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SPLINE FUNCTIONS ON THE CIRCLE: CARDINAL L-SPLINES REVISITED

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1. Introduction.

Although the literature on splines has grown vastly during the last decade [**11**], the study of polynomial splines on the circle seems to have suffered neglect. The first to study the subject in depth seem to be Ahlberg, Nilson and Walsh [**1**]. Almost at the same time I. J. Schoenberg [**8**] studied the problem of interpolation at the roots of unity by splines and its relation to quadrature on the circle. For discrete polynomial splines on the circle we refer to [**5**]. M. Golomb [**3**] also considers interpolation by a class of "spline" functions in the complex plane but his point of view is based on minimum norm properties of spline functions. Perhaps the reason for this neglect may be attributed to the fact that one can pass from the circle to the line by means of the transformation $z \rightarrow \exp 2\pi ix$. This changes the problem on the circle into periodic interpolation on the line with the difference that instead of interpolation by piecewise polynomial, we now consider piecewise exponential polynomials with complex exponents.

Recently J. Tzimbalario [**13**] has brought out the close affinity of the problem of cardinal trigonometric interpolation and that of interpolating at the roots of unity on the circle and has given a unified treatment using the ideas on cardinal L-splines in [**6**].

The object of this note is to bring this affinity into better perspective by considering piecewise functions of the form $\sum_0^n c_\nu z^{\lambda_\nu}$ which is suggestive of the Müntz approximation theorem (D. J. Newman [**7**]). Such polynomials have been termed "incomplete" polynomials by G. G. Lorentz. In this case we are led to investigate cardinal L-splines with complex exponents occuring in conjugate pairs. Our result in this context extends some of the results of [**6**] which treats L-splines with real exponents.

The possibility of such an extension was first envisaged by Schoenberg ([**9**] p. 274) and later investigated by Tzimbalario and Sharma [**12**] for the case of trigonometric splines. Our treatment is based upon ideas from [**6**] and [**9**]. It does not require, as in [**13**], a lemma of Ahlberg, Nilson and Walsh concerning the sign of $\operatorname{Re} p(z)$ on the locus $\operatorname{Im} p(z) = 0$ for a polynomial with real negative zeros. (See [**13**] for its precise formulation.)

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In Section 2 we formulate the interpolation problem on the circle and state our main result. In Section 3 we provide a necessary and sufficient condition for the solution of the interpolation problem. Section 4 is devoted to some formulae from cardinal L-splines and their relationship to interpolation on the circle. Here we formulate in two different ways the uniqueness criterion of Section 3. Section 5 deals briefly with the derivation of the B-spline representation. In Section 6 we obtain results about the zeros of cardinal L-spline with complex exponents thereby extending some of the results of [**6**]. Section 7 deals with a proof of Theorem 1 of Section 2. We turn to an application of these results to quadrature in Section 8 where we also determine the generalized monospline of least L_p-norm.

2. Statement of the problem and the main result. Let U denote the unit circle $|z| = 1$ and let n, k be positive integers $k > n \geqq 1$. Let $\Lambda = \{\lambda_0 < \lambda_1 < \ldots < \lambda_n\}$ be a set of $n + 1$ integers which is a subset of $\{0, 1, \ldots, k - 1\}$. We shall denote by π_Λ the class of polynomials $P(z)$ given by

$$(2.1)\qquad P(z) = \sum_0^n c_\nu z^{\lambda_\nu}.$$

Let $\omega = e^{i\alpha}$ $(\alpha = 2\pi/k)$ be the kth root of unity. We shall consider the class $S_{\Lambda,n}{}^k(=\mathscr{S})$ of Λ-splines $S(z)$ defined by the following two conditions:

(i) $S(z) \in C^{n-1}(U)$

(ii) $S(z)|_{A_\nu} = P_\nu(z) \in \pi_\Lambda$, $\nu = 0, 1, \ldots, k - 1$

where A_ν denotes the arc of the circle $(\omega^\nu, \omega^{\nu+1})$. We can now formulate the

Problem. Let $\psi = e^{i\alpha\tau}$, $0 \leqq \tau < 1$ be a point on the circle. For what choice of k, Λ, τ does the interpolation problem I.P.:

$$(2.2)\qquad S(\psi\omega^j) = w_j,\ j = 0, 1, \ldots, k - 1$$

have a unique solution $S \in \mathscr{S}_{\Lambda,n}{}^k$?

For a history of the problem and for an elegant solution thereof when $\Lambda = \{0, 1, \ldots, n - 1\}$ we refer to [**8**]. We are able to solve the problem only when $\lambda_j + \lambda_{n-j}$ is independent of j. We shall prove

THEOREM 1. *Let $0 \leqq \lambda_0 < \lambda_1 \ldots < \lambda_n \leqq k - 1$ be integers such that $\lambda_j + \lambda_{n+j}$ is independent of j $(j = 0, 1, \ldots, n)$ then the* I.P *has a unique solution except when $\lambda_0 + \lambda_n + k$ is even and either of the following two cases holds:*

I. $\tau = 0$ *and n is an even integer,*

II. $\tau = \frac{1}{2}$ *and n is an odd integer.*

We get more information from

THEOREM 2. *In each of the cases of non-uniqueness in Theorem* 1, *the* I.P *admits a solution* $S(z) \in \mathscr{S}$ *if and only if* w_j *satisfy the relation*

$$(2.3) \qquad \sum_{j=0}^{k-1} \omega^{-hj} w_j = 0, \quad h = (k + \lambda_0 + \lambda_n)/2.$$

3. A uniqueness criterion for I.P. We begin with

LEMMA 1. *For every integer* r, $0 \leqq r \leqq k - 1$ *there exists a non-trivial solution* $S_r(z) \in \mathscr{S}$ (*unique up to a multiplicative constant*) *which satisfies*

$$(3.1) \qquad S_r(\omega z) = \omega^r S_r(z),\ z \in U.$$

The functions $S_0, \ldots, S_{k-1}$ *form a basis for* $\mathscr{S}$.

