $\Theta(x)$ have been evaluated. For the more general relationship of Equation (4) explicit formulas for the basal stresses are probably difficult to find, but asymptotic re-

presentations for small $|x|$ can be found with reasonable effort. The expressions have the general form

$$f(\bar{x},1) = \begin{cases} \sum\limits_{\upsilon=-1}^{\infty} a^f_{\upsilon/2}\ \bar{x}^{\upsilon/2} & , \ \text{as} \ \bar{x} \to 0^+, \\ \sum\limits_{\upsilon=-1}^{\infty} b^f_{\upsilon/2}\ \bar{x}^{\upsilon/2} & , \ \text{as} \ \bar{x} \to 0^-, \end{cases} \quad (5)$$

in which f stands for either shear stress, overburden, or longitudinal stress Θ, Σ_y, and Σ_x. Notice that υ may have the value -1, so that the stresses may have a square-root singularity. Indeed, calculations show that the shear stress is regular as $x = 0^-$ is approached from the temperate side, but becomes singular when $x = 0^+$ is approached from the cold side. The singularity is of square-root type and can be integrated. When integrated to lowest order one obtains

$$\Theta(\bar{x},1) = \begin{cases} a^\Theta_0 & , \ \text{as} \ \bar{x} \to 0^+, \\ b^\Theta_{-\frac{1}{2}}\ |\bar{x}|^{-\frac{1}{2}} & , \ \text{as} \ \bar{x} \to 0^-, \end{cases} \quad (6)$$

with coefficients which are functions of C_0, C_∞, and δ. Detailed analysis shows that $b^\Theta_{-\frac{1}{2}}$ is finite and differs from zero for all values of C_0/C_∞ in the interval $[0,1]$. Consequently *the square-root singularity persists even in case the coefficient function $C(x)$ is continuous at $x = 0$.*

Explicit calculations are too elaborate for arbitrary values of the transition length δ, but become much simplified when $\delta = \pi^{-1}$. This value corresponds to a transition length which is about one-third of the glacier thickness. This should be borne in mind. The basal shear stress reads

$$\Theta(\bar{x},1) = \begin{cases} -(\frac{C_\infty}{\sqrt{\beta}} - \frac{C_\infty - C_0}{\sqrt{\beta + \pi}})\ \frac{1}{\sqrt{\pi \bar{x}}} + 0(1), \ \text{as} \ \bar{x} \to 0^+, \\ -(\frac{C_\infty}{\sqrt{\beta}} - \frac{C_\infty - C_0}{\sqrt{\beta + \pi}})\ \frac{1}{4 C_0 \beta} - (C_\infty - C_0) + 0(|x|^{\frac{1}{2}}), \ \text{as} \ \bar{x} \to 0^-, \end{cases} \quad (7)$$

where β is a function of C_0 and C_∞ and can be read off from Table I. This table shows the product $C_\infty \beta$ as a function of $(C_0 - C_\infty)$; see also Figure 3. It should also be noticed that in the limit $C_0 = C_\infty \to \infty$ Equation (7) breaks down and results must be obtained from the earlier investigation (Hutter and Olunloyo 1980).

TABLE I. COEFFICIENT β AS A FUNCTION OF THE PARAMETER $(1 - \dfrac{C_0}{C_\infty})$ AND RESIDUE INDICATING THE ACCURACY OF THE ROOT

$1 - \dfrac{C_0}{C_\infty}$	β	Residue
0.0	3.181 1	0.000 2
0.1	3.179 3	-0.000 3
0.2	3.177 7	0.000 2
0.3	3.176 0	-0.000 7
0.4	3.174 6	0.000 0
0.5	3.173 3	-0.000 3
0.6	3.172 0	0.000 3
0.7	3.170 7	-0.000 4
0.8	3.169 6	0.000 0
0.9	3.168 5	-0.000 2
1.0	3.167 5	0.000 0

31
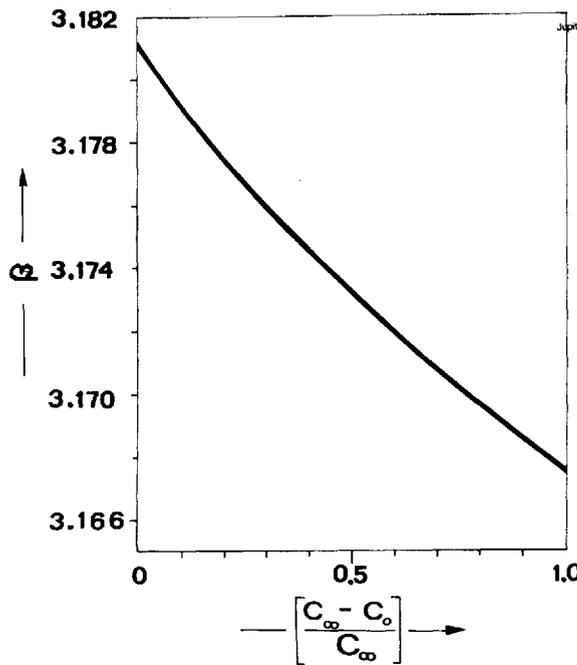
Fig. 3. Functional dependence of the coefficient β as a function of $(1-\frac{C_o}{C_\infty})$.

