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HOMOMORPHISMS OF LIE ALGEBRAS OF ALGEBRAIC GROUPS AND ANALYTIC GROUPS

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ABSTRACT. Let $\phi\colon \mathcal{L}(G) \longrightarrow \mathcal{L}(H)$ be a Lie algebra homomorphism from the Lie algebra of G to the Lie algebra of H in the following cases: (i) G and H are irreducible algebraic groups over an algebraically closed field of characteristic 0, or (ii) G and H are linear complex analytic groups. In this paper, we present some equivalent conditions for ϕ to be a differential in the above two cases. That is, ϕ is the differential of a morphism of algebraic groups or analytic groups as appropriate.

In the algebraic case, for example, it is shown that ϕ is a differential if and only if ϕ preserves nilpotency, semisimplicity, and integrality of elements. In the analytic case, ϕ is a differential if and only if ϕ maps every integral semisimple element of $\mathcal{L}(G_0)$ into an integral semisimple element of $\mathcal{L}(H_0)$, where G_0 and H_0 are the universal algebraic subgroups of G and H. Via rational elements, we also present some equivalent conditions for ϕ to be a differential up to coverings of G in the algebraic case, and for ϕ to be a differential up to *finite* coverings of G in the analytic case.

If G is an algebraic group, an element of $\mathcal{L}(G)$ is called *integral* (resp. *rational*) if all its eigenvalues under the representations of $\mathcal{L}(G)$ associated with representations of G belong to $\mathbf{Z}$ (resp. $\mathbf{Q}$), or equivalently, if all its eigenvalues for its derivation action on the algebra of polynomial functions of G belong to $\mathbf{Z}$ (resp. $\mathbf{Q}$). We adopt the following notation and conventions: "algebraic group" means affine algebraic group, and "analytic group" means connected Lie group. $\mathcal{L}(G)$ is the Lie algebra of the group G. If G is algebraic, $\mathcal{A}[G]$ is its (Hopf) algebra of polynomial functions, $X(G)$ is its character group, G_u is its unipotent radical, and G_1 is its identity component. If τ is an element of $\mathcal{L}(G)$, then τ_n and τ_s are the nilpotent and semisimple (Jordan) parts of τ.

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1. Lie algebras of algebraic groups.

Unless otherwise stated, all algebraic groups are assumed to be over an algebraically closed field F. Let G be an algebraic group. We view the elements of its Lie algebra $\mathcal{L}(G)$ as left invariant derivations on its algebra $\mathcal{A}[G]$ of polynomial functions, *i.e.*, derivations that commute with the translation G action given by: $(f \cdot x)(y) = f(xy)$. For every element τ of $\mathcal{L}(G)$, we let E_τ (relative to G) be the set of all eigenvalues for the derivation action of τ on $\mathcal{A}[G]$. If the base field F is of characteristic 0, an element τ of $\mathcal{L}(G)$ is called *integral* (resp. *rational*) if E_τ is contained in $\mathbf{Z}$ (resp. $\mathbf{Q}$) where $\mathbf{Z}$ is the ring of rational integers of F, and $\mathbf{Q}$ is the prime field of F.

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If the base field F is of prime characteristic p, then τ is called *integral* if E_τ is contained in F_p, the prime field of F. For every subring A of F, we define $\mathcal{L}_A(G)$ as the set of all elements τ of $\mathcal{L}(G)$ for which $E_\tau \subset A$.

Let V be a polynomial G-module, *i.e.*, a locally finite G-module whose associated representative functions belong to $\mathcal{A}[G]$ (*cf.* [2, p. 41]). Then the differential of the representation of G on V makes V into an $\mathcal{L}(G)$-module. The basic example of a polynomial G-module is $\mathcal{A}[G]$ under the translation G-action given by: $(x \cdot f)(y) = f(yx)$. In this case, the associated $\mathcal{L}(G)$-module structure is given by the above derivation action of $\mathcal{L}(G)$ on $\mathcal{A}[G]$.

LEMMA 1.1. *Let τ be an element of $\mathcal{L}(G)$.*

(1) If V is a polynomial G-module, then the eigenvalues for the action of τ on V belong to E_τ. In particular, if $\rho\colon G \longrightarrow H$ is a morphism of algebraic groups, then $E_{\rho^0(\tau)} \subset E_\tau$ where ρ^0 is the differential of ρ.

