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On stable commutator length of non-filling curves in surfaces

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We give a new proof of rationality of stable commutator length (scl) of certain elements in surface groups: those represented by curves that do not fill the surface. Such elements always admit extremal surfaces for scl. These results also hold more generally for non-filling 1–chains.

Keywords: Stable commutator length; surface; linear programming; bounded cohomology

1. Introduction

This paper concerns the computation of stable commutator length (scl) in fundamental groups of closed orientable surfaces. It is of considerable interest to know whether scl is always rational in these groups, or more generally, whether the scl norm is piecewise rational linear (in the sense of [4]).

In more concrete terms, let Σ be a closed orientable surface and $\gamma\colon\ S^1 \to \Sigma$ a null-homologous loop. We are interested in identifying efficient maps of surfaces into Σ, with boundary mapping to a positive power of γ. 'Efficient' means that the ratio of the topological complexity of the surface to the power of γ on the boundary is as small as possible. The infimal value of this ratio is the stable commutator length of γ. In some cases, there may be a surface realizing this infimal value; these are called *extremal surfaces* for scl, and when they exist scl is rational.

Calegari [2] has shown that one can always find an *immersed* surface with boundary a power of γ, and such surfaces are the most efficient among those in their relative homology classes. However, there are infinitely many relative classes, and one cannot draw any immediate conclusions about the existence of an extremal surface for γ.

In the present paper, we give a new proof the following theorem of Calegari, which establishes rationality in the special case of curves γ that do not fill Σ. It appears as Example 4.51 in [3], as an application of Theorem 4.47. We state the result here for integral 1–chains, i.e. immersed multicurves.

Theorem 1.1 (Calegari). *Let Σ be a closed orientable surface of genus > 1 and $\gamma\colon\ \coprod S^1 \to \Sigma$ a null-homologous integral* 1*-chain which does not fill Σ. Then, the* *stable commutator length of γ is rational and computable. Moreover, there exists an extremal surface for γ.*

It follows immediately that *rational* 1–chains supported on non-filling multicurves also have rational stable commutator length. Indeed, we find that scl is piecewise rational linear on 'non-filling' rational subspaces of the space of 1–chains; that is, subspaces spanned by rational 1–chains $c_1, \ldots, c_k$ such that the union of their supports is a non-filling multicurve.

Other rationality results

The list of groups known to have rational scl, excluding those with scl $\equiv 0$, is fairly short. Such groups include:

- free groups [4]
- fundamental groups of non-compact Seifert fibred 3–manifolds [4]
- free products of free abelian groups [5]
- free products of cyclic groups [13]
- amalgamated products of free abelian groups [12]
- Baumslag–Solitar groups, alternating elements only [7]
- free products of groups in which scl $\equiv 0$ [8]
- generalized Baumslag–Solitar groups [9]

Surface groups are conspicuously absent from this list, despite being among the simplest and most well-understood one-relator groups.

It is worth noting that finitely presented groups do not always have rational scl; a counterexample was given by Zhuang in [14]. If one allows non-finitely presented groups, then in fact every non-negative real number occurs as scl of some element of a small cancellation group [11].

Methods

The general outline of the argument is similar to the case of free groups as presented in [1]. The stable commutator length of γ is the infimum of $-\chi(S)/2n(S)$ over all admissible surfaces S mapping to Σ. First, we show that the infimum can be taken over the much smaller subset $\mathscr{T}(\gamma)$ of 'taut' admissible surfaces—these will be described below and defined precisely in § 3.

Taut surfaces can be encoded as integer vectors via a map $v\colon\ \mathscr{T}(\gamma) \to \mathbb{R}^k$, and the function $-\chi(S)/2n(S)$ factors through this map as $\mathscr{T}(\gamma) \xrightarrow{v} \mathbb{R}^k \to \mathbb{R}$ where the second map is a ratio $A(v)/B(v)$ of linear functions. The computation of $\mathrm{scl}(\gamma)$ is then reduced to a linear programming problem after showing that the image of v in $\mathbb{R}^k$ fills (in a suitable sense) the integer points of a finite-sided rational polyhedron in $\mathbb{R}^k$.

The two main steps are *encoding*, which entails defining $\mathscr{T}(\gamma)$ and the function v; and *reassembly*, in which specific vectors are shown to be in the image of v. Our main assumption throughout the paper is that Σ contains a subsurface Σ_1 with non-trivial topology that is disjoint from the image of γ. This property is needed for both steps.

To make a surface S taut, one needs to put it into a normal form that can be encoded as a finite integer vector. Following Gabai [**10**], we arrange that away from a region containing the double points of γ, the preimage in S maps by a branched immersion. We arrange further that the remaining parts of S fall into finitely many configurations, and all branch points lie above Σ_1. The branch points are now the main obstacle to having a finite encoding, but these can be removed by passing to finite-sheeted covers, using the non-trivial topology of Σ_1.

For reassembly we show that every integer vector in the rational polyhedron has a multiple that encodes a taut surface. When building this surface out of pieces, a phenomenon involving branch points arises. A taut surface cannot be built directly, but one can build one with branch points. The presence of branch points means that the linear functional on $\mathbb{R}^k$ reporting $\chi(S)$ does not report this number correctly. Once again, using finite coverings of Σ_1, we are able to eliminate these branch points and construct a taut surface with the desired encoding.

In contrast, Calegari's approach is quite different. He proves theorem 1.1 by viewing surface groups as fundamental groups of graphs of groups in which vertex groups are free and edge groups are cyclic. If γ is non-filling then it avoids an essential annulus, and one may encode Σ as a graph of groups with one edge. The proof then proceeds by reducing the computation of scl to computing scl in each vertex group, using the assumption that γ is supported in the vertex groups.

2. Preliminaries

Stable commutator length

We start with basic working definitions for stable commutator length of group elements and of 1–chains. See [**3**, Chapter 2] for more information.

Definition 2.1 (scl of integral 1–chains). *Let X be a space with fundamental group G.*

An integral 1–chain over G is a finite formal sum $c = \sum_i g_i$ with $g_i \in G$. Let $\gamma_i\colon S^1 \to X$ be a loop representing g_i for each i and let $\gamma\colon \coprod_i S^1 \to X$ be the map given by $\coprod_i \gamma_i$. An admissible surface for γ (or for c) is a compact oriented surface S together with a map $f\colon S \to X$ such that

- *$\partial S \neq \emptyset$ and S has no sphere or disk components*
- *the restriction $f|_{\partial S}$ factors through γ; that is, there is a commutative diagram*

$$\begin{array}{ccc} \partial S & \hookrightarrow & S \\ \downarrow & & \downarrow f \\ \coprod_i S^1 & \xrightarrow{\coprod_i \gamma_i} & X \end{array}$$

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- *the restriction of the map* $\partial S \to \coprod_i S^1$ *to each connected component of* ∂S *is a map of positive degree*
- *there is a positive integer* $n(S)$ *such that for each component of* $\coprod_i S^1$*, the preimage in* ∂S *maps to it by total degree* $n(S)$*.*

The stable commutator length (scl) of c *is defined by*

$$\operatorname{scl}(c) = \inf_S \frac{-\chi(S)}{2n(S)}$$

where the infimum is taken over all admissible surfaces for γ*. If no admissible surface exists, we define* $\operatorname{scl}(c) = \infty$*.*

Note: the third bullet is not included in Calegari's definition of admissible surface in [3]. However, Proposition 2.13 of [3] shows that including it does not change the meaning of scl.

