# Appendix $B$
# The rotation functions

A state of angular momentum $J$, and $z$ component of angular momentum $m$, is transformed under a rotation by the Euler angles $\alpha, \beta, \gamma$ according to (see Edmonds (1960) p. 54)

$$D(\alpha,\beta,\gamma)\,|Jm\rangle = \sum_{m'=-J}^{J} |Jm'\rangle\,\langle Jm'|\,D(\alpha,\beta,\gamma)\,|Jm\rangle \qquad (B.1)$$

where the rotation operator is

$$D(\alpha,\beta,\gamma) \equiv e^{i\alpha J_z}\,e^{i\beta J_y}\,e^{i\gamma J_z} \qquad (B.2)$$

i.e. a rotation by angle $\gamma$ about the $z$ axis, followed by a rotation by $\beta$ about the $y$ axis, followed by a further rotation by $\alpha$ about the $z$ axis.

Since the eigenvalue of $J_z$ is $m$, the matrix elements of $D(\alpha,\beta,\gamma)$ can be written

$$\langle Jm'|\,D(\alpha,\beta,\gamma)\,|Jm\rangle \equiv \mathscr{D}^J_{m'm}(\alpha,\beta,\gamma) \qquad (B.3)$$

$$= e^{im'\alpha}\,d^J_{m'm}(\beta)\,e^{im\gamma} \qquad (B.4)$$

where the rotation matrices are defined by

$$d^J_{m'm}(\beta) \equiv \langle Jm'|\,e^{i\beta J_y}\,|Jm\rangle \qquad (B.5)$$

These matrix elements can readily be evaluated for $J = \frac{1}{2}$ by substituting the Pauli matrix for $J_y$ and expanding the exponential (see ($B$.19) below), and then higher $J$ values can be derived using the Clebsch–Gordan series (see for example Wigner (1959) p. 167). It is found that (Edmonds (1960) p. 57)

$$d^J_{m'm}(\beta) = \left[\frac{(J+m')!\,(J-m')!}{(J+m)!\,(J-m)!}\right]^{\frac{1}{2}} \sum_\sigma \binom{J+m}{J-m'-\sigma}\binom{J-m}{\sigma}(-1)^{J-m'-\sigma}$$
$$\times\left(\cos\frac{\beta}{2}\right)^{2\sigma+m'+m}\left(\sin\frac{\beta}{2}\right)^{2J-2\sigma-m'-m} \qquad (B.6)$$

If the scattering plane is taken to be the $x$–$z$ plane, then the angle $\beta$ here corresponds to the scattering angle $\theta$ between the directions of motion in the initial and final states, and it is more convenient to write the rotation matrices as functions of $z \equiv \cos\theta$ rather than $\theta$. Also for two-particle helicity states $m'$ and $m$ correspond to the helicity differences $\lambda$ and $\lambda'$ defined in (4.4.15). So we shall usually replace ($B$.6) by $d^J_{\lambda\lambda'}(z)$ from now on.

The functions defined by ($B$.6) satisfy the symmetry relations

$$\left.\begin{aligned} d^J_{\lambda\lambda'}(z) &= (-1)^{\lambda-\lambda'} d^J_{-\lambda-\lambda'}(z) = (-1)^{\lambda-\lambda'} d^J_{\lambda'\lambda}(z) \\ d^J_{\lambda\lambda'}(\pi-\theta) &= (-1)^{J-\lambda} d^J_{-\lambda\lambda'}(-\theta) = (-1)^{J-\lambda} d^J_{\lambda'-\lambda}(\theta) \end{aligned}\right\} \qquad (B.7)$$

The expression ($B$.6) can be rewritten in terms of Jacobi polynomials $P^{(a,b)}_c(z)$ as

$$d^J_{\lambda\lambda'}(z) = \left[\frac{(J+\lambda)!\,(J-\lambda)!}{(J+\lambda')!\,(J-\lambda')!}\right]^{\frac{1}{2}} \left(\frac{1-z}{2}\right)^{\frac{1}{2}(\lambda-\lambda')} \left(\frac{1+z}{2}\right)^{\frac{1}{2}(\lambda+\lambda')} P^{(\lambda-\lambda',\,\lambda+\lambda')}_{J-\lambda}(z) \qquad (B.8)$$

but this is only valid for non-negative values of $\lambda-\lambda'$ and $\lambda+\lambda'$. Other values can be obtained from ($B$.8) using the symmetry relations ($B$.7), which may be incorporated by writing