(2) If S is an algebraic subgroup of G such that $\tau \in \mathcal{L}(S)$, then E_τ relative to G coincides with E_τ relative to S.

(3) If G is an algebraic subgroup of Aut V *where V is a finite dimensional F-space, then E_τ coincides with the $\mathbf{Z}$-module generated by the eigenvalues of τ on V.*

PROOF. (1) This follows immediately from the fact that every finite dimensional polynomial representation G-module is isomorphic with a G-submodule of a direct sum of copies of $\mathcal{A}[G]$ [13, Lemma 3.5].

(2) By the first part, E_τ relative to G is contained in E_τ relative to S. For the other inclusion, we may assume that τ is semisimple because the eigenvalues of any linear operator on a finite-dimensional vector space are precisely the eigenvalues of its semisimple part. In this case, the polynomial S-module $\mathcal{A}[S]$ is isomorphic with a τ-submodule of $\mathcal{A}[G]$, because $\mathcal{A}[S]$ is an S-homomorphic image of $\mathcal{A}[G]$. Hence E_τ relative to S is contained in E_τ relative to G.

(3) By tensoring and dualization of polynomial G-modules, (1) implies that E_τ is a $\mathbf{Z}$-module because, in tensoring, eigenvalues add; and in dualization, eigenvalues turn into their negatives. Hence E_τ contains the $\mathbf{Z}$-module V_τ generated by the eigenvalues of τ on V. Conversely, since every finite-dimensional polynomial representation of G can be constructed from its faithful representation on V by the process of forming tensor products, direct sums, subrepresentations, quotients and duals [13, p. 25], it follows that $E_\tau \subset V_\tau$ since the eigenvalues of τ in a quotient G-module of V are contained in that V as seen in the proof of (2). Hence $E_\tau = V_\tau$.

LEMMA 1.2. *Let T be a torus and let A be a subring of the base field F containing 1. Then $\mathcal{L}_A(T)$ is an A-form of $\mathcal{L}(T)$, i.e., $\mathcal{L}_A(T)$ is an A-module and $\mathcal{L}(T) = \mathcal{L}_A(T) \otimes F$.*

PROOF. Let τ be an element of $\mathcal{L}(T)$ and let τ_1 be the differentiation of $\mathcal{A}[T]$ corresponding to (the derivation) τ. Then $\tau(f) = \tau_1(f) \cdot f$ for every element f of $X(T)$. Since $X(T)$ spans $\mathcal{A}[T]$, it follows that $E_\tau \subset A$ if and only if $\tau_1\big(X(T)\big) \subset A$.

Now we shall view the elements of $\mathcal{L}(T)$ as differentiations of $\mathcal{A}[T]$. Let $\varepsilon\colon \mathcal{L}(T) \longrightarrow \mathrm{Hom}_{\mathbf{Z}}\big(X(T), F\big)$ be the F-linear map where for each τ in $\mathcal{L}(T)$, $\varepsilon(\tau)$ is the restriction of τ

to $X(T)$. In fact, $\varepsilon(\tau) \in \mathrm{Hom}_{\mathbf{Z}}\big(X(T), F\big)$, because τ is a differentiation. Since T is a torus, $X(T)$ spans $\mathcal{A}[T]$, so ε is injective. Moreover, every map h from $X(T)$ to F has a unique extension to an F-linear map $\hat{h}$ from $\mathcal{A}[T]$ to F. Clearly, $\hat{h}$ is a differentiation if and only if $h \in \mathrm{Hom}_{\mathbf{Z}}\big(X(T), F\big)$. This shows that ε is surjective.

Hence $\varepsilon\colon \mathcal{L}(T) \longrightarrow \mathrm{Hom}_{\mathbf{Z}}\big(X(T), F\big)$ is an F-linear isomorphism. Now if $\tau \in \mathcal{L}(T)$, we have seen above that $E_\tau \subset A$ if and only if τ maps $X(T)$ into A. Hence ε restricts to an A-module isomorphism $\mathcal{L}_A(T) \longrightarrow \mathrm{Hom}_{\mathbf{Z}}\big(X(T), A\big)$. Since $X(T)$ is a free $\mathbf{Z}$-module and 1 belongs to A, it follows that $\mathrm{Hom}_{\mathbf{Z}}\big(X(T), A\big)$ is an A-form of $\mathrm{Hom}_{\mathbf{Z}}\big(X(T), F\big)$. Consequently, via ε, $\mathcal{L}_A(T)$ is an A-form of $\mathcal{L}(T)$.