Proof. It is easily seen that the conditions

$$(3.2) \qquad \omega^j P^{(j)}(\omega) = \omega^r P^{(j)}(1),\ j = 0, 1, \ldots, n - 1$$

determine up to a constant a unique $P \in \pi_\Lambda$. When $r \notin \Lambda$, we may choose the normalization

$$(3.3) \qquad \omega^n P^{(n)}(\omega) - \omega^r P^{(n)}(1) = 1$$

and for $r \in \Lambda$, $P(z) = z^r$ satisfies (3.2). Clearly P may be extended from the arc $(1, \omega)$ to the unit circle by means of (3.1) and the resulting extension $S_r(z)$ is in $\mathscr{S}$. The linear independence of $S_0, \ldots S_{k-1}$ follows directly from (3.1).

We now show that these functions span $\mathscr{S}$.

To this end, let S be any element of $\mathscr{S}$. Then

$$S = \sum_{\nu=0}^{k-1} \tilde{S}_\nu, \quad \tilde{S}_\nu(z) = \frac{1}{k} \sum_{l=0}^{k-1} \omega^{-\nu l} S(\omega^l z).$$

Since $\tilde{S}_\nu(\omega z) = \omega^\nu \tilde{S}(z)$, the first half of Lemma 1 implies the existence of constants d_ν such that $\tilde{S}_\nu(z) = d_\nu S_\nu(z)$. This proves that $S_0, S_1, \ldots, S_{k+1}$ form a basis for $\mathscr{S}$.

LEMMA 2. *The* I.P *has a unique solution in* $\mathscr{S}$ *if and only if* $S_\nu(\psi) \neq 0$, $\nu \notin \Lambda$.

Proof. Let $S = \sum_0^{k-1} d_\nu S_\nu$. Then $S(\psi\omega^j) = 0$, $j = 0, 1, \ldots, k - 1$ if and only if $d_\nu S_\nu(\psi) = 0$ for $\nu = 0, 1, \ldots, k - 1$. Since $S_\nu(\psi) = \psi^\nu \neq 0$ for $\nu \in \Lambda$, the lemma is proved.

Note that if $S_\nu(\psi) \neq 0$, then $S_\nu(z)/S_\nu(\psi)$ interpolates z^ν at $\psi\omega^j (j = 0, 1, \ldots, k - 1)$.

4. Properties of $A_n(x; \lambda|T)$, $T = \{t_0, \ldots t_n\}$. Cardinal L-splines are related to a given differential equation

$$(4.1) \quad p_{n+1}(D)y = \prod_0^n (D - t_\nu)y = 0, \quad D = \frac{d}{dx},$$

where T denotes the zero set of $p_{n+1}(x)$. On each interval $(\nu, \nu + 1)$, ν an integer, the L-splines are piecewise exponential polynomials, i.e., a solution to (4.1) but globally in $C^{n-1}(\mathbf{R})$. Let $e^T = \{e^t | t \in T\}$ and let $A_n(x; \lambda|T)$, $\lambda \notin e^T$ be the unique element of the form $\sum_0^n c_\nu e^{xt_\nu}$ which satisfies

$$(4.2) \quad A_n^{(\nu)}(1; \lambda|T) = \lambda A_n^{(\nu)}(0; \lambda|T) + \delta_{\nu n}, \nu = 0, 1, \ldots, n.$$

We shall write $A_n(x; \lambda)$ for brevity when there is no ambiguity. $A_n(x; \lambda)$ is a rational function in λ and is an exponential polynomial in x. We list below some of the known properties of $A_n(x; \lambda)$. (See [**6**] where they are given for T real, but these algebraic properties hold also for T complex):

(a) The extension of $A_n(x; \lambda)$ from $[0, 1]$ to $\mathbf{R}$ by the equation

$$(4.3) \quad A_n(x + 1; \lambda) = \lambda A_n(x; \lambda), x \in \mathbf{R}$$

is an L-spline.

(b) On $[0, 1]$,

$$(4.4) \quad A_n(x; \lambda) = [t_0, t_1, \ldots, t_n]\, e^{xt}/(e^t - \lambda)$$

where $[t_0, \ldots t_n]f(t)$ denotes the divided difference of f at the nodes $t_0, \ldots t_n$ with respect to the parameter t.