Using homogeneity properties of scl*, the definition extends to rational chains. If* $c = \sum_i c_i g_i$ *with* $c_i \in \mathbb{Q}$ *then* mc *is integral for some* m*, and we may define* $\operatorname{scl}(c) = 1/m \operatorname{scl}(mc)$*. Extending by continuity, it is also defined for real chains. See [3, Section 2.6].*

Definition 2.2 (scl of group elements). *For any* $g \in G$ *we define* $\operatorname{scl}(g)$ *by considering* g *as an integral* 1*–chain consisting of one element. The above definition, in this case, specializes to the standard definition of* $\operatorname{scl}(g)$*. Note that admissible surfaces for* g *exist if and only if* $g^k \in [G, G]$ *for some* $k \neq 0$*, and so* $\operatorname{scl}(g)$ *is finite exactly when the latter occurs.*

Definition 2.3. *Let* c *be an integral* 1*–chain over* G*. An extremal surface for* c *is an admissible surface* S *for* c *that realizes the infimum in the definition of* $\operatorname{scl}(c)$*. If* c *is a rational* 1*–chain, an extremal surface for* c *is just an extremal surface for any* mc *that is integral. Extremal surfaces need not exist, but when they do,* $\operatorname{scl}(c)$ *must of course be rational.*

Let $C_1(G)$ be the space of 1–chains, which is a vector space over $\mathbb{R}$ with basis G. Let $B_1(G) \subset C_1(G)$ be the subspace of boundaries of 2–chains, i.e. the kernel of the quotient map $C_1(G) \to H_1(G; \mathbb{R})$. Let $B_1^H(G)$ be the quotient of $B_1(G)$ by the subspace H spanned by elements of the form $g^n - ng$ and $g - hgh^{-1}$ for all $g, h \in G$, $n \in \mathbb{Z}$. Then the function scl is a pseudo-norm on $B_1^H(G)$. Moreover, whenever G is hyperbolic, scl is a geniune norm on $B_1^H(G)$ [6].

Surfaces

Let Σ be a closed surface. We say that a multicurve $\gamma \colon \coprod S^1 \to \Sigma$ is *non-filling* if it is homotopic to an immersion in general position such that some component of the complement of the image is not a disk (or equivalently, Σ contains an essential simple closed curve disjoint from the image of γ). Correspondingly, an integral 1–chain or group element in $G = \pi_1(\Sigma)$ is *non-filling* if it is represented by a non-filling multicurve.

Definition 2.4. *A map $f\colon\ S \to \Sigma$ of orientable surfaces is compressible if there is an essential simple closed curve $C \subset S$ such that $f(C)$ is nullhomotopic in Σ. If f is not compressible it is called incompressible.*

When f is compressible, one can replace an annular neighbourhood of C by two disks and extend f over these disks, thus obtaining $f'\colon\ S' \to \Sigma$ of smaller complexity.

Coverings

Throughout this paper curves and surfaces typically have orientations, and by the *degree* of a map we always mean the homological degree.

Suppose $p\colon\ X \to Y$ is a finite-sheeted covering of oriented manifolds. Partition the connected components of X as $X^+ \sqcup X^-$ such that p is orientation-preserving on X^+ and orientation-reversing on X^-. We define the *positive degree* of p to be the number of sheets of X^+ and the *negative degree* of p to be the number of sheets of X^-. Thus the sum of these numbers is the total number of sheets of p and their difference is the homological degree.

Next we record some basic covering lemmas for surfaces.

Lemma 2.5. *Let S be a compact connected oriented surface having $p \geqslant 1$ boundary components, with $S \ncong D^2$.*

(1) *For any $q < p$ boundary components $C_1, \ldots, C_q$ of S and integers $n_1, \ldots, n_q \geqslant 1$, there is a connected regular finite-sheeted cover $S' \to S$ such that every component of $\partial S'$ mapping to C_i covers with degree n_i.*

(2) *For any integer $n \geqslant 1$ there is a connected finite-sheeted cover $S' \to S$ such that every component of $\partial S'$ covers its image with degree n.*

Proof. For (1), let $x_i \in H_1(S)$ be the element represented by C_i, for $1 \leqslant i \leqslant q$. This set extends to a basis $\{x_1, \ldots, x_n\}$ of $H_1(S)$, because $q < p$. Let N be the least common multiple of the numbers n_i. Let $\varphi\colon\ H_1(S) \to \mathbb{Z}/N\mathbb{Z}$ send x_i to N/n_i for $1 \leqslant i \leqslant q$ (the values on the other basis elements are irrelevant). Now $S' \to S$ is the cover corresponding to the kernel of the composition $\pi_1(S) \to H_1(S) \to \mathbb{Z}/N\mathbb{Z}$. Since $\varphi(x_i)$ has order n_i in $\mathbb{Z}/N\mathbb{Z}$ and the cover is regular, the boundary $\partial S'$ behaves as desired.

For (2), suppose first that $p > 1$. Choose boundary components $C_1, \ldots, C_{p-1}$ of ∂S and a basis $\{x_i\}$ of $H_1(S)$ as before. Let $\varphi\colon\ H_1(S) \to (\mathbb{Z}/n\mathbb{Z})^{p-1}$ send x_i to the ith generator $(0, \ldots, 1, \ldots, 0)$ for $1 \leqslant i \leqslant p-1$. Then φ sends the class represented by the last boundary component of S to $(-1, \ldots, -1)$. Since every boundary component maps to an element of order n, the cover $S' \to S$ corresponding to the kernel of $\pi_1(S) \to H_1(S) \to (\mathbb{Z}/n\mathbb{Z})^{p-1}$ has the desired property.

If ∂S has one component, first let $S'' \to S$ be any connected 2–sheeted cover (which exists since $S \ncong D^2$). It will have 2 boundary components, each mapping by degree 1, because $\chi(S'')$ is even. Apply the preceding case to get $S' \to S''$, and take the composition $S' \to S'' \to S$. $\square$

It will be convenient to have a version of this lemma for disconnected surfaces. By a finite-sheeted cover of a disconnected surface, we mean a cover for which the number of sheets is the same over every component.

Lemma 2.6. *Let S be a compact oriented surface where each component has at least 1 boundary component and no component is homeomorphic to D^2. For any set of boundary components $C_1, \ldots, C_q$ of S which does not contain all the boundary components of any component of S and integers $n_1, \ldots, n_q \geqslant 1$, there is a finite-sheeted cover $S' \to S$ such that every component of $\partial S'$ mapping to C_i covers with degree n_i.*

Proof. Let $S_1, \ldots, S_m$ be the components of S. We first apply lemma 2.5(1) to each S_i to obtain an N_i-fold cover $S_i' \to S_i$ with the correct degree on boundary components. Let $N = \mathrm{lcm}(N_1, \ldots, N_m)$. Then the cover of S consisting of N/N_i copies of S_i has the desired properties. □

We are also interested in the case where we allow branched covers with prescribed degrees on the boundaries. The flexibility of degrees on boundaries is much larger for branched covers.