LEMMA 1.3. *Let $\phi\colon \mathcal{L}(T_1) \longrightarrow \mathcal{L}(T_2)$ be a Lie algebra homomorphism of Lie algebras of tori. Then ϕ is the differential of a morphism of algebraic groups if and only if ϕ maps every integral element of $\mathcal{L}(T_1)$ into an integral element of $\mathcal{L}(T_2)$.*

PROOF. If ϕ is the differential of a morphism of algebraic groups then, by Lemma 1.1, ϕ preserves integral elements. Conversely, suppose $\phi\big(\mathcal{L}_I(T_1)\big) \subset \mathcal{L}_I(T_2)$ where I is the subring of F generated by 1. For $i = 1, 2$, let $\varepsilon_i\colon \mathcal{L}_I(T_i) \longrightarrow \mathrm{Hom}_{\mathbf{Z}}\big(X(T_i), I\big)$ be the canonical isomorphisms discussed in the proof of Lemma 1.2. Since ϕ maps $\mathcal{L}_I(T_1)$ into $\mathcal{L}_I(T_2)$, ϕ induces a $\mathbf{Z}$-module homomorphism $\hat{\phi}\colon \mathrm{Hom}_{\mathbf{Z}}\big(X(T_1), I\big) \longrightarrow \mathrm{Hom}_{\mathbf{Z}}\big(X(T_2), I\big)$ where $\hat{\phi} = \varepsilon_2 \circ \phi \circ \varepsilon_1^{-1}$. Since each $X(T_i)$ is a finitely generated free $\mathbf{Z}$-module, it follows that $\hat{\phi}$ is the dual map of a $\mathbf{Z}$-module homomorphism $\alpha\colon X(T_2) \longrightarrow X(T_1)$ (α is not unique in prime characteristic). Now, by the character theory of tori, α is the transpose of a morphism $\rho\colon T_1 \longrightarrow T_2$ of algebraic groups. From definitions, it is evident that the following diagram commutes

$$\begin{array}{ccc} \mathcal{L}_I(T_1) & \xrightarrow{\rho^0} & \mathcal{L}_I(T_2) \\ \big\downarrow{\scriptstyle \varepsilon_1} & & \big\downarrow{\scriptstyle \varepsilon_2} \\ \mathrm{Hom}_{\mathbf{Z}}\big(X(T_1), I\big) & \xrightarrow{\hat{\rho}} & \mathrm{Hom}_{\mathbf{Z}}\big(X(T_2), I\big) \end{array}$$

where $\hat{\rho}$ is the dual map of the transpose $\rho^t\colon X(T_2) \longrightarrow X(T_1)$. Moreover, $\hat{\rho} = \hat{\phi}$ and $\hat{\phi} = \varepsilon_2 \circ \phi \circ \varepsilon_1^{-1}$. Hence ρ^0 agrees with ϕ on $\mathcal{L}_I(T_1)$. But this last is an I-form of $\mathcal{L}(T_1)$ by Lemma 1.2. Hence $\rho^0 = \phi$.

COROLLARY 1.4. *Let A be a toral subalgebra of $\mathcal{L}(G)$, i.e., A is an abelian subalgebra of semisimple elements of $\mathcal{L}(G)$. Then A is the Lie algebra of a torus of G if and only if A has a basis of integral elements of $\mathcal{L}(G)$.*

PROOF. Let L be a maximal toral subalgebra of $\mathcal{L}(G)$ containing A. Then, in any characteristic, L is the Lie algebra of a maximal torus of G ([7, p. 62], [11, p. 1054]). Hence, by Lemma 1.1, we may assume that G is a torus.