(c) If $\lambda = |\lambda|e^{iu}$, $-\pi < u \leqq \pi$, $\delta_l = \ln|\lambda| + i(u + 2\pi l)$, $l = 0, \pm 1, \pm 2, \ldots,$ then

$$(4.5) \quad A_n(x; \lambda) = \sum_{-\infty}^{\infty} \frac{e^{(x-1)\delta_l}}{p_{n+1}(\delta_l)} = \lambda^{x-1} \sum_{-\infty}^{\infty} \frac{e^{2\pi i l x}}{p_{n+1}(\delta_l)}.$$

(d) If the zeros of $P_{n+1}(x)$ are symmetric about the origin, i.e., $T = -T$, then

$$(4.6) \quad A_n(1 - x; \lambda|T) = (-1)^{n-1}\lambda^{-1}A_n(x; \lambda|T).$$

If we set

$$(4.7) \quad \Pi_n(\lambda; x|T) = \Pi_n(\lambda; x) = r(\lambda)A_n(x; \lambda|T), \quad r(\lambda) = \prod_0^n (e^{t_\nu} - \lambda)$$

then $\Pi_n(\lambda; x)$ is a polynomial of degree n in λ, if $x \in (0, 1)$ and of degree $n - 1$ if $x = 0$.

We now choose $T = i\alpha\Lambda$ where $\Lambda = \{\lambda_0, \ldots, \lambda_n\}$. Then for $\lambda \notin e^T$,

$$(4.8) \quad A_n(x; \lambda) = [i\alpha\lambda_0, \ldots, i\alpha\lambda_n]\, e^{xt}/(e^t - \lambda).$$

Choose $\lambda = e^{ir\alpha} = \omega^r$, $r \notin \Lambda$ so that in (4.5) $\delta_l = i(r\alpha + 2\pi l)$, and set

(4.9) $\tilde{A}_n(z; \omega^r | \Lambda) = A_n(x; \omega^r), z = e^{i\alpha x}$.

Then from (4.5) we have for $z \in A_0$

$$(4.10) \quad \tilde{A}_n(z; \omega^r | \Lambda) = \omega^{-r}(i\alpha)^{-n-1} \sum_{-\infty}^{\infty} \frac{z^{ks+r}}{\prod_{\nu=0}^{n} (ks + r - \lambda_\nu)}$$

and because of (4.3), $\tilde{A}_n(z; \omega^r | \Lambda) \in \mathscr{S}$ and satisfies

(4.11) $\tilde{A}_n(\omega z; \omega^r | \Lambda) = \omega^r \tilde{A}_n(z; \omega^r | \Lambda)$.

According to Lemma 1 we have

$$S_r(z) = \tilde{A}_n(z; \omega^r | \Lambda).$$

When $\Lambda = \{0, 1, \ldots, n - 1\}$, S_r has been called an *r-flower* by Schoenberg [**8**] in view of its rotational symmetry. In this case on using (4.8) with $\lambda = \omega^r$ and taking $r = n + 1$ we get except for a non-zero constant factor a simpler derivation of formula (6.17) in [**8**] which gives the polynomial component of an r-flower on the arc A_0.

Using the above formulae we now reformulate Lemma 2 in a more useful-form.

LEMMA 3. *The* I.P. *is solvable if and only if one of the following* (*equivalent*) *conditions is satisfied:*

(4.12) $\Pi_n(\omega_r; \tau | i\alpha\Lambda) \neq 0, r \notin \Lambda, \tau \in [0, 1)$

$$(4.13) \quad N_r \equiv \sum_{-\infty}^{\infty} \frac{\psi^{ks+r}}{\prod_{\nu=0}^{n} (ks + r - \lambda_\nu)} \neq 0, r \notin \Lambda, \psi = e^{i\alpha\tau}.$$

5. B-splines for $\mathscr{S}$. Condition (4.12) above is reminiscent of Schoenberg's approach in [**8**] for $\Lambda = \{0, 1, \ldots, n - 1\}$ and may be derived in a similar fashion from the B-spline representation for $S(z) \in \mathscr{S}$. To this end we define the polynomial $\varphi_\Lambda(z) \in \Pi_\Lambda$ by

$$\varphi_\Lambda(z) = \begin{vmatrix} z^{\lambda_0} & z^{\lambda_1} & \ldots & z^{\lambda_n} \\ 1 & 1 & \ldots & 1 \\ \lambda_0 & \lambda_1 & \ldots & \lambda_n \\ & \ldots & \ldots & \\ \lambda_0^{n-1} & \lambda_1^{n-1} & \ldots & \lambda_n^{n-1} \end{vmatrix} \div V[\lambda_0, \lambda_1, \ldots, \lambda_{n-1}]$$

where $V[\lambda_0, \ldots, \lambda_{n-1}]$ denotes the Vandermondian. $\varphi_\Lambda(z)$ has the property that the coefficient of z^{λ_n} is unity and that $\varphi_\Lambda^{(\nu)}(1) = 0, \nu = 0, 1, \ldots, n - 1$. With these polynomials we can easily form the B-splines $M_\Lambda(z)$ for the class $\mathscr{S}$. We can also show as in [**8**] that $M_\Lambda(z)$ has as its support $\bigcup_{i=0}^{n} A_i$ and that every $S(z) \in \mathscr{S}$ can be written uniquely in the form

$$S(z) = \sum_{0}^{k-1} a_\nu M_\Lambda(z\omega^{-\nu})$$

where on the arc A_j we have

$$M_\Lambda(z) = \sum_{0}^{j} d_\nu \varphi_\Lambda(z\omega^{-\nu})_+, \; j = 0, 1, \ldots, k$$

where the coefficients d_ν are given by the identity

$$\sum_{0}^{k} d_\nu x^\nu = \prod_{0}^{n} (x - \omega^{-\lambda_\nu}).$$

When $n = 1$, it is easy to see that $\Lambda = \{\lambda_0, \lambda_1\}$, $\varphi_\Lambda(z) = z^{\lambda_1} - z^{\lambda_0}$ and

$$d_0 + d_1 x + d_2 x^2 = c(x - \omega^{-\lambda_0})(x - \omega^{-\lambda_1}).$$

Also in this case

$$M_\Lambda(z) = \begin{cases} \omega^{-\lambda_1-\lambda_0}(z^{\lambda_1} - z^{\lambda_0}), & z \in A_0 \\ -z^{\lambda_1}\omega^{-2\lambda_1} + z^{\lambda_0}\omega^{-2\lambda_0}, & z \in A_1 \\ \quad 0 \text{ elsewhere.} \end{cases}$$