Lemma 2.7. *Let S be a compact connected oriented surface having $p \geqslant 1$ boundary components. Let $C_1, \ldots, C_p$ be the boundary components of S, let n be a natural number, and let $\lambda_1, \ldots, \lambda_p$ be partitions of n. Then there is an n-fold branched cover $S' \to S$ such that the degrees of the components over C_i are precisely the values in the partition λ_i.*

Note that S' is not necessarily connected.

Proof. We start with the cover $S'' \to S$ where S'' is n copies of S. Let A, B be distinct copies of C_i in S''. Let α, β be copies of the same small arc from a point on C_i into the interior of S. Cut S'' along α and β and glue the right side of β to the left side of α and vice versa. This new surface, in a canonical way, admits a branched cover to S where the degrees over C_i are 2, 1, 1, …, 1. By using this kind of surgery to connect boundary components of S'', it is clear that we can construct a branched cover $S' \to S$ where the degrees over C_i are an arbitrary partition of n. □

3. Mapping surfaces to surfaces

Setup

Let Σ be a closed oriented surface of genus greater than one. Let $\gamma\colon \coprod S^1 \to \Sigma$ be a null-homologous 1–chain that does not fill Σ. We can put γ into general position (smoothly immersed, transverse intersections, no triple points) such that at least one complementary component is not a disk.

Let Σ_0 be one such complementary component. Let $B \subset \Sigma_0$ be an embedded closed disk and let $\lambda_1, \ldots, \lambda_k$ be disjoint properly embedded arcs in $\Sigma_0 - \mathrm{int}(B)$ with endpoints on ∂B such that the inclusion of $B \cup \bigcup_i \lambda_i$ into Σ_0 is a homotopy equivalence. Let $L \subset (\Sigma_0 - \mathrm{int}(B))$ be the closure of a tubular neighbourhood of $\bigcup_i \lambda_i$, so that $\Sigma_1 = B \cup L$ is compact subsurface homotopy equivalent to Σ_0.

Next let $\mu_1, \ldots, \mu_\ell$ be disjoint properly embedded arcs in $\Sigma - \mathrm{int}(\Sigma_1)$ with endpoints on $\partial B - L$ such that the μ_i cut each component of $\Sigma - \Sigma_1$ into a single disk and the μ_i do not cross any double points of γ. We also require that the arcs μ_i are transverse to the immersed submanifold $\gamma(\coprod S^1)$. Let $M \subset (\Sigma - \mathrm{int}(\Sigma_1))$ be the closure of a tubular neighbourhood of $\bigcup_i \mu_i$, disjoint from L, so that $\Sigma_2 = \Sigma_1 \cup M$ contains no double point of γ and $\gamma(\coprod S^1) \cap M$ consists of fibres of M. Let D be the closure of $\Sigma - \Sigma_2$; this is a disjoint union of embedded closed disks. Finally, define $\Sigma_3 = \Sigma - \mathrm{int}(\Sigma_1) = D \cup M$.

Each component of $\coprod S^1$ maps by γ to a path which alternates between embedded oriented arcs in D, called *turn arcs*, and arcs crossing M. Let $\tau_1, \ldots, \tau_m$ be the turn arcs. See figure 1.

Definition 3.1 (Turn paths). *A turn path is a loop $S^1 \to D$ which is a concatenation of paths, alternating between turn arcs and immersed paths in ∂D. Immersions $S^1 \to \partial D$ are included. We consider two turn paths to be the same if they differ by an orientation-preserving reparametrization of S^1. See figure 2.*

Each component of the oriented boundary of D is a concatenation of arcs, alternating between arcs in the boundary of M and arcs in the boundary of Σ_1. Let $\alpha_1, \ldots, \alpha_n$ be the arcs of ∂D which lie in the boundary of Σ_1.

A turn path is taut if it uses each turn arc at most once and each arc from $\{\alpha_i\} \cup \{\overline{\alpha}_i\}$ at most once (where $\overline{\alpha}_i$ is the reverse of α_i). Note that there are only finitely many taut turn paths.

Definition 3.2. *An admissible map $f\colon S \to \Sigma$ is taut if*

(1) *$f\colon f^{-1}(\Sigma_2) \to \Sigma_2$ is an immersion*

(2) *$f^{-1}(D)$ is a disjoint union of disks,*

(3) *for each component E of $f^{-1}(D)$, the boundary map $f|_{\partial E}$ is a taut turn path.*

The set of taut admissible maps for γ will be denoted by $\mathscr{T}(\gamma)$. The disks in (2) and (3) will be called turn disks.

The main result of this section is the following.

Proposition 3.3. *Suppose $f\colon S \to \Sigma$ is an admissible map. Then there exists a taut admissible map $f'\colon S' \to \Sigma$ such that $-\chi(S')/2n(S') \leqslant -\chi(S)/2n(S)$.*

Corollary 3.4. *There is an equality*

$$\mathrm{scl}(\gamma) = \inf_{S \in \mathscr{T}(\gamma)} \frac{-\chi(S)}{2n(S)}.$$

The next lemma provides the first step towards proposition 3.3. It achieves all the desired properties except for tautness of the turn paths in item (3).

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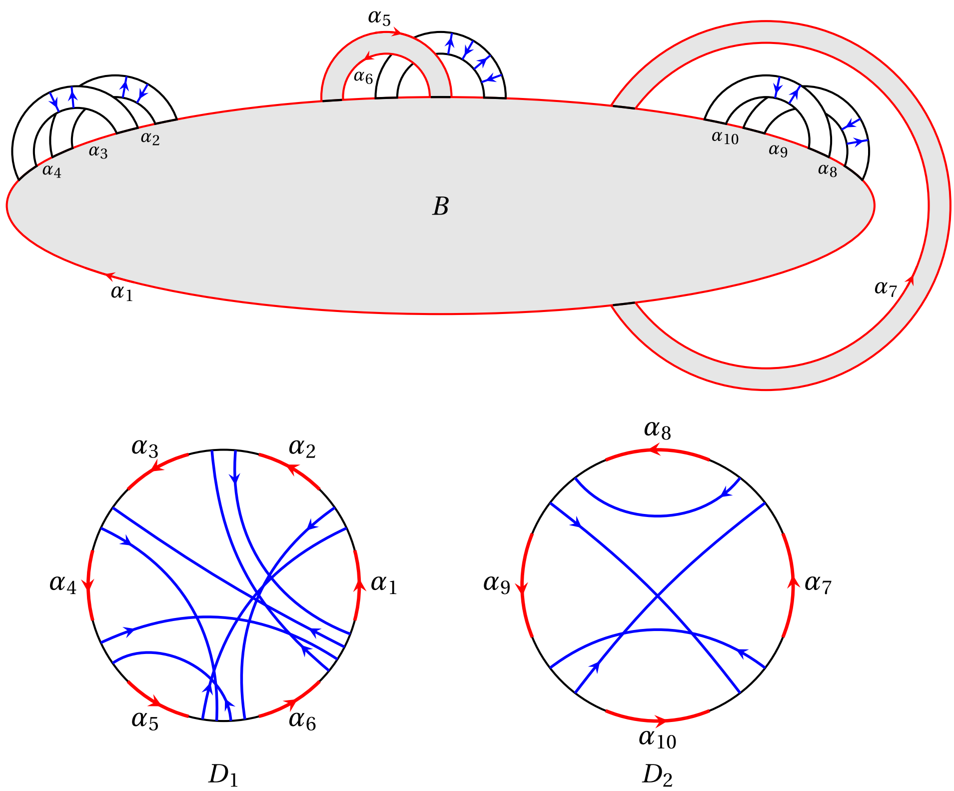

Figure 1. A surface Σ and its handle decomposition $\Sigma = B \cup (L \cup M) \cup D$. The shaded part is the subsurface $\Sigma_1 = B \cup L$. The null-homologous 1–cycle γ is shown in blue. All self-intersections of γ lie inside $D = D_1 \cup D_2$. Outside of D, γ meets only the 1–handles M and crosses them transversely. The arcs of γ crossing D are called turn arcs. The arcs α_i are the maximal sub-arcs of ∂D that meet $\partial \Sigma_1$.