Now suppose that A has a basis of integral elements. Then, by Lemma 1.2, $A \cap \mathcal{L}_I(G)$ is an I-form of A. Let D be a torus of dimension $\dim_F A$. Then we may identify $\mathcal{L}(D)$ with A, and $\mathcal{L}_I(D)$ with $A \cap \mathcal{L}_I(G)$. Now the injection $i\colon \mathcal{L}(D) \longrightarrow \mathcal{L}(G)$ maps $\mathcal{L}_I(D)$ into $\mathcal{L}_I(G)$. Hence, by Lemma 1.3, i is the differential of an algebraic group homomorphism

$\rho: D \longrightarrow G$. Since ρ^0 is injective, it follows that $\mathcal{L}\big(\rho(D)\big) = \rho^0\big(\mathcal{L}(D)\big) = A$ for reasons of dimension. This means that A is the Lie algebra of a torus of G whenever A has a basis of integral elements of $\mathcal{L}(G)$. The converse is also true by Lemmas 1.1 and 1.2.

Recall that a sub Lie algebra A of $\mathcal{L}(G)$ is called *algebraic* if it is the Lie algebra of an algebraic subgroup of G.

THEOREM 1.5. *Let $\phi: \mathcal{L}(G) \longrightarrow \mathcal{L}(H)$ be a Lie algebra homomorphism of Lie algebras of irreducible algebraic groups over an algebraically closed field F of characteristic* 0, *and let T be a maximal torus of G. Then the following are equivalent.*

(1) ϕ is the differential of a morphism algebraic groups.

(2) ϕ preserves nilpotency, semisimplicity, and integrality of elements.

(3) ϕ maps every (nilpotent) element of $\mathcal{L}(G_u)$ into a nilpotent element of $\mathcal{L}(H)$, and ϕ maps every integral (semisimple) element of $\mathcal{L}(T)$ into an integral semisimple element of $\mathcal{L}(H)$.

(4) The graph of ϕ is an algebraic sub Lie algebra of $\mathcal{L}(G \times H)$, and ϕ maps every integral element of $\mathcal{L}(T)$ into an integral element of $\mathcal{L}(H)$.

PROOF. (1) implies (2) by Lemma 1.1, and (2) implies (3) trivially. Now we show that (3) implies (4). Let R be the radical of G and let D be the identity component of $R \cap T$. Since T is a maximal torus of G, it follows that D is a maximal torus of R and $\mathcal{L}(R) = \mathcal{L}(G_u) + \mathcal{L}(D)$ (semi-direct). Since each of $\mathcal{L}(G_u)$ and $\phi\big(\mathcal{L}(G_u)\big)$ consists of nilpotent elements, so does the graph of ϕ on $\mathcal{L}(G_u)$. Hence it is an algebraic sub Lie algebra of $\mathcal{L}(G) \times \mathcal{L}(H)$. Moreover, $\mathcal{L}(D)$ has a basis of integral (semisimple) elements by Lemma 1.2, and we are given that ϕ maps every integral (semisimple) element of $\mathcal{L}(T)$ into an integral semisimple element of $\mathcal{L}(H)$. Thus the graph of ϕ on $\mathcal{L}(D)$ has a basis of integral semisimple elements of $\mathcal{L}(G) \times \mathcal{L}(H)$, so this graph is the Lie algebra of a torus of $G \times H$ by Corollary 1.4. Now the graph of ϕ on the sum $\mathcal{L}(R) = \mathcal{L}(G_u) + \mathcal{L}(D)$, is the semi-direct sum of two algebraic sub Lie algebras of $\mathcal{L}(G) \times \mathcal{L}(H)$. Hence it is an algebraic sub Lie algebra of $\mathcal{L}(G) \times \mathcal{L}(H)$. But this sub Lie algebra coincides with the radical of graph ϕ (on $\mathcal{L}(G)$). Hence graph ϕ is an algebraic sub Lie algebra of $\mathcal{L}(G) \times \mathcal{L}(H)$, so (3) implies (4).