If we define convolution $f * g$ by

$$f * g(z) = \int_U f(z\zeta^{-1}) g(\zeta) \frac{d\zeta}{\zeta},$$

then we can see that $M_\Lambda(z) = M_{\{\lambda_0\}} * M_{\{\lambda_1\}} * \ldots * M_{\{\lambda_n\}}$ where $M_{\{\lambda_\nu\}} = z^{\lambda_\nu}$ on A_0 and 0 elsewhere.

The Fourier series for $M_\Lambda(z)$ is given by

$$(5.1) \qquad M_\Lambda(z) \sim \frac{1}{2\pi i} \sum_{-\infty}^{\infty} b_\mu z^\mu$$

where

$$(5.2) \qquad \begin{cases} b_\mu = \prod_{l=0}^{n} \dfrac{\omega^{\lambda_l - \mu} - 1}{\lambda_l - \mu}, \; \mu \notin \Lambda \\ b_{\lambda_\nu} = \dfrac{2\pi i}{k} \prod_{l \neq \nu} \dfrac{\omega^{\lambda_l - \lambda_\nu} - 1}{\lambda_l - \lambda_\nu}, \; \mu = \lambda_\nu \in \Lambda. \end{cases}$$

Since $\{M_\Lambda(\psi\omega^j)\}_0^{k-1}$ is a periodic sequence, we write

$$M_\Lambda(\psi\omega^j) = \sum_{\nu=0}^{k-1} \zeta_\nu{}^\Lambda \omega^{\nu j}, \; j = 0, 1, \ldots, k-1$$

so that

$$\zeta_\nu{}^\Lambda = \frac{1}{k} \sum_{j=0}^{k-1} M_\Lambda(\psi\omega^j)\omega^{-\nu j}, \; \nu = 0, 1, \ldots, k-1.$$

Using (5.1) we have

$$\zeta_\nu{}^\Lambda = K \sum_{s=-\infty}^{\infty} b_{\nu+ks}\psi^{ks}, \; K \text{ a non zero constant.}$$

For $\nu \in \Lambda$ say $\nu = \lambda_r$, we have from (5.2)

$$\zeta_{\lambda_r}{}^\Lambda = \psi^{\epsilon\lambda_r} b_{\lambda_r} \neq 0.$$

For $\nu \notin \Lambda$, we get from (5.2),

$$\zeta_\nu{}^\Lambda = K \prod_{l=0}^{n} (1 - \omega^{\lambda_l - \nu}) N_\nu$$

where

$$(5.3) \qquad N_\nu = \sum_{s=-\infty}^{\infty} \frac{\psi^{ks}}{\prod_{l=0}^{n} (\nu + ks - \lambda_l)}.$$

Following the reasoning in [**8**] we would be led to another derivation of (4.12) of Lemma 3.

A direct examination of (4.13) of Lemma 3 in the general case seems to be difficult. In the following section we prepare the ground for proving Theorem 1 by using (4.12).

6. Zeros of $\Pi_n(\lambda; x|T)$, $T = \{t_0, t_1, \ldots, t_n\}$. If T is real the properties of the zeros of $\Pi_n(\lambda; x)$ are known [**6**]. When T is complex but $p_{n+1}(x) = \Pi_0{}^n(x - t_\nu)$ is real, we can still prove

LEMMA 4. *If the polynomial* $p_{n+1}(x)$ *is real with* $|\text{Im } t_j| < \pi$, *then for any* $\lambda < 0$, *the exponential polynomial* $\Pi_n(\lambda; x)$ *has exactly one simple zero in* $[0, 1]$.

Proof. Observe that if $(D - \alpha)(D - \bar{\alpha})g = 0$ with $|\text{Im } \alpha| < \pi$, then $g(x)$ can have at most one zero in $[0, 1]$.

Assuming that $P_{n+1}(x)$ has a complex zero α, it follows from the fact that $|\text{Im } t_j| < \pi$ that

$$(6.1) \qquad p_{n+1}(D) = (D - \alpha)(D - \bar{\alpha})h(D)$$

with

$$(6.2) \qquad \begin{cases} h(D) = D_{n-1}D_{n-2} \ldots D_2 D_1 \\ D_j f = w_j D(w_j{}^{-1} f), \; w_j > 0, \; x \in [0, 1]. \end{cases}$$

The existence of a decomposition as in (6.2) follows from the identities $(D - t)y = e^{tx}D(e^{-tx}y)$ and $(D^2 + 1)y = (D - \tan x)(D + \tan x)y$. We may use one or the other according as $g(x)$ has a real zero t or a pair of conjugate zeros.