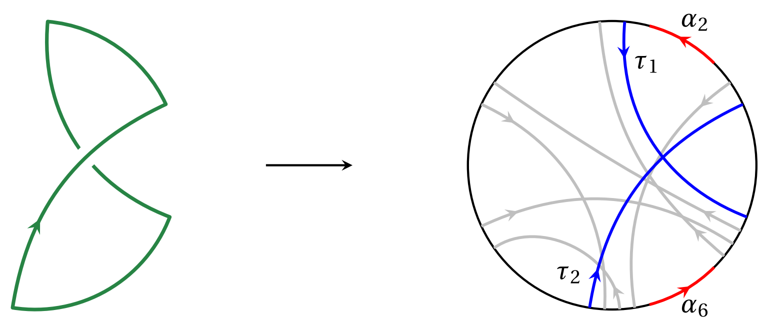

Figure 2. A taut turn path in D_1 which runs over τ_1, $\overline{\alpha}_6$, τ_2, and α_2. Every turn path extends to a map of a disk; in this case the disk is twisted.

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LEMMA 3.5. *Suppose $f\colon S \to \Sigma$ is an admissible map which is incompressible. Then f is homotopic to an admissible map $g\colon S \to \Sigma$ such that*

(1) *$g\colon g^{-1}(\Sigma_2) \to \Sigma_2$ is an immersion*

(2) *$g^{-1}(D)$ is a disjoint union of disks,*

(3) *for each component E of $g^{-1}(D)$, the boundary map $g|_{\partial E}$ is a turn path.*

The boundary maps $f|_{\partial S}$ and $g|_{\partial S}$ differ only by reparametrization.

Proof. We proceed as in Step 1 of [**10**, Theorem 2.1]. Homotope f rel boundary to f_1 which is transverse to the disk B. We then have that each component of $f_1^{-1}(B)$ is a disk mapping homeomorphically onto B. Next, let $T = S - \mathrm{int}(f_1^{-1}(B))$. Since $f_1|_{\partial T}$ is transverse to the arcs λ_i and μ_i, we may homotope f_1 rel $\partial S \cup f_1^{-1}(B)$ to be transverse to these arcs. Among all maps f_2 homotopic to f rel ∂S that are transverse to B and transverse to the arcs λ_i and μ_i on $S - \mathrm{int}(f_2^{-1}(B))$, let f_2 minimize the *complexity*, which is the pair $\left(\left|\pi_0\left(f_2^{-1}(B)\right)\right|, \left|\pi_0\left(f_2^{-1}\left(\bigcup_i \lambda_i \cup \bigcup_i \mu_i\right)\right)\right|\right)$ ordered lexicographically.

Since f_2 is incompressible, no component of $f_2^{-1}(\lambda_i)$ or $f_2^{-1}(\mu_i)$ is an essential closed curve. No component is an inessential closed curve either. Suppose C is such a curve mapping to λ_i, say. For a sufficiently close arc λ_i' that is parallel to λ_i, there is a curve C' parallel to C mapping to λ_i' that bounds a disk $E \subset S$, with C in the interior of E. By re-defining the map on $\mathrm{int}(E)$ to map into λ_i' one obtains a map of smaller complexity; the new map is homotopic to f_2 rel ∂S because $\pi_2(\Sigma) = 0$.

Thus each component of $f_2^{-1}(\lambda_i)$ is an arc with endpoints mapping to $\partial\lambda_i$, and each component of $f_2^{-1}(\mu_i)$ is an arc with endpoints mapping to $\partial\mu_i \cup \{\mu_i \cap \gamma(\coprod S^1)\}$. Note that in the second case, both endpoints cannot map to the same point $z \in \{\mu_i \cap \gamma(\coprod S^1)\}$, by orientation considerations. There is a consistent transverse orientation on $f_2^{-1}(\mu_i)$, but $f_2|_{\partial S}$ always passes through z in the same direction, being a positive power of γ.

Since complexity is minimized, both endpoints cannot map to the same point of $\partial\mu_i$; otherwise, $\left|\pi_0\left(f_2^{-1}\left(\bigcup_i \lambda_i \cup \bigcup_i \mu_i\right)\right)\right|$ could be reduced, and then $\left|\pi_0\left(f_2^{-1}(B)\right)\right|$ as well by homotopy rel ∂S to a new map which is still transverse to $B \cup \{\mu_i\} \cup \{\lambda_i\}$. Hence the endpoints map to distinct points. The same reasoning applies to components of $f_2^{-1}(\lambda_i)$. Hence the endpoints of every arc in $f_2^{-1}(\lambda_i)$ or $f_2^{-1}(\mu_i)$ map to distinct points.

Now, by a further homotopy rel $\partial S \cup f_2^{-1}(B)$, the components of $f_2^{-1}(\lambda_i)$ and $f_2^{-1}(\mu_i)$ can be tightened so that each maps by an immersion; let f_3 be the resulting map. There are compatible tubular neighbourhoods on which f_3 is a bundle map, and by another homotopy to f_4 (which expands along fibres of the tubular neighbourhoods L and M) we can ensure that each component of $f_4^{-1}(L)$ and $f_4^{-1}(M)$ maps by an immersion. Now $f_4\colon f_4^{-1}(\Sigma_2) \to \Sigma_2$ is an immersion, and f_4 satisfies (1).

Next consider a component E of $f_4^{-1}(D)$. Each boundary component of ∂E maps by a concatenation of paths that alternate between turn arcs and paths in ∂D; moreover these latter paths are part of the boundary map of $f_4\colon f_4^{-1}(\Sigma_2) \to \Sigma_2$ and hence are immersions. Thus ∂E maps to D by turn paths and f_4 satisfies (3).

Next, suppose some component E of $f_4^{-1}(D)$ is not a disk. Then there is a simple closed curve $C \subset \mathrm{int}(E)$ which is essential in E. Since $f_4(C)$ is nullhomotopic in Σ, we have that C bounds a disk $E' \subset S$. Since $\pi_2(\Sigma) = 0$ there is a homotopy rel $S - \mathrm{int}(E')$ from f_4 to f_5 such that $f_5(E') \subset D$. By performing finitely many homotopies of this type, we arrive at g such that each component of $g^{-1}(D)$ is a disk and g agrees with f_4 on $g^{-1}(\Sigma - \mathrm{int}(D)) = g^{-1}(\Sigma_2)$. In particular, g satisfies (1) and (2).