Finally, suppose (4) holds. Then graph ϕ is the Lie algebra of an irreducible algebraic subgroup $\hat{G}$ of $G \times H$. Let $f: \hat{G} \longrightarrow G$ and $\rho: \hat{G} \longrightarrow H$ be the projection morphisms to the factors of G and H. Then f^0 and ρ^0 are the projection morphisms of graph ϕ on the factors of $\mathcal{L}(G) \times \mathcal{L}(H)$. Thus f^0 is an isomorphism and $(f^0 \times \rho^0)\big(\mathcal{L}(\hat{G})\big) = \text{graph}\,\phi$. Hence $\rho^0 = \phi \circ f^0$ and $f: \hat{G} \longrightarrow G$ is a covering, *i.e.*, a surjective (separable) morphism with finite kernel.

Now we show that $\ker f \subset \ker \rho$. Since $\rho^0 = \phi \circ f^0$, it follows that ϕ preserves semisimple elements. Thus ϕ maps every integral semisimple element of $\mathcal{L}(T)$ into an integral semisimple element of $\mathcal{L}(H)$. It follows from Lemma 1.3 and Corollary 1.4 that ϕ is the differential of a morphism $h: T \longrightarrow H$ of algebraic groups. Consequently, if $\hat{T}$ is the maximal torus of $\hat{G}$ such that $f(\hat{T}) = T$, then ρ^0 agrees with $h^0 \circ f^0$ on $\mathcal{L}(\hat{T})$. Hence ρ agrees with $h \circ f$ on $\hat{T}$, because $\hat{T}$ is connected. Thus, the intersection of $\ker f$ with $\hat{T}$

is contained in $\ker\rho$. But $f\colon \hat{G} \longrightarrow G$ is a covering. Hence $\ker f$ is a finite closed central subgroup of $\hat{G}$ which necessarily consists of semisimple elements. But such a subgroup of $\hat{G}$ is contained in every maximal torus of $\hat{G}$. Hence $\ker f \subset \hat{T}$, so $\ker f \subset \ker\rho$. Now ρ induces a morphism $\bar{\rho}\colon G \longrightarrow H$ of algebraic groups whose differential is ϕ, because $\rho^0 = \phi \circ f^0$, so (4) implies (1). This completes the proof of Theorem 1.5.

REMARK. The equivalence of (1) and (2) in Theorem 1.5 may fail over arbitrary fields of characteristic 0 mainly because an irreducible abelian reductive algebraic group G may fail to be a torus whence $\mathcal{L}(G)$ may fail to have a basis of integral elements.

For example, let $G = \mathrm{SO}(2, \mathbf{R})$ be the group of all orthogonal 2×2 real matrices of determinant 1. Then its Lie algebra consists of all skew-symmetric 2×2 real matrices. In particular, $\mathcal{L}(G)$ has no non-zero integral elements. Moreover, if $H = \mathbf{R}^*$, the multiplicative group of non-zero real numbers, and if $\phi\colon \mathcal{L}(G) \longrightarrow \mathcal{L}(H)$ is any non-trivial Lie algebra homomorphism, then ϕ satisfies condition (2) of Theorem 1.5 although there are no non-trivial morphisms from G into H.

DEFINITION 1.6. Let $\phi\colon \mathcal{L}(G) \longrightarrow \mathcal{L}(H)$ be a Lie algebra homomorphism of Lie algebras of algebraic groups. Then ϕ is called a *differential up to coverings of* G if there exists a covering $f\colon \hat{G} \longrightarrow G$, and a morphism $\rho\colon \hat{G} \longrightarrow H$ of algebraic groups such that $\rho^0 = \phi \circ f^0$.

THEOREM 1.7. *Let $\phi\colon \mathcal{L}(G) \longrightarrow \mathcal{L}(H)$ be a Lie algebra homomorphism of Lie algebras of irreducible algebraic groups over an algebraically closed field F of characteristic* 0, *and let D be a maximal torus of the radical of G. Then the following are equivalent.*

(1) ϕ is a differential up to coverings of G (see Definition 1.6)

(2) ϕ preserves nilpotency, semisimplicity, and rationality of elements.

(3) ϕ maps every (nilpotent) element of $\mathcal{L}(G_u)$ into a nilpotent element of $\mathcal{L}(H)$, and ϕ maps every rational (semisimple) element of $\mathcal{L}(D)$ into a rational semisimple element of $\mathcal{L}(H)$.

(4) The graph of ϕ is an algebraic sub Lie algebra of $\mathcal{L}(G \times H)$.