Now suppose to the contrary that $\Pi_n(\lambda; x)$ has more than one zero in $[0, 1)$ counting multiplicities. Then according to (4.3) and (4.7), $A_n(x; \lambda)$ has at least $2n$ zeros in $[0, n)$. Thus $g(x) \equiv h(D)A_n(x; \lambda)$ has at least $2n - (n - 1) = n + 1$ zeros in $[0, n)$, whence it follows that there is at least one interval $[j, j + 1), 0 \leqq j < n$ where $g(x)$ has at least two zeros. Since $h(D)$ is a differential operator with constant coefficient and $A_n(x; \lambda)$ satisfies (4.3), the function $g(x)$ has at least two zeros in $[0, 1)$. However (6.1) implies that $(D - \alpha)(D - \bar{\alpha})g = 0$ and our initial remark says that $g(x)$ can have at most one zero in $[0, 1)$. This contradiction completes the proof.

If $p_{n+1}(x)$ has only real zeros, we may proceed as before and use the fact that if $(D - \alpha_1)(D - \alpha_2)g = 0$ (α_1, α_2 real), then $g(x)$ has at most one zero in $(-\infty, \infty)$. This case is proved by Micchelli ([**6**] Corollary 2.2, p. 210).

THEOREM 3. *If the polynomial* $p_{n+1}(x) = \prod_0^n (x - t_j)$ *is real and if* $|\mathrm{Im}\, t_j| < \pi$ $(j = 0, 1, \ldots, n)$, *then for any fixed* $x \in [0, 1)$, *the polynomial* $\Pi_n(\lambda; x)$ *given by* (4.7) *has all its zeros real and negative.*

Moreover these zeros are simple and increasing functions of $x \in (0, 1)$, *i.e.*,

$$(6.3) \qquad \pi_n(\lambda_j(x); x) = 0, \ -\infty < \lambda_1(x) < \ldots < \lambda_n(x) < 0, \ \lambda_j'(x) > 0.$$

If $\mu_1 < \ldots < \mu_{n-1} < 0$ *are the zeros of* $\Pi_n(\lambda; 0)$ *then* $\lambda_1(0) = -\infty$, $\lambda_j(0) = \mu_{j-1}, j = 2, \ldots, n$ *and* $\lambda_n(1) = 0$, $\lambda_j(1) = \mu_j, j = 1, \ldots n - 1$.

Remark. If all the zeros t_j are real, Theorem 3 is proved in [**6**] p. 222. It is also shown there that when T is real, the zeros of $A_n(x; \lambda|T)$ interlace the zeros of $A_{n-1}(x; \lambda|T - \{t_n\})$. This fact leads to the useful information that each $\lambda_j(x)$ is a strictly decreasing function of each t_ν. For the special case $T = \{0, t, 2t, \ldots, nt\}$, t real the monotonicity of λ_j as a function of t has interesting ramifications. It is related to the problem of finding best bounds on the local mesh ratio of knots which will allow interpolation by bounded splines to bounded data on the real line ([**2**], [**6**]). No such results are known when T is complex with $T = \bar{T}$.

When T is real, another derivation of Theorem 3 is given by Schoenberg [**9**] which is based upon the total positivity properties of B-splines for cardinal L-splines and upon results regarding the generating function of Polya frequency functions. This method can be adapted to the case when T is complex and $T = \bar{T}$ as is shown below.

The proof of Theorem 3 follows the approach of Schoenberg [**9**] almost verbatim. The difference rests on Lemma 4.

Proof. We shall use the following representation for $\Pi_n(\lambda; x)$ ([**6**] p. 222).

$$(6.4) \qquad \Pi_n(\lambda; x) = (-1)^n \sum_{\nu=0}^{n} Q_{n+1}{}^*(\nu + 1 - x)\lambda^\nu$$

where $Q_{n+1}^*(x)$ is the forward B-spline with respect to the operator

$$(-1)^{n+1}p_{n+1}(-D) = \prod_0^n (D + t_j).$$

Since $p_{n+1}(-x)$ is also real and $|\text{Im}\,(-t_j)| < \pi$, $p_{n+1}(-D)$ has a Polya factorization. More precisely

$$(6.5) \qquad p_{n+1}(-D)f = w_nD(w_n^{-1}w_{n-1}D)\ldots(w_1^{-1}w_0D)(w_0^{-1}f), w_j > 0, \forall j.$$

From a theorem of Karlin ([**4**] p. 527, Theorem 4.1), the sequence $\{Q_{n+1}^*(\nu + 1 - x)\}_0^\infty$ is a Polya frequency sequence for all $x \in [0, 1)$. It follows from a theorem of Schoenberg (Theorem 5.3 p. 412 [**4**]) that $\Pi_n(\lambda; x)$ has all its roots real and negative.

Inequality (6.3) (the simplicity of these zeros) follows from Lemma 4 on repeating the reasoning in Schoenberg ([**9**] Lemma 2 and Theorem 3, p. 260).

Remark. The result of Theorem 2 can be succinctly summarized by saying that the map $x \to \lambda(x)$ where $\lambda(x) = \lambda_j(x)$, $x \in (j - 1, j]$, $j = 1, \ldots, n$ is a C^{n-1} map of $(0, n]$ onto $(-\infty, 0]$. The smoothness of this map follows because $A_n(x; \lambda)$ is a cardinal L-spline.