Fig. 4. Plot of the coefficient of the singular term in the stress expansion as given in the Equations (7) and (8).

For the longitudinal normal stress and the overburden pressure, the singular behaviour is more sensitive to variations in boundary data than is the case for the shear tractions. When the perfect slip condition is applied at the temperate side, both stress components Σ_x and Σ_y are singular at $x = 0$ with square-root singularities. On the other hand, when the viscous sliding law (1) is used, the singularity in the longitudinal stress Σ_x persists while overburden pressure is now finite. This is an indication that with a decrease in singularity in boundary data smoother stress fields should be obtained. To be sure, asymptotic representations for Σ_x and Σ_y are

$$\Sigma_x(\bar{x},1)= \cot\gamma \ + \begin{cases} B + 0(\bar{x}^2) & \text{as } \bar{x} \to 0^+, \\ -2(\frac{C_\infty}{\sqrt{\beta}} - \frac{C_\infty - C_0}{\sqrt{\beta+\pi}}) \frac{1}{\sqrt{\pi|\bar{x}|}} - \frac{\bar{x}}{C_0}, & \text{as } \bar{x} \to 0^-, \end{cases} \quad (8)$$

and

$$\Sigma_y(\bar{x},1)= \cot\gamma \ + \begin{cases} B + 0(\bar{x}^2), & \text{as } \bar{x} \to 0^+, \\ -\frac{\bar{x}}{C_0} + 0(\bar{x}^2), & \text{as } \bar{x} \to 0^-, \end{cases} \quad (9)$$

with

$$B = \frac{1}{4(2C_\infty - C_0)\sqrt{\beta}} \frac{1}{\sqrt{\beta}} (\frac{C_\infty}{\sqrt{\beta}} - \frac{C_\infty-C_0}{\sqrt{\beta+\pi}}) \left[\frac{1}{2} + \beta\bar{x} - 2^{3/2}\sqrt{\pi} \exp\{-\beta x/2\} D_{-3}(2^{\frac{1}{2}}\beta^{\frac{1}{2}}\bar{x}^{\frac{1}{2}}) \right] \quad (10)$$

where $D_{-3}(.)$ is the parabolic cylinder function. Of particular interest is the coefficient of the $x^{-1/2}$ term in Equations (7) and (8) because it determines the dominant parts of the stresses Θ and Σ_x. Values for this coefficient are plotted as shown in Figure 4. This graph is for the coefficient of Σ_x which is twice as large as the corresponding coefficient for Θ. The singular behaviour expressed by square-root singularities occurring in Equations (7) and (8) points at a possibility of failure or removal of basal rock; in fact, square-root singularities are also expected in the principal stresses at the base so that from a theoretical point of view both ice

and rock must fail if a failure criterion under tensile stress is assumed. Asymptotic expressions for the principal stresses Σ_1 and Σ_2 can be obtained by using the formula

$$\Sigma_{1,2} = \frac{\Sigma_x + \Sigma_y}{2} \pm \left[(\frac{\Sigma_x - \Sigma_y}{2})^2 + \tau^2 \right]^{\frac{1}{2}}$$

and upon substitution of the foregoing expansions. The dominant term is

$$\Sigma_{1,2} \cong 2b_{-\frac{1}{2}}^\Theta |\bar{x}|^{-\frac{1}{2}}, \quad \text{as } \bar{x} \to 0^{\pm}.$$

It is interesting to observe that the principal stresses are singular on both sides of the discontinuity of boundary data. In contrast to the stresses, forces remain finite, however, because it is possible to integrate the singularities. Therefore integrating the stresses over a small length ϵ yields (in physical dimensions):

$$F_1 = 2 \cdot 2\rho g |\sin\gamma| Db_{-\frac{1}{2}}^\Theta (\frac{\epsilon}{D})^{\frac{1}{2}}$$

as a typical value for the tensile force acting perpendicularly to a possible failure plane. This tensile force arises for both positive and negative slope γ but acts on planes of different orientations in these instances.

A typical value for $b_{-\frac{1}{2}}^\Theta$ is about 1 or 2 so that with $\rho=9\times10^2$ kg m^{-3}, $g=9.8$ m s^{-2}, $\sin\gamma = 10^{-2}$, $D=10^2$m and $\epsilon=10^{-1}$m, $F_1 = 2\times10^3$ N. Hence the average maximal tensile stress acting on an element of 10^{-2}m^2 area in a possible failure plane is about $\sigma_1 = 2\times10^5$ Pa. This is in the order of the failure strength of ice under uniaxial tension and is probably also sufficiently high for rocks, provided they contain a sufficient amount of microcracks. Of course, on a smaller scale, mean stresses are larger.