Lastly, $f_4^{-1}(\partial D) \supseteq g^{-1}(\partial D)$ and f_4 and g agree on $g^{-1}(\partial D)$ (and on ∂S), so the boundary paths of $g^{-1}(D)$ are unchanged and g satisfies (3). For the last statement, note that $f|_{\partial S} = f_3|_{\partial S}$ and $f_4|_{\partial S} = g|_{\partial S}$, and the change from $f_3|_{\partial S}$ to $f_4|_{\partial S}$ simply reparametrizes along ∂S. $\square$

The map g given by the previous lemma may be quite badly behaved on $g^{-1}(D)$, with branch points, folding, twisting, etc. This will mostly not be a concern for us; what matters most is the boundary maps on $g^{-1}(D)$. The next lemma achieves tautness of these boundary maps at the expense of creating branch points over Σ_1.

Lemma 3.6. *Suppose $g\colon S \to \Sigma$ is an admissible map which satisfies the conclusions of lemma 3.5. Then there is an admissible map $g_1\colon S_1 \to \Sigma$ with $n(S_1) = n(S)$ and $\chi(S) \leqslant \chi(S_1)$ such that*

(1) *$g_1\colon g_1^{-1}(\Sigma_2) \to \Sigma_2$ is a branched immersion with all branch points in the interior of $g_1^{-1}(\Sigma_1)$,*

(2) *$g_1^{-1}(D)$ is a disjoint union of disks,*

(3) *for each component E of $g_1^{-1}(D)$, the boundary map $g_1|_{\partial E}$ is a taut turn path,*

(4) *g_1 is an immersion on every component of $g_1^{-1}(\Sigma_3)$ that does not meet ∂S_1.*

Proof. Let E be a component of $g^{-1}(D)$. There are two types of moves we will use to achieve tautness in item (3). Suppose first that ∂E contains two or more instances of the turn arc τ_i. Delete the interior of E, cut the midpoints of the two instances of τ_i and glue the first half of one to the second half of the other and vice versa. This splits ∂E into two components, which can be filled by two disks mapping to D, increasing $\chi(S)$. After finitely many moves of this type, there are no repeated instances of turn arcs in components of $\partial g^{-1}(D)$. The new boundary of S still maps by positive powers of γ with degree $n(S)$.

Now suppose that the arc $\alpha \in \{\alpha_i\} \cup \{\overline{\alpha}_i\}$ appears two or more times in ∂E. These two occurrences are also part of $\partial g^{-1}(\Sigma_1)$; denote them by α' and α''. Let U be an open neighbourhood of α in Σ_1 that is evenly covered by g, and let U', U'' be the corresponding neighbourhoods of α' and α'' in $g^{-1}(\Sigma_1)$. Let β be a small arc in U from the midpoint of α to an interior point of U. Let β', β'' be the lifts of β to U' and U'' respectively. Now delete the interior of E and cut along β' and β''. Rejoin the left side of β' to the right side of β'' and vice versa. This creates an index two branch point in $g^{-1}(\Sigma_1)$ and splits ∂E into two components, which can be filled with two disks mapping to D. The Euler characteristic $\chi(S)$ does not change, and neither does $n(S)$, because this move takes place in the interior of S.

After finitely many moves of this second type, we arrive at $g_0\colon S_1 \to \Sigma$ satisfying (1)–(3).

Let E be a component of $g_0^{-1}(D)$ which does not meet ∂S. Then the map $g_0|_{\partial E}$ is a homeomorphism to $\partial D'$ for some component D' of D by (3). For each such E, we replace $g_0|_E$ by a homeomorphism $E \to D'$ extending $g_0|_{\partial E}$ to get a new map $g_1\colon S_1 \to \Sigma$. Note that g_1 still satisfies (1)–(3). By (1) and the above, g_1 also satisfies (4). $\square$

For the last step of the argument, we use finite covers to eliminate the branch points over Σ_1. This step depends on Σ_1 having non-trivial topology. Note that conclusions (1) and (2) of this lemma are not required for proposition 3.3 but will be useful in a later section.

Lemma 3.7. *Suppose $g_1\colon S_1 \to \Sigma$ is an admissible map which satisfies the conclusions of lemma 3.6. Let r be the number of sheets of the branched cover $g_1\colon g_1^{-1}(\Sigma_1) \to \Sigma_1$. Then there is a taut admissible map $g_2\colon S_2 \to \Sigma$ and a positive integer N such that*

(1) $n(S_2) = Nn(S_1)$,

(2) $\chi(g_2^{-1}(\Sigma_3)) = N\chi(g_1^{-1}(\Sigma_3))$ *and* $\chi(g_2^{-1}(\Sigma_1)) = Nr\chi(\Sigma_1)$,

(3) $-\chi(S_2)/2n(S_2) \leqslant -\chi(S_1)/2n(S_1)$.

Proof. Let T_3'' be the union of all components of $g_1^{-1}(\Sigma_3)$ which meet ∂S_1. Let

$$d = \mathrm{lcm}\{\text{degree of } C' \xrightarrow{g_1} C \mid C', C \text{ are components of } \partial T_3'', \partial\Sigma_3 \text{ resp.}\}$$

By lemma 2.6, there is a finite-sheeted cover $T_3' \to T_3''$ such that every component of $\partial T_3'$ covers a component of $\partial\Sigma_3$ with degree $\pm d$ under the composition $T_3' \to T_3'' \xrightarrow{g_1} \Sigma_3$. (The orientation of a covering surface will be induced by the base unless indicated otherwise.) Let $\Sigma_{3,1}, \ldots, \Sigma_{3,\ell}$ be the components of Σ_3. By lemma 2.5(2), there are connected finite-sheeted covers $T_{3,i}' \to \Sigma_{3,i}$ and $T_1' \to \Sigma_1$ such that every component of $\partial T_{3,i}'$ and $\partial T_1'$ covers a component of $\partial\Sigma_3$ with degree $\pm d$.

Let s_i^+, s_i^- be the positive and negative degrees by which $g_1^{-1}(\Sigma_{3,i}) - T_3''$ covers $\Sigma_{3,i}$. Let r^+ and r^- be the positive and negative degrees respectively of the branched cover $g_1\colon g_1^{-1}(\Sigma_1) \to \Sigma_1$. By taking multiple copies of $T_{3,i}'$ for all i and multiple copies of T_3' and T_1', and by orienting the copies of $T_{3,i}'$ and T_1' appropriately, we can find covers $h_3\colon T_3 \to T_3''$, $h_{3,i}\colon T_{3,i} \to \Sigma_{3,i}$, and $h_1\colon T_1 \to \Sigma_1$ such that, for some positive integer N,

- the positive (resp. negative) degree of $h_{3,i}$ is Ns_i^+ (resp. Ns_i^-),
- h_3 has degree N,
- h_1 has positive degree Nr^+ and negative degree Nr^-.