COROLLARY 1. *If $T = \bar{T}$ and if $|\text{Im}\,t_\nu| < \pi$, $\nu = 0, 1, \ldots, n$, then for any complex number a the roots of $A_n(x; \lambda|a + T)$ are $e^a\lambda_1(x), \ldots, e^a\lambda_n(x)$ where $\{\lambda_\nu(x)\}_1^n$ are the zeros of $A_n(x; \lambda|T)$.*

This corollary is an immediate consequence of Theorem 3 and follows easily on using the identity

$$[a + t_0, a + t_1, \ldots, a + t_n]f(t) = [t_0, \ldots, t_n]f(t + a)$$

and the representation (4.4) which yields

$$A_n(x; \lambda|a + T) = e^{a(x-1)}A_n(x; e^{-a}\lambda|T).$$

THEOREM 4. *For any set Λ of $n + 1$ real numbers $\lambda_0 \leqq \lambda_1 \leqq \ldots \leqq \lambda_n$ with $\lambda_n - \lambda_0 < 2\pi$ and with $\lambda_j + \lambda_{n-j}$ independent of j $(j = 0, 1, \ldots, n)$, the polynomial $\pi_n(\lambda; x|i\Lambda)$ has distinct roots on the ray*

$$(6.6) \qquad \{-\rho e^{i(\lambda_0+\lambda_n)/2}|\rho > 0\}.$$

Moreover $\Pi_n(\lambda; x|i\Lambda)$ vanishes for some $\lambda \in U$ if and only if $x = 0$ and n is even or $x = \frac{1}{2}$ and n is odd and in each of these cases $\Pi(\lambda; xi\Lambda)$ vanishes only for

$$(6.7) \qquad \lambda = e^{i(\pi+(\lambda_0+\lambda_n)/2)}.$$

Proof. The first part of the theorem follows from Corollary 1 and Theorem 3, since $i\Lambda = a + T$, where $a = i(\lambda_0 + \lambda_n)/2$ and

$$T = \{i(\lambda_0 - \lambda_n)/2, i(\lambda_1 - \lambda_{n-1})/2, \ldots, i(\lambda_n - \lambda_0)2\}.$$

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The second part follows from the observation that $T = \bar{T} = -T$ which implies, because of (4.3) and (4.6), that

$$A_n(\tfrac{1}{2}; -1|T) = 0 \text{ for } n \text{ odd}, \; A_n(0; -1|T) = 0 \text{ for } n \text{ even}.$$

COROLLARY 2. *Given any m real numbers $\{\mu_\nu\}_1^m$ with $\max_\nu|\mu_\nu| < \pi$ and data $\{y_\nu\}_{-\infty}^{\infty}$ of power growth γ, there is a unique cardinal L-spline $S(x)$ of power growth γ corresponding to the differential operator $D\prod_{j=1}^m (D^2 + \mu_j^2)$, satisfying*

$$S(\nu + \beta) = y_\nu, \; \nu = 0, \neq 1, \neq 2, \ldots .$$

provided $\beta \in (0, 1)$. When $\beta = 0$, this result fails.

Proof. According to [**6**], the cardinal L-spline interpolation problem above has a unique bounded solution if and only if

$$\Pi_{2m}(\lambda; \beta|i\Lambda) \neq 0, \; \Lambda = \{\pm\mu_j|j = 1, \ldots, m\} \cup \{0\},$$

for $\lambda \in U$. Since Λ satisfies the conditions of Theorem 4, the corollary is seen to follow immediately from it.

Example 1. In Theorem 4, set $\lambda_\nu = (l + \nu)\eta$, l real, $|\eta| < 2\pi/n$. Then $A_n(x; \lambda)$ has distinct zeros on the ray

$$\{-\zeta e^{+i(l+n/2)\eta}|\zeta > 0\}.$$

This case was proved by Tzimbalario [**13**].

Example 2. If we take $\lambda_\nu = \nu\eta$, with $n\eta < 2\pi$ in Theorem 4, we obtain from the representation (4.4) with $x = 0$ that

$$(6.8) \qquad A_n(0; \lambda) = [0, i\eta, \ldots, in\eta]\frac{1}{e^t - \lambda},$$

has $n - 1$ distinct zeros on the ray $\{-\zeta \exp(in\eta/2)|\zeta > 0\}$. The second part of Theorem 4 implies that when n is odd, (6.8) will not vanish for $\lambda = -1$, while for n even, (6.8) will vanish for $\lambda = -1$ only when $\eta = 0$. If we set $e^{i\eta} = q$, then for $\eta \neq 0$ an easy computation shows that $A_n(0; -1)$ becomes $R_n(q)$ (except for a non-zero factor) where

$$R_n(q) = \sum_{\nu=0}^{n} \binom{n}{\nu} (-1)^\nu \frac{1}{q^\nu + 1}.$$

Thus we get the result that $R_n(q)$ is not zero on the arc of the unit circle $|z| = 1$ given by $|\arg q| < 2\pi/n$ when n is odd unless $q \neq 1$. When n is even, $R_n(q)$ will vanish only when $\eta = 0$, i.e., $q = 1$. In [**6**] it was shown that when q is real, $R_n(q)$ has exactly $n - 1$ simple positive zeros occurring in reciprocal pairs. For the importance of $R_n(q)$ in spline interpolation see [**2**] and [**6**].