$$d^J_{\lambda\lambda'}(z) = (-1)^{\Lambda} \left[\frac{(J+M)!\,(J-M)!}{(J+N)!\,(J-N)!}\right]^{\frac{1}{2}} \xi_{\lambda\lambda'}\, P^{(|\lambda-\lambda'|,\,|\lambda+\lambda'|)}_{J-M} \qquad (B.9)$$

where

$$M \equiv \max\{|\lambda|, |\lambda'|\}, \quad N \equiv \min\{|\lambda|, |\lambda'|\}, \quad \Lambda \equiv \tfrac{1}{2}(\lambda-\lambda'-|\lambda-\lambda'|) \qquad (B.10)$$

and the 'half-angle factor' is defined by

$$\xi_{\lambda\lambda'}(z) \equiv \left(\frac{1-z}{2}\right)^{\frac{1}{2}|\lambda-\lambda'|} \left(\frac{1+z}{2}\right)^{\frac{1}{2}|\lambda+\lambda'|} \qquad (B.11)$$

Equation ($B$.9) is a very convenient representation because for integer $J-M$ the Jacobi function is an entire function of $z$, so the only possible singularities of $d^J_{\lambda\lambda'}(z)$ in $z$ stem from the behaviour of the half-angle factor ($B$.11) at $z = \pm 1$.

However, we shall also wish to continue in $J$, and for this purpose it is more useful to re-express ($B$.9) in terms of the hypergeometric function (see Andrews and Gunson 1964)

$$d^J_{\lambda\lambda'}(z) = (-1)^{\Lambda} \left[\frac{(J+M)!\,(J-M+|\lambda-\lambda'|)!}{(J-M)!\,(J+M-|\lambda-\lambda'|)!}\right]^{\frac{1}{2}} \frac{1}{|\lambda-\lambda'|!}$$

$$\times\, \xi_{\lambda\lambda'}(z)\, F(-J+M, J+M+1, |\lambda-\lambda'|+1; (1-z)/2) \qquad (B.12)$$

The hypergeometric function is an entire function of $J$, so the only singularities stem from the square bracket, when the factorial functions have poles for negative integer values of their arguments.

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Also from the asymptotic form of the hypergeometric function we find

$$d^J_{\lambda\lambda'}(z) \xrightarrow[z\to\infty]{} (-1)^A$$
$$\times \frac{(2J)!}{[(J+M)!\,(J-M+|\lambda-\lambda'|)!\,(J-M)!\,(J+M-|\lambda-\lambda'|)!]^{\frac{1}{2}}}$$
$$\times\, \xi_{\lambda\lambda'}(z)\left(\frac{z}{2}\right)^{J-M}(1+O(z^{-2}))+O(z^{-J-1}) \qquad (B.13)$$

so, since $\xi_{\lambda\lambda'}(z) \sim z^M$ from $(B.11)$

$$d^J_{\lambda\lambda'}(z) \sim z^J, \quad \text{for} \quad J > -\tfrac{1}{2}$$

for $J-v \neq$ integer $< M$ where $d^J_{\lambda\lambda'}$ vanishes ($v \equiv 0/\frac{1}{2}$ for physical $J =$ integer/half-odd-integer, i.e. for even/odd fermion number).

These functions also satisfy the orthogonality relations

$$\int_{-1}^{1} d^J_{\lambda\lambda'}(z)\, d^{J'}_{\lambda\lambda'}(z)\, \mathrm{d}z = \delta_{JJ'}\frac{2}{2J+1} \qquad (B.14)$$

$$\tfrac{1}{2}\sum_J (2J+1)\, d^J_{\lambda\lambda'}(z)\, d^J_{\lambda\lambda'}(z') = \delta(z-z') \qquad (B.15)$$

$$\sum_\lambda d^J_{\lambda\lambda'}(z)\, d^J_{\lambda\lambda''}(z) = \delta_{\lambda'\lambda''} \qquad (B.16)$$

Some useful special values are

$$d^J_{m0}(z) = \left[\frac{(J-m)!}{(J+m)!}\right]^{\frac{1}{2}} P^m_J(z) \qquad (B.17)$$

$$d^J_{00}(z) = P_J(z) \qquad (B.18)$$

for integer $J$, and

$$d^{\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}}(z) = \frac{1+z}{2} = \cos\tfrac{1}{2}\theta, \quad d^{\frac{1}{2}}_{\frac{1}{2}-\frac{1}{2}} = \frac{1-z}{2} = \sin\tfrac{1}{2}\theta \qquad (B.19)$$