We now show that we can glue the above pieces together to get a taut admissible surface S_2. Note that every component of $\partial\Sigma_{3,i}$ is covered by $\partial T_3''$ with positive and negative degrees $r^+ - s_i^+$ and $r^- - s_i^-$ respectively. Thus, under $g_1 \circ h_3$, every component of $\partial\Sigma_{3,i}$ is covered by $N(r^+ - s_i^+)/d$ curves by degree d and $N(r^- - s_i^-)/d$ curves by degree $-d$. Each component of $\partial\Sigma_{3,i}$, under $h_{3,i}$, is covered by Ns_i^+/d curves by degree d and Ns_i^-/d curves by degree $-d$. Each component of $\partial\Sigma_{3,i}$, under h_1, is covered by Nr^+/d curves by degree $-d$ and Nr^-/d curves by degree d. Thus, by gluing T_1 to $T_3 \cup \bigcup_i T_{3,i}$ along the preimage of $\partial\Sigma_3$ appropriately respecting orientations, we obtain an oriented compact surface S_2 and a map $g_2\colon\ S_2 \to \Sigma$, which, by construction, is a taut admissible map.

It is clear that $n(S_2) = Nn(S_1)$, so it remains to compute the Euler characteristic of various surfaces.

$$
\begin{aligned}
\chi(g_2^{-1}(\Sigma_3)) &= \chi(T_3) + \sum_i \chi(T_{3,i}) \\
&= N\chi(T_3'') + \sum_i N(s_i^+ + s_i^-)\chi(\Sigma_{3,i}) \\
&= N\chi(T_3'') + \sum_i N\chi(g_1^{-1}(\Sigma_{3,i}) - T_3'') \\
&= N\chi(g_1^{-1}(\Sigma_3)) \\
\chi(g_2^{-1}(\Sigma_1)) &= N(r^+ + r^-)\chi(\Sigma_1) = Nr\chi(\Sigma_1)
\end{aligned}
$$

Finally, note that $\chi(g_1^{-1}(\Sigma_1)) \leqslant r\chi(\Sigma_1)$ since $g_1\colon\ g_1^{-1}(\Sigma_1) \to \Sigma_1$ is a branched cover, and so

$$
\chi(S_2) = \chi(g_2^{-1}(\Sigma_3)) + \chi(g_2^{-1}(\Sigma_1)) = N\chi(g_1^{-1}(\Sigma_3)) + Nr\chi(\Sigma_1) \geqslant N\chi(S_1).
$$

□

Proof of proposition 3.3. Start with an admissible surface $f\colon\ S \to \Sigma$. We may assume that f is incompressible, since performing a compression reduces $-\chi(S)$ without changing $n(S)$ or the property of being admissible. Now apply lemmas 3.5, 3.6, and 3.7. □

4. Encoding surfaces

Let TP denote the set of all taut turn paths. For each taut turn path $p \in \mathrm{TP}$, we define the sides of p. Recall that $p\colon\ S^1 \to D$ is an immersed path which alternates between turn arcs and immersed paths in ∂D. Since the intersection of γ with Σ_2 lies in M, the turn path p alternates between arcs from $\{\tau_i\} \cup \{\alpha_i\} \cup \{\overline{\alpha}_i\}$ and arcs in $\partial M - \Sigma_1$. A *side* of p is an ordered pair $s \in (\{\tau_i\} \cup \{\alpha_i\{\cup\{\overline{\alpha}_i\})^2$ such that p traverses the first arc in the ordered pair, then a part of $\partial M - \Sigma_1$, and then the second arc in the ordered pair. For any side s, we let $m_s \subseteq \partial M$ denote the middle arc. See figure 3. Note that since p is taut, it traverses any side at most once.

Let $\mathcal{S}$ denote the set of all ordered pairs of arcs from $\{\tau_i\} \cup \{\alpha_i\} \cup \{\overline{\alpha}_i\}$ which are a side for some taut turn path p. Note that $\mathcal{S}$ does not contain all ordered pairs. E.g., $\mathcal{S}$ cannot contain any pairs of the form $(\alpha_i, \overline{\alpha}_j)$ since a taut turn path maps as an immersion to ∂D. E.g., any pair $(\tau_i, \tau_j) \in \mathcal{S}$ must have one arc oriented into M and one oriented out from M on the same side of M.

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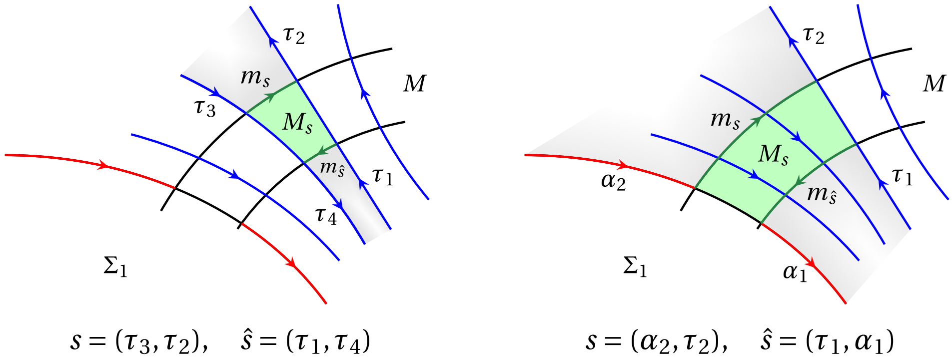

Figure 3. Two examples of dual pairs of sides $(s, \hat{s})$. Geometrically, each side encodes a portion of the boundary of a possible turn disk (shaded grey in the figure) mapping to D.

For any side $s \in \mathcal{S}$, there is a unique side $\hat{s} \in \mathcal{S}$ such that m_s, $m_{\hat{s}}$ and two unique arcs from $(\gamma \cup \partial\Sigma_1) \cap M$ bound a subrectangle of M. Call $\hat{s}$ the *dual side* of s, and let $M_s \subseteq M$ be the bounded subrectangle. Note that, here, we are using the fact that M contains no self-intersections of γ. Also, note that $M_s = M_{\hat{s}}$. See figure 3 again.

The encoding

Let $V = \mathbb{R}^{\mathrm{TP}} \oplus \mathbb{R}^2$ with coordinates $\{x_p\}$, $\{r^+, r^-\}$. We define a function

$$v \colon \mathscr{T}(\gamma) \to V$$

as follows. Let $f \colon S \to \Sigma$ be a taut admissible map. For each $p \in \mathrm{TP}$, define $x_p(S)$ to be the number of components E of $f^{-1}(D)$ such that $f|_{\partial E} = p$. Let $S_1 = f^{-1}(\Sigma_1)$. As mentioned above, since $f(\partial S) \cap \Sigma_1 = \emptyset$, the map $f|_{S_1} \colon S_1 \to \Sigma_1$ is a covering map. Define $r^+(S)$ to be the positive degree of $f|_{S_1}$ and $r^-(S)$ the negative degree. Finally let $v(S) = ((x_p(S)), r^+(S), r^-(S))$.