7. Proof of theorem 1. According to Lemma 3 and Theorem 4 (with Λ replaced by $\alpha\Lambda$), the I.P is solvable when n is even and $\tau \neq 0$ or when n is odd and $\tau \neq \frac{1}{2}$. In the exceptional cases, i.e., n even and $\tau = 0$ or n odd and $\tau = \frac{1}{2}$, the I.P is solvable if and only if

$$(7.1) \quad e^{i(\lambda_0+\lambda_n/2)\alpha} e^{i\pi} \neq \omega^\nu, \ (\nu = 0, 1, \ldots, k-1).$$

This implies that the only zero of $\Pi_n(\lambda; 0|i\alpha\Lambda)$ on the unit circle, viz. $\exp\,(i(\lambda_0 + \lambda_n)\alpha/2 + i\pi)$ is not equal to a root of unity. Since $\alpha = 2\pi/k$, $\omega = e^{i\alpha}$, it follows from (7.1) that

$$(7.2) \quad \lambda_0 + \lambda_n + k - 2\nu \not\equiv 0 \pmod{2k}, \ \nu = 0, 1, \ldots, k-1.$$

If $\lambda_0 + \lambda_n + k$ is odd, (7.2) is valid so that the I.P is solvable. This completes the proof of Theorem 1.

Proof of Theorem 2. If $\lambda_0 + \lambda_n + k$ is even, we shall show that (2.2) fails only for $\nu = (k + \lambda_0 + \lambda_n)/2 = h$ (say). Let us again verify what we saw earlier (see the proof of Lemma 2) that if $\nu = \lambda_r$, (7.2) is equivalent to

$$\lambda_{n-r} - \lambda_r + k \not\equiv 0 \pmod{2k}$$

since $\lambda_r + \lambda_{n+r} = \lambda_0 + \lambda_n$. This is obviously true since $|\lambda_{n-r} - \lambda_r| < k - 1$.

If $\lambda_r < \nu < \lambda_{r+1}$ for some r, then we have

$$\lambda_{n-r-1} - \lambda_{r+1} + k < \lambda_0 + \lambda_n + k - 2\nu < \lambda_{n-r} - \lambda_r + k.$$

Since λ_ν's are $<k$, we get

$$1 < \lambda_0 + \lambda_n + k - 2\nu < 2k - 1$$

so that (7.2) is true for $\lambda_r < \nu < \lambda_{r+1}$.

If $\nu > \lambda_n$, then

$$-k < \lambda_0 + \lambda_n - k + 2 < \lambda_0 + k - 2\nu < \lambda_0 - \lambda_n - 2 + k < k.$$

Thus $\lambda_0 + \lambda_n + k - 2\nu \equiv 0 \pmod{2k}$ if and only if $\lambda_0 + \lambda_n + k - 2\nu = 0$. Thus if $\lambda_0 + \lambda_n + k$ is even, there is exactly one ν for which inequality (7.2) fails. From Lemma 3 and the preceding discussion it follows that $S_j(\psi) = 0$ if and only if $j = h$.

If $S(z) \in \mathscr{S}$ satisfies (2.2), then by Lemma 1,

$$S(z) = \sum_0^{k-1} c_j S_j(z)$$

so that

$$\sum_{j=0}^{k-1} c_j S_j(\psi\omega^\nu) = \sum_{j=0}^{k-1} c_j \omega^{\nu j} S_j(\psi) = w_j, j = 0, 1, \ldots, k-1.$$

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Therefore

$$c_j S_j(\psi) = \sum_{\nu=0}^{k-1} \omega_\nu^{-\nu j}$$

from which Theorem 2 easily follows.

Remark. We do not know what happens if $\lambda_j + \lambda_{n-j}$ is not independent of j. In that case the above method does not seem to work.

8. Monosplines and quadrature formulae. Consider the quadrature formula

$$(8.1) \qquad \int_U f(z)dz = \sum_{j=0}^{k-1} c_j f(\omega^j) + Rf$$

where the unit circle U is described counter-clockwise. The requirement that the remainder Rf vanishes when $f = z^{\lambda_\nu}$, $\nu = 0, 1, \ldots, n$, is equivalent to the relations

$$(8.2) \qquad \int_U z^{\lambda_\nu} dz = 0 = \sum_{j=0}^{k-1} c_j \omega^{j\lambda_\nu}, \ \nu = 0, 1, \ldots, n.$$

We assume that $n + 1 < k$ in which case there are $k - n - 1$ free parameters in (8.1).

In order to bring out the connection of the quadrature formula (8.1) satisfying (8.2) with the class of splines $\mathscr{S}$, we introduce some differential operators. Set

$$\mathscr{L}_\nu f = D(z^{-(\lambda_\nu - \lambda_{\nu-1} - 1)} f), \ \nu = 1, 2, \ldots, n$$
$$\mathscr{L}_0 f = D(z^{-\lambda_0} f)$$
$$\mathscr{L}_\nu^* f = z^{-(\lambda_\nu - \lambda_{\nu-1} - 1)} Df, \quad \mathscr{L}_0^* f = z^{-\lambda_0} Df, \ \nu = 1, 2, \ldots, n.$$

Then

$$\mathscr{L}_n \mathscr{L}_{n-1} \ldots \mathscr{L}_0 z^{\lambda_n + 1} = \prod_{j=0}^{n} (\lambda_n - \lambda_j + 1).$$

We shall prove

LEMMA 5. *If*

$$(8.3) \qquad K(z) = \frac{z^{\lambda_n + 1}}{\prod_0^n (\lambda_n - \lambda_\nu + 1)} - S(z), \ S(z) \in \mathscr{S},$$

and if $f(z) \in C^{n+1}(U)$, *then*

$$(8.4) \qquad \int_U f(z)dz = \sum_{j=0}^{k-1} C_j f(\omega^j) + (-1)^{n+1} \int_U K(z) \mathscr{L}_0^* \mathscr{L}_1^* \ldots \mathscr{L}_n^* f\, dz.$$

