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REES MATRIX COVERS FOR A CLASS OF SEMIGROUPS WITH LOCALLY COMMUTING IDEMPOTENTS

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Abstract McAlister proved that every regular locally inverse semigroup can be covered by a regular Rees matrix semigroup over an inverse semigroup by means of a homomorphism which is locally an isomorphism. We generalize this result to the class of semigroups with local units whose local submonoids have commuting idempotents and possessing what we term a 'McAlister sandwich function'.

Keywords: Rees matrix covers; locally inverse semigroups; semigroups with local units; semigroups with commuting idempotents

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1. Introduction

This is the first of two papers in which we generalize McAlister's theory of locally inverse regular semigroups, developed in [**7**] and [**8**], to a class of non-regular semigroups.

Recall that a locally inverse regular semigroup is a regular semigroup S in which each local submonoid eSe, where e is an idempotent, is inverse or, equivalently, in which the idempotents in each local submonoid commute. In this paper, we replace regular semigroups by semigroups S having *local units*: this means that for each $s \in S$ there exist idempotents $e, f \in S$ such that $es = s = sf$. Thus our aim is to generalize McAlister's results to semigroups S with local units which have 'locally commuting idempotents', in the sense that the idempotents in each submonoid commute.

In this paper, we concentrate on generalizing the results McAlister obtained in [**7**], where locally inverse regular semigroups are described in the following way. To begin with, he showed (in [**6**]) that the regular elements of a Rees matrix semigroup over an inverse semigroup form a locally inverse regular semigroup; he called these semigroups 'regular Rees matrix semigroups over inverse semigroups'. Then he proved that every locally inverse regular semigroup is a locally isomorphic image of a regular Rees matrix semigroup over an inverse semigroup. Recall that a *local isomorphism* is a homomorphism $\theta\colon S \to T$, which is injective when restricted to local submonoids.

To achieve our generalization, we have to assume that our semigroups are equipped with what we term a 'McAlister sandwich function', something which is automatic in

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the regular case. In our second paper [**3**], we completely characterize semigroups with local units having locally commuting idempotents which are endowed with a McAlister sandwich function.

In the non-regular case, we have to be careful about what we mean by a 'local isomorphism'. In Lemma 1.3 of [**7**], McAlister proves that in the regular case local isomorphisms are in fact injective on every subset of the form aSb, where $a, b \in S$. For semigroups which are not necessarily regular, the concept of a *strict local isomorphism* was devised in [**9**]; this is a function $\theta \colon S \to T$, which is injective on every subset of the form aSb, where a and b are any elements such that $a \in Sa$ and $b \in bS$. Evidently, local isomorphisms between regular semigroups are equivalent to strict local isomorphisms. It is easy to check that a strict local isomorphism between semigroups with local units is the same thing as a homomorphism which is injective on all subsets of the form eSf, where e and f are idempotents: this is the form of the definition of 'strict local isomorphism' that we shall use in this paper.

2. Properties of regular elements

Let S be an arbitrary semigroup. The set $\mathrm{Reg}(S)$ of regular elements of S will play an important role in our work, although it need not be a subsemigroup. If $A \subseteq S$ then $E(A)$ will denote the set of idempotents in A. If $s \in S$ then $V(s)$ denotes the set of all inverses of s. Let $e, f \in E(S)$. Then the *sandwich set* $S(e, f)$ is defined to be the set $fV(ef)e$. Thus the sandwich set is non-empty precisely when ef is regular. It is easy to check (or see Nambooripad [**10**]) that if $S(e, f)$ is non-empty then

$$h \in S(e, f) \Leftrightarrow h^2 = h, \quad fhe = h \quad \text{and} \quad ehf = ef.$$

Nambooripad [**11**] showed that a natural partial order could be defined on any regular semigroup; independently, Hartwig [**1**] showed that the regular elements of any semigroup could be naturally ordered. We use Nambooripad's form of the definition, but follow Hartwig in applying it to the regular elements of any semigroup. Specifically, let S be an arbitrary semigroup. A relation $\leqslant$ is defined on the set $\mathrm{Reg}(S)$ as follows. Let $s, t \in S$. Then $s \leqslant t$ if and only if $R_s \leqslant R_t$ and $s = ft$ for some $f \in E(R_s)$. We include the proof of the following result for completeness.

Proposition 2.1. *Let S be an arbitrary semigroup. Then the relation $\leqslant$ is a partial order on the set of regular elements of S.*

Proof. Let $s \in \mathrm{Reg}(S)$. Then, by assumption, there exists $s' \in V(s)$. Thus $s = (ss')s$. Hence $R_s = R_s$, $s = (ss')s$ and $ss' \in E(R_s)$. It follows that $s \leqslant s$, and so $\leqslant$ is reflexive.

Suppose that $s \leqslant t$ and $t \leqslant s$. Then $R_s = R_t$ and there exist idempotents $e, f \in E(R_s) = E(R_t)$ such that $s = ft$ and $t = es$. But $f \in R_s = R_t$ implies that $ft = t$. Thus $s = t$, and $\leqslant$ is antisymmetric.

Finally, suppose that $s \leqslant t$ and $t \leqslant v$. Then $R_s \leqslant R_t$ and $R_t \leqslant R_v$ and there are idempotents $f \in E(R_s)$ and $e \in E(R_t)$ such that $s = ft$ and $t = ev$. Clearly, $R_s \leqslant R_v$ and $s = (fe)v$. But $R_f \leqslant R_e$ and so $fe \in E(R_s)$. Hence $s \leqslant v$, and $\leqslant$ is transitive. □

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The relation $\leqslant$ is called the *Hartwig–Nambooripad order* [**1**, **10**] or the *natural partial order* defined on the regular elements. Observe that if e and f are idempotents then $e \leqslant f$ precisely when $e = ef = fe$, which is the usual order on the idempotents of a semigroup. There are a number of alternative ways of characterizing this order; the proofs of the following can all be deduced from [**11**]. Again, we include proofs for the sake of completeness.

Proposition 2.2. *Let S be a semigroup and let $s, t \in \operatorname{Reg}(S)$. Then the following are equivalent.*

(i) $s \leqslant t$.

(ii) *For each $f \in E(R_t)$ there exists $e \in E(R_s)$ such that $e \leqslant f$ and $s = et$.*

(iii) *For each $f' \in E(L_t)$ there exists $e' \in E(L_s)$ such that $e' \leqslant f'$ and $s = te'$.*

(iv) *There exist idempotents e and f such that $s = et = tf$.*

Proof. (i) $\Rightarrow$ (ii). Let $s \leqslant t$. Then by definition, $R_s \leqslant R_t$ and $s = it$ for some $i \in E(R_s)$. Let $f \in E(R_t)$. Then $R_i = R_s \leqslant R_t = R_f$, and so $R_i \leqslant R_f$. In particular, $fi = i$. Put $e = if$, then it is easy to check that $e^2 = e$, $e \leqslant f$ and $i \