Matching equations

The vector $v(S)$ will satisfy two kinds of equations. Broadly speaking, for each turn path p with a side s, there must be a corresponding turn path p' (possibly equal to p) with a matching dual side $\hat{s}$. Moreover, r^+ and r^- should match with the number of α_i's and $\overline{\alpha}_i$'s appearing in turn paths of S. We define some linear maps. For each side $s \in \mathcal{S}$, we define $d_s \colon V \to \mathbb{R}$ as

$$d_s((x_p), r^+, r^-) = \sum_{p \text{ has } s} x_p.$$

For each $\alpha \in \{\alpha_i\} \cup \{\overline{\alpha}_i\}$, define $d_\alpha \colon V \to \mathbb{R}$ as

$$d_\alpha((x_p), r^+, r^-) = \sum_{\substack{\alpha \text{ is a} \\ \text{subarc of } p}} x_p.$$

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For each τ_i, define $d_{\tau_i}\colon V \to \mathbb{R}$ by

$$d_{\tau_i}((x_p), r^+, r^-) = \sum_{\substack{\tau_i \text{ is a} \\ \text{subarc of } p}} x_p.$$

LEMMA 4.1. *Let* (f, S) *be a taut admissible surface. Then,* $((x_p), r^+, r^-) = v(S)$ *satisfies the following matching equations:*

(1) $d_s((x_p), r^+, r^-) = d_{\hat{s}}((x_p), r^+, r^-)$ *for all* $s \in \mathcal{S}$.

(2) $d_{\alpha_i}((x_p), r^+, r^-) = r^+$ *for all* i.

(3) $d_{\overline{\alpha}_i}((x_p), r^+, r^-) = r^-$ *for all* i.

(4) $d_{\tau_i}((x_p), r^+, r^-) = d_{\tau_j}((x_p), r^+, r^-)$ *for all* i, j.

Proof. Let M' be a component of M and let N' be a component of $N = f^{-1}(M)$ mapping to M'. By tautness, the map $f\colon N' \to M'$ is an immersion. Since M' is simply connected, N' is compact, and ∂S maps exclusively to γ, the surface N' is a topological rectangle with two edges mapping to ∂D and two edges mapping to $(\gamma \cup \partial\Sigma_1) \cap M$. The map $f|_{N'}$ may be orientation reversing or preserving.

Let E', E'' be the two components of $f^{-1}(D)$ bordering N'. Then N' borders E', E'' of E in the middle of sides in $f|_{\partial E'}$, $f|_{\partial E''}$. Since f is an immersion from $f^{-1}(\Sigma_2)$ to Σ_2 and from ∂S to Σ (as a 1-manifold), the map $f|_{\partial E'}$ (resp. $f|_{\partial E''}$) is determined near the intersection of $\partial N'$ and $\partial E'$ (resp. $\partial N'$ and $\partial E''$). Thus, $f|_{N'}$ determines precisely one side each of the turn paths $f|_{\partial E'}$, $f|_{\partial E''}$ and those sides must be dual. Conversely, the side of $f|_{\partial E'}$ at $\partial E' \cap \partial N'$ determines the image of $f(N')$ and the side of $f|_{\partial E''}$ at $\partial E'' \cap \partial N'$ which must be dual. Since $f|_{\partial E}$ is taut for any component E of $f^{-1}(D)$, each arc τ_i, α_j, $\overline{\alpha}_j$ is traversed at most once and thus each side appears in $f|_{\partial E}$ at most once. Thus, $d_s(v(S))$ is the total number of times s appears as a side in f, and the above argument shows this is equal to the total number of times $\hat{s}$ appears as a side in f which is $d_{\hat{s}}(v(S))$.

For the second equality, first note that $r^+(S)$ equals the positive degree of the map $f\colon f^{-1}(\alpha_i) \to \alpha_i$ where we view $f^{-1}(\alpha_i)$ as a subarc of the boundary of $f^{-1}(\Sigma_3)$. This is equal to the number of times that the boundary of $f^{-1}(D)$ traverses α_i, and since f is taut, this is $d_{\alpha_i}(v(S))$. The argument for the third equation is similar. Again by tautness, $d_{\tau_i}(v(S)) = n(S)$ for all i, and the last equality holds. □

Degree and Euler characteristic

Recall from the preceding proof that $d_{\tau_i}(v(S)) = n(S)$ for each turn arc τ_i and all taut admissible surfaces S. Thus we have the factorization

$$\mathscr{T}(\gamma) \xrightarrow{v} V \xrightarrow{d_{\tau_i}} \mathbb{R}$$

of the degree function $n(S)$ through v via any of the linear maps d_{τ_i}.

The Euler characteristic $\chi(S)$ also factors through v in a similar fashion. First we need a formula for $\chi(S)$ based on the combinatorics of turn disks.

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Definition 4.2. *For $p \in \mathrm{TP}$ let*

$$\kappa(p) = 1 - \frac{1}{2}(\#\ \textit{sides of}\ p).$$

Let $F_\chi\colon \mathbb{R}^{\mathrm{TP}} \oplus \mathbb{R}^2 \to \mathbb{R}$ be the linear map

$$F_\chi((x_p), r^+, r^-) = \Big(\sum_{p\in\mathrm{TP}} \kappa(p) x_p\Big) + \chi(\Sigma_1)(r^+ + r^-).$$

Lemma 4.3. *If S is a taut admissible surface, then $\chi(S) = F_\chi(v(S))$.*

Proof. Given a taut admissible surface $f\colon S \to \Sigma$, the subsurface $S_3 = f^{-1}(\Sigma_2 - \mathrm{int}(\Sigma_1))$ consists of turn disks mapping to D and 1-handles mapping by immersions to M where each 1-handle attaches to the middle arcs of a dual pair of sides. It is clear then that

$$\chi(S) = \Big(\sum_{p\in\mathrm{TP}} \kappa(p) x_p(S)\Big).$$

Let $S_1 = f^{-1}(\Sigma_1)$. By definition of taut surface, S_1 maps to Σ_1 by an immersion and since $S_1 \cap \partial S = \emptyset$, the map is a covering map. Therefore $\chi(S_1) = \chi(\Sigma_1)(r^+(S) + r^-(S))$. Since $S = S_1 \cup S_3$ and S_1, S_3 meet along full boundary components of S_1 and S_3, we have $\chi(S) = \chi(S_1) + \chi(S_3)$ and the lemma follows. □

Remark 4.4. Having explained our encoding of surfaces, we can relate this encoding to that used to determine stable commutator length in a free group. Our 1–chain γ lies in the union of the D_i and bands M. If we collapse each D_i to a point and the bands in M to edges, we obtain a 1–chain in a graph. However, only the turn paths which traverse no α_i correspond to turn circuits as defined in [**1**], and of course, there is nothing analogous to the role of Σ_1 and its matching equations with turn paths.

5. Reassembly

Given a taut admissible surface, we associate the vector $v(S)$ which encodes it and satisfies matching equations. Now, we would like to show that we can construct a taut admissible surface with the right Euler characteristic and degree $(n(S))$ from a vector satisfying the matching equations. In general, it's not clear that this is possible for arbitrary positive integral vectors satisfying the matching equations. However, it is possible after multiplying the vector by a sufficiently large integer. In fact, we make the statement for positive rational vectors.