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Proof. It is easy to verify that

$$\int_U K(z)(\mathscr{L}_0^*\mathscr{L}_1^* \dots \mathscr{L}_n^* f)(z)dz$$
$$= (-1)^n \sum_{j=0}^{k-1} \int_{\omega^j}^{\omega^{j+1}} (\mathscr{L}_{n-1} \dots \mathscr{L}_0 K)(z) z^{-(\lambda_n - \lambda_{n-1} - 1)} f'(z)dz.$$

Since $H(z) \equiv z^{-(\lambda_n-\lambda_{n-1}-1)}(\mathscr{L}_{n-1} \dots \mathscr{L}_0 K)(z)$ has discontinuities at $z = \omega^j$ ($j = 0, 1, \dots, n - 1$), the coefficients C_j in (8.4) are given by $H(\omega_-{}^j) - H(\omega_+{}^j)$.

Formula (8.4) establishes a correspondence between the function $K(z)$ which we will call a Λ-monospline and quadrature formulae (8.1) satisfying (8.2).

A natural problem which now arises is to find a Λ-monospline which has least L_p-norm $1 \leqq p \leqq \infty$. We seek to minimize

$$(8.5) \qquad \|K\|_{L_p}{}^P = \int_0^{2\pi} |K(z)|^p |dz|$$

over all Λ-monosplines of order n. The quadrature formula which corresponds to a solution of (8.5) will be called an *optimal* quadrature formula.

We shall prove

THEOREM 5. *Let* $1 \leqq n + 1 < k$. *Among all quadrature formulae of the form* (8.1) *which are exact for* $f \in \Pi_\Lambda$, *an optimal quadrature formula is*

$$(8.6) \qquad \int_U f(z)dz = \frac{2\pi i}{k} \lambda_* \sum_{j=0}^{k-1} \omega^{(\lambda_n - n+1)j} f(\omega^j) + Rf$$

with

$$Rf = (-1)^{n+1} \int_U K_*(z)(\mathscr{L}_0^* \dots \mathscr{L}_n^* f(z))dz$$

where the kernel $K_*(z)$ *is given by*

$$K_*(z) = \frac{1}{\prod_0^n (r - \lambda_\nu)} \{z^r - \lambda_* S_r(z)\},\ r \equiv (\lambda_n + 1) \bmod k$$

and λ_* *minimizes the integral*

$$\int_0^\alpha |1 - \lambda z^{-r} S_r(z)|^p d\theta,\ z = e^{i\theta},$$

over all $\lambda \in \mathbf{C}$.

Proof. Let $K(z)$ be any Λ-monospline. Set

$$\tilde{K}(z) = \frac{1}{k}\sum_{l=0}^{k-1} \omega^{-lr} K(\omega^l z).$$

Then $\tilde{K}(z)$ is a Λ-monospline satisfying the functional equation

$$\tilde{K}(\omega z) = \omega^r \tilde{K}(z).$$

It is easy to see that $\|\tilde{K}\|_{L_p} \leqq \|K\|_{L_p}$ and thus in minimizing (8.5) it is sufficient to restrict ourselves to Λ-monosplines which are r-flowers.

According to Lemma 1, we can write

$$\tilde{K}(z) = \frac{1}{\prod_0^n (r - \lambda_\nu)} \{z^r - \lambda S_r(z)\}, \lambda \in \mathbf{C}.$$

Using this equation the theorem easily follows.

Remark. When $p = 2$ it is a straightforward matter to compute λ_*. It should also be possible to find λ_* when $p = 1, \infty$. In the case $\lambda_\nu = \nu$ considered by Schoenberg [**8**], Theorem 5 leads to a solution of a question raised at the end of his paper.

9. Conclusion. It is of some interest to point out that Theorem 1 is only a sufficient condition for the I.P to be uniquely solvable in the class $\mathcal{S}$. It can be easily proved that if k is even, $\psi = 1$, $\lambda_0 = 0$, $\lambda_1 = 1$, $\lambda_2 = 4$, then the number N_r of Lemma 3 given by

$$N_r = \sum_{s=-\infty}^{\infty} \frac{1}{(ks + r)(ks + r - 1)(ks + r - 4)},$$

is not equal to zero. This is easily seen on rewriting

$$N_r = \sum_{s=0}^{\infty} \frac{1}{(ks + r)(ks + r - 1)(ks + r - 4)} - \sum_{s=0}^{\infty} \frac{1}{(ks + k - r + 4)(ks + k - r + 1)(ks + k - r)} = I_1 - I_2.$$

A comparison of the terms in I_1 and I_2 shows that for $2r \leqq k + 2$, $I_1 > I_2$ each term in I_1 being greater than the corresponding term in I_2. Similarly for $2r \geqq k + 2$, $I_1 < I_2$. Thus the only case to be checked is $2r = k + 3$ which cannot occur since k is even.

Thus although $\lambda_0 + \lambda_2 \neq 2\lambda_1$, the criterion (4.13) of Lemma 3 applies and the I.P is uniquely solvable. It would be interesting to obtain a proof of Theorem 1 independent of the theory of Cardinal L-splines and to see if the conditions on Λ can be further relaxed.

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