(5) The restriction of ϕ to the Lie algebra of the radical of G preserves nilpotency, semisimplicity, and rationality of elements.

The proof of Theorem 1.7 follows immediately from the proof of Theorem 1.5 in view of Lemma 1.1 and the following lemma.

LEMMA 1.8. *Let $f\colon G \longrightarrow H$ be a covering of algebraic groups and suppose that the base field F is of characteristic* 0. *Then $f^0\big(\mathcal{L}_{\mathbf{Q}}(G)\big) = \mathcal{L}_{\mathbf{Q}}(H)$.*

PROOF. $f^0\big(\mathcal{L}_{\mathbf{Q}}(G)\big) \subset \mathcal{L}_{\mathbf{Q}}(H)$ by Lemma 1.1. Conversely, suppose $f^0(x) = y$ where $y \in \mathcal{L}_{\mathbf{Q}}(H)$. To prove that x is rational, it suffices to prove that its semisimple part x_s is rational (see the proof of Lemma 1.1). Also y is rational if and only if y_s is rational. Thus we may assume that x and y are semisimple. In this case, x (resp. y) lies in the Lie algebra of a torus of G (resp. H) by Corollary 1.4. Consequently, by Lemma 1.1 we may assume that G and H are tori. Since f is a covering, $\ker f$ is a group of finite order, n say.

Thus, if $\theta \in X(H)$, θ^n induces a polynomial character on G. Consequently the **Z**- module homomorphism $\hat{f}$: $\mathrm{Hom}_{\mathbf{Z}}\big(X(G), \mathbf{Q}\big) \longrightarrow \mathrm{Hom}_{\mathbf{Z}}\big(X(H), \mathbf{Q}\big)$ induced from f by dualizing the transpose of f is surjective. Now if we consider the commutative diagram

$$\begin{array}{ccc} \mathcal{L}_{\mathbf{Q}}(G) & \xrightarrow{f^0} & \mathcal{L}_{\mathbf{Q}}(H) \\ \downarrow{\scriptstyle \varepsilon_1} & & \downarrow{\scriptstyle \varepsilon_2} \\ \mathrm{Hom}_{\mathbf{Z}}\big(X(G), \mathbf{Q}\big) & \xrightarrow{\hat{f}} & \mathrm{Hom}_{\mathbf{Z}}\big(X(H), \mathbf{Q}\big) \end{array}$$

as in the proof of Lemma 1.2, we see immediately via the isomorphism ε_1 and ε_2 that f^0 is surjective because $\hat{f}$ is surjective.

2. **Lie algebras of complex analytic groups.** Throughout this section, all complex analytic groups are assumed to have faithful (finite dimensional) complex analytic representations. Let G be a complex analytic group. An analytic subgroup P of G is called *reductive* if every complex analytic representation of P is semisimple. Now we recall from [10] the following notion with some of its properties.

DEFINITION 2.1. The *universal algebraic subgroup* G_0 of G is the subgroup of G generated by its commutator subgroup $[G, G]$ and all reductive analytic subgroups of G.

(2a) [10, Proposition 1]. G_0 is the unique maximal normal analytic subgroup of G that is algebraic under all analytic representations of G.

(2b) [10, Corollary 3]. The analytic structure of G induces a unique algebraic group structure on G_0 in such a way that every analytic morphism from G to a complex algebraic group H restricts to a morphism of complex algebraic groups from G_0 to H.

We shall need the notion of a nucleus of G. This is a closed simply connected solvable normal subgroup K of G such that G/K is reductive.

(2c) G has a nucleus if and only if G has a faithful representation; if G has a nucleus K then $G = K \cdot P$ (semi-direct) for any maximal reductive analytic subgroup P of G [6, Section 2].

THEOREM 2.2. *Let ϕ: $\mathcal{L}(G) \longrightarrow \mathcal{L}(H)$ be a Lie algebra homomorphism of Lie algebras of (faithfully representable) complex analytic groups. Let G_0 and H_0 be the universal algebraic subgroups of G and H. Then the following are equivalent.*

(1) ϕ is the differential of an analytic morphism from G to H.