4. SOME FINAL REMARKS

The purpose of the above linear analysis is twofold: first, it can be used to substantiate the supposition that stress concentrations at the melting point of the base in a polythermal glacier are sufficiently high to tear off from the base rock pieces as seen in field observations; second, the analysis gives information about the nature of this stress concentration, as it depends on the variation of the sliding coefficient C. We reported that for the sliding law obeying Equation (4) stress singularities persist even when $C(x)$ is continuous (but not differentiable) at $x = 0$. This does not imply that stresses which are large but bounded at $x = 0$ cannot exist, but that bounded stress would require a smoother transition of the sliding coefficient from cold to temperate sides. Morland (private communication) suggests such a smoother transition by requiring a viscous sliding condition to be valid everywhere at the boundary, whereby the sliding coefficient C would approach infinity as the temperate side is approached. We have not analysed this case, and the problem is a global one, "local" stresses in the basal layer are likely to be concentrated, and induce bed failure. This points clearly at the difficulty associated with the sliding condition. Nothing is known at present about the sliding law close to the cold-temperate transition zone and the associated stress concentrations.

Removal of rock from the base can also be explained by mechanisms other than stress concentrations due to abrupt changes in boundary data only. Röthlisberger and Iken (1981) give such an alternative based on a regelation argument when basal cavities are formed. Their model explains how pieces of basal rock, that have already broken loose, are transported by the ice; our model suggests why the basal rock fractures into pieces.

Finally it should be mentioned that our calculations can only be regarded as approximations because effects of regelation and non-linearities of the creep behaviour have been omitted. It is unlikely that these effects can be included in a quantitative analysis. Yet we think that our model gives orders of magnitude correctly.

REFERENCES

Engelhardt H F, Harrison W D, Kamb W B 1978 Basal sliding and conditions at the glacier bed as revealed by bore-hole photography. *Journal of Glaciology* 20(84): 469-508

Fowler A C 1979[a] A mathematical approach to the theory of glacier sliding. *Journal of Glaciology* 23(89):131-141

Fowler A C 1979[b] The use of a rational model in the mathematical analysis of a poly-thermal glacier. *Journal of Glaciology* 24(90):443-456

Harrison W D 1972 Temperature of a temperate glacier. *Journal of Glaciology* 11(61): 15-29

Hutter K, Olunloyo V O S 1980 On the distribution of stress and velocity in an ice strip, which is partly sliding over and partly adhering to its bed, using a Newtonian viscous approximation. *Proceedings of the Royal Society of London* A 373 (1754): 385-403

Kamb W B 1970 Sliding motion of glaciers: theory and observation. *Reviews of Geophysics and Space Physics* 8(4):673-728

Lliboutry L A 1975 Loi de glissement d'un glacier sans cavitation. *Annales de Géophysique* 31(2): 207-225

Lliboutry L A 1979 Local friction laws for glaciers: a critical review and new openings. *Journal of Glaciology* 23(89): 67-96

Morland L W 1976[a] Glacier sliding down an inclined wavy bed. *Journal of Glaciology* 17(77): 447-462

Morland L W 1976[b] Glacier sliding down an inclined wavy bed with friction. *Journal of Glaciology* 17(77): 463-477

Morris E M 1979 The flow of ice, treated as a Newtonian viscous liquid, around a cylindrical obstacle near the bed of a glacier. *Journal of Glaciology* 23(89): 117-129

Müller F 1976 On the thermal regime of a high-Arctic valley glacier. *Journal of Glaciology* 16(74): 119-133

Nye J F 1969 A calculation on the sliding of ice over a wavy surface using a Newtonian viscous approximation. *Proceedings of the Royal Society of London* A 311(1506): 445-467

Nye J F 1970 Glacier sliding without cavitation in a linear viscous approximation. *Proceedings of the Royal Society of London* A 315(1522): 381-403

Robin G de Q 1976 Is the basal ice of a temperate glacier at the pressure melting point? *Journal of Glaciology* 16(74):183-196

Röthlisberger H, Iken A 1981 Plucking as an effect of water-pressure variations at the glacier bed. *Annals of Glaciology* 2: 57-62

Theakstone W H 1967 Basal sliding and movement near the margin of the glacier Østerdalsisen, Norway. *Journal of Glaciology* 6(48): 805-816

Vivian R A, Bocquet G 1973 Subglacial cavitation phenomena under the Glacier d'Argentière, Mont Blanc. *Journal of Glaciology* 12(66): 439-451

Weertman J 1979 The unsolved general glacier sliding problem. *Journal of Glaciology* 23(89): 97-115