We shall also make use of the second-type rotation functions $e^J_{\lambda\lambda'}(z)$, analogous to the second-type Legendre functions $Q_l(z)$ introduced in Appendix $A$ (see Andrews and Gunson 1964). They are defined in terms of the second-type Jacobi functions $Q^{(a,b)}_c(z)$ by

$$e^J_{\lambda\lambda'}(z) = (-1)^{A+\lambda-\lambda'}\left[\frac{(J+M)!\,(J-M)!}{(J+N)!\,(J-N)!}\right]^{\frac{1}{2}} \xi_{\lambda\lambda'}(z)\, Q^{(|\lambda-\lambda'|,\,|\lambda+\lambda'|)}_{J-M}(z) \qquad (B.20)$$

For integer $J-M \geqslant 0$ they are related to the $d^J_{\lambda\lambda'}$ by the generalized Neumann relation

$$\xi_{\lambda\lambda'}(z)\, e^J_{\lambda\lambda'}(z) = \frac{1}{2}\int_{-1}^{1}\frac{\mathrm{d}z'}{z-z'}\, d^J_{\lambda\lambda'}(z')\,\xi_{\lambda\lambda'}(z') \qquad (B.21)$$

and they satisfy the symmetry properties

$$e^J_{\lambda\lambda'}(z) = (-1)^{\lambda-\lambda'} e^J_{-\lambda-\lambda'}(z) = (-1)^{\lambda-\lambda'} e^J_{\lambda'\lambda}(z) \qquad (B.22)$$

$$e^J_{\lambda\lambda'}(-z) = (-1)^{J-\lambda+1} e^J_{\lambda-\lambda'}(z) \qquad (B.23)$$

Equation ($B$.20) can be re-expressed in terms of the hypergeometric function as

$$e^J_{\lambda\lambda'}(z) = (-1)^A \frac{1}{(2J+1)!} [(J+M)!\,(J-M)!\,(J+N)!\,(J-N)!]^{\frac{1}{2}} \xi^{-1}_{\lambda\lambda'}(z)$$
$$\times \frac{1}{2}\left(\frac{z-1}{2}\right)^{-J-1+M} F\left(J-M+1, J-M+|\lambda-\lambda'|+1, 2J+2, \frac{2}{1-z}\right) \qquad (B.24)$$

which gives the $J$-plane singularities directly, and since $F \to 1$ as $z \to \infty$ the asymptotic behaviour is

$$e^J_{\lambda\lambda'}(z) \underset{z\to\infty}{\longrightarrow} (-1)^{\frac{1}{2}(\lambda-\lambda')} \frac{1}{(2J+1)!}$$
$$\times [(J+M)!\,(J-M)!\,(J+N)!\,(J-N)!]^{\frac{1}{2}} \frac{1}{2}\left(\frac{z}{2}\right)^{-J-1} \qquad (B.25)$$

and, cf. ($A$.31),

$$e^J_{\lambda\lambda'}(z) \underset{J\to\infty}{\longrightarrow} \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{e^{\pm i\pi(\lambda-\lambda')}}{J^{\frac{1}{2}}} \frac{1}{(z^2-1)^{\frac{1}{4}}} e^{-(J+\frac{1}{2})\zeta(z)}, \quad \arg J < \pi \qquad (B.26)$$

where $\zeta(z) \equiv \log[z+(z^2-1)^{\frac{1}{2}}]$, and we use $\pm$ for $\mathrm{Im}\{z\} \gtrless 0$.

For half-odd-integer values of $J-v$ they obey the symmetry relation

$$e^J_{\lambda\lambda'}(z) = (-1)^{\lambda-\lambda'} e^{-J-1}_{\lambda\lambda'}(z) \qquad (B.27)$$

Also, analogous to ($A$.18), there is the relation

$$\frac{d^J_{\lambda\lambda'}(z)}{\sin\pi(J-\lambda)} = \frac{e^J_{\lambda\lambda'}(z)}{\pi\cos\pi(J-\lambda)} - \frac{e^{-J-1}_{-\lambda-\lambda'}(z)}{\pi\cos\pi(J-\lambda)} \qquad (B.28)$$

and we find

$$e^J_{\lambda\lambda'}(z) \approx \frac{d^{J_0}_{\lambda\lambda'}(z)}{J-J_0} \qquad (B.29)$$

for $J \to$ integer $(J_0 - v)$ when $J_0 < -M$, and similar poles or $(J-J_0)^{-\frac{1}{2}}$ factors for integer $(J_0-v)$, $-M \leqslant J_0 < M$, from ($B$.24) (Andrews and Gunson 1964).