(2) $\phi\big(\mathcal{L}(G_0)\big) \subset \mathcal{L}(H_0)$ and the restriction of ϕ to $\mathcal{L}(G_0)$ is the differential of an algebraic morphism from G_0 to H_0.

(3) ϕ maps every integral semisimple element of $\mathcal{L}(G_0)$ into an integral semisimple element of $\mathcal{L}(H_0)$.

(4) For some maximal (complex) torus T of G, ϕ maps every integral (semisimple) element of $\mathcal{L}(T)$ into an integral semisimple element of $\mathcal{L}(H_0)$.

(5) *For some closed analytic subgroup A of G such that G/A is a simply connected homogeneous space, the restriction of ϕ to $\mathcal{L}(A)$ is a differential.*

PROOF. If $f: P \longrightarrow H$ is a morphism of analytic groups where P is reductive, then $f(P) \subset H_0$. By (2b), this shows that (1) implies (2). Lemma 1.1 shows that (2) implies (3), while (3) implies (4) trivially.

Now suppose that (4) holds. Let P be a maximal reductive analytic subgroup of G containing T. Since $\phi\big(\mathcal{L}(T)\big) \subset \mathcal{L}(H_0)$ and $\mathcal{L}(P)$ is contained in $\mathcal{L}(T) + [\mathcal{L}(G), \mathcal{L}(G)]$, it follows that $\phi\big(\mathcal{L}(P)\big) \subset \mathcal{L}(H_0)$. By (2b), we may view P as an algebraic group and T as a maximal (algebraic) torus of P. Since ϕ maps every integral (semisimple) element of $\mathcal{L}(T)$ into an integral semisimple of $\mathcal{L}(H_0)$, Theorem 1.5 implies that the restriction of ϕ to $\mathcal{L}(P)$ is a differential. But G/P is a simply connected homogeneous space by (2c). Hence (4) implies (5).

Finally, we show that (5) implies (1). Let $\rho: \hat{G} \longrightarrow G$ be the universal analytic covering of G. Then $\phi \circ \rho^0$ is the differential of an analytic morphism $f: \hat{G} \longrightarrow H$. To prove that ϕ is a differential, it suffices to show that $\ker \rho \subset \ker f$. Let $\hat{A}$ be the connected component of $\rho^{-1}(A)$. Then ρ induces a covering morphism $\hat{\rho}: \hat{G}/\hat{A} \longrightarrow G/A$ of homogeneous spaces. Since G/A is simply connected, $\hat{\rho}$ must be a homeomorphism, so $\ker \rho \subset \hat{A}$. Since (5) holds, we have a morphism, α say, from A to H whose differential agrees with ϕ on $\mathcal{L}(A)$. Thus $\alpha^0 \circ \rho^0$ agrees with f^0 on $\mathcal{L}(\hat{A})$. But $\hat{A}$ is connected. Hence $\alpha \circ \rho$ agrees with f on $\hat{A}$. Thus $\ker \rho \subset \ker f$ because $\ker \rho \subset \hat{A}$. Consequently, f induces a morphism from G to H whose differential is ϕ, so (5) implies (1), and the proof is complete.

Similarly, using Theorem 1.7, we can prove the following:

THEOREM 2.3. *Let $\phi: \mathcal{L}(G) \longrightarrow \mathcal{L}(H)$ be as in Theorem 2.2. Let G_0 and H_0 be the universal algebraic subgroups of G and H. Then the following are equivalent.*

(1) *ϕ is the differential up to finite coverings of G, i.e., there exists a finite analytic covering $f: A \longrightarrow G$, and a morphism $\rho: A \longrightarrow H$ of analytic groups such that $\rho^0 = \phi \circ f^0$.*

(2) *$\phi\big(\mathcal{L}(G_0)\big) \subset \mathcal{L}(H_0)$ and restriction of ϕ to $\mathcal{L}(G_0)$ is the differential up to (algebraic) coverings of G_0.*

(3) *ϕ maps every rational semisimple element of $\mathcal{L}(G_0)$ into a rational semisimple element of $\mathcal{L}(H_0)$.*

(4) *For some maximal (complex) torus D of the radical G, ϕ maps every rational (semisimple) element of $\mathcal{L}(D)$ into a rational semisimple element of $\mathcal{L}(H_0)$.*

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