Lemma 5.1. *Suppose a nonzero, nonnegative vector $((x_p), r^+, r^-) \in \mathbb{Q}^{\mathrm{TP}} \oplus \mathbb{Q}^2$ satisfies the matching equations (1)–(4) in lemma 4.1. Then, there is a taut admissible surface S such that*

$$\frac{\chi(S)}{n(S)} = \frac{F_\chi((x_p), r^+, r^-)}{d_{\tau_1}((x_p), r^+, r^-)}.$$

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Proof. We can, without loss of generality, assume that the vector is integral. The basic idea is as follows. We take x_p turn disks for each p whose boundary will map to the turn path p. We then attach 1-handles to dual sides. This surface, call it say S_3, then has boundary components some of which map to powers of components of γ and some to components of $\partial\Sigma_3$ with varying degrees. We then want to glue in a (probably disconnected) surface to latter boundary components which will map to Σ_1 by a covering map. The difficulty is that the part of ∂S_3 mapping to $\partial\Sigma_3 = \partial\Sigma_1$ may not extend to a covering map of Σ_1 depending on the topology of Σ_1 and the degrees of the maps from components of ∂S_3 to components of $\partial\Sigma_1$. Lemmas 2.7 and 3.7 will fix this problem at the cost of scaling $((x_p),\, r^+,\, r^-)$ by some positive integer.

For each turn path $p \in \mathrm{TP}$, we let E_p be an oriented disk and $f_p\colon\ E_p \to D$ a map such that $f_{\partial E_p} = p$ (where ∂E_p has its boundary orientation). If p does not traverse any turn arcs, then $p\colon\ S^1 \to \partial D$ is an immersion; in this case, we insist f_p is a homeomorphism. Note that if p traverses $\overline{\alpha}_i$'s, then f_p is orientation-reversing. For each side $s \in \mathcal{S}$, we let N_s be an oriented disk which is a copy of M_s and $f_s\colon\ N_s \to M_s$ the canonical homeomorphism. We orient N_s as follows. If s has a turn τ_i, then we orient N_s so that the boundary orientation is consistent with the orientation of τ_i. If s has an α_i, then we orient N_s such that f_s is orientation-preserving and if s has an $\overline{\alpha}_i$, then we orient N_s such that f_s is orientation-reversing.

Now, we construct our initial surface. We take x_p copies of E_p and $d_s((x_p),\, r^+,\, r^-)$ copies of N_t. (Recall that $M_s = M_{\hat{s}}$ and thus $N_s = N_{\hat{s}}$. By the matching equations, we are taking a well-defined number of copies of N_s.) We attach the E_p together using the N_s in the obvious way such that f_p and f_s extend to a map on the oriented surface constructed. Call this surface S_3 and the map f_3. Some boundary components of S_3 map to γ and others to $\partial\Sigma_3 = \partial\Sigma_1$. By the matching equations, the positive degree of the map ∂S_3 to any component of $\partial\Sigma_3$ is r^+ and the negative degree is r^-. By applying lemma 2.7 separately to the positive and negative degrees, there is a branched cover $S_1 \to \Sigma_1$ such that the degree partition over any component of $\partial\Sigma_3$ is precisely the negative of that of S_3. By gluing boundary components of S_3 to S_1 which map to the same component of $\partial\Sigma_3$ with matching degrees (i.e. equal but opposite), we obtain a surface S' and a map $g'\colon\ S' \to \Sigma$ satisfying the hypotheses of lemma 3.6.

By lemma 3.7, there is a positive integer N and a taut admissible surface $g\colon\ S \to \Sigma$ satisfying $n(S) = Nn(S') = Nd_{\tau_1}((x_p),\, r^+,\, r^-)$ and $\chi(g^{-1}(\Sigma_3)) = N\chi(S_3)$ and $\chi(g^{-1}(\Sigma_1)) = N(r^+ + r^-)\chi(\Sigma_1)$. We then have

$$\begin{aligned}\chi(S) = \chi(g^{-1}(\Sigma_3)) + \chi(g^{-1}(\Sigma_1)) &= N\chi(S_3) + N(r^+{+}r^-)\chi(\Sigma_1)\\ &= NF_\chi((x_p), r^+, r^-).\end{aligned}$$

The lemma follows. □

6. Linear programming and scl

We are now ready to prove that the stable commutator length of the nonfilling 1-chain is the solution of a linear programme. We define the region $R \subseteq \mathbb{R}^{\mathrm{TP}} \oplus \mathbb{R}^2$ to be the subset defined by the following linear equations and inequalities.

- the matching equations from lemma 4.1
- $d_{\tau_1}((x_p), r^+, r^-) = 1$
- $x_p \geqslant 0$ for all $p \in \mathrm{TP}$ and $r^+ \geqslant 0$ and $r^- \geqslant 0$

LEMMA 6.1. *The stable commutator length of γ is equal to*

$$\inf \left\{ -\frac{1}{2} F_\chi((x_p), r^+, r^-) \mid ((x_p), r^+, r^-) \in \mathbb{Q}^{\mathrm{TP}} \oplus \mathbb{Q}^2 \cap R \right\} \tag{6.1}$$

Proof. Recall from corollary 3.4 that

$$\mathrm{scl}(\gamma) = \inf_{S \in \mathscr{T}(\gamma)} \frac{-\chi(S)}{2n(S)}.$$

Let S be a taut admissible surface. Then, by lemma 4.1, $v(S) \in \mathbb{Z}^{\mathrm{TP}} \oplus \mathbb{Z}^2$ satisfies the matching equations. Moreover, $n(S) = d_{\tau_1}(v(S))$ and, by lemma 4.3, $\chi(S) = F_\chi(v(S))$. By linearity, $d_{\tau_1}(v(S)/n(S)) = 1$ and

$$-\frac{\chi(S)}{2n(S)} = -\frac{1}{2} \frac{F_\chi(v(S))}{n(S)} = -\frac{1}{2} F_\chi \left(\frac{v(S)}{n(S)} \right).$$

Thus, $\mathrm{scl}(\gamma) \geqslant$ (6.1). On the other hand, by lemma 5.1, for any vector $((x_p), r^+, r^-)$ satisfying the matching equations and $d_{\tau_1}((x_p), r^+, r^-) = 1$, there is a taut admissible surface S such that

$$-\frac{\chi(S)}{2n(S)} = -\frac{1}{2} \frac{F_\chi((x_p), r^+, r^-)}{d_{\tau_1}((x_p), r^+, r^-)} = -\frac{1}{2} F_\chi((x_p), r^+, r^-).$$

Thus (6.1) $\geqslant \mathrm{scl}(\gamma)$. □

We are now ready to prove the main theorem.

Proof of theorem 1.1. Since $v(S) \in R$ for any taut admissible surface S, the region R is non-empty. Since R is defined by equations and inequalities with rational coefficients, $R \cap \mathbb{Q}^{\mathrm{TP}} \oplus \mathbb{Q}^2$ is dense in R. By lemma 6.1 and the fact that stable commutator length is nonnegative, $-1/2F_\chi$ is bounded below on R. Since R is finite-dimensional and defined by finitely many equations, minimizing the linear function $-1/2F_\chi$ on R is a linear programme and therefore has an optimal solution at a vertex of R. Since R is defined by equations with rational coefficients, all vertices are rational points, and so by lemma 6.1, $\mathrm{scl}(\gamma)$ is rational. □

REMARK 6.2. The proofs of theorem 1.1 and lemma 5.1 also establish that γ has an extremal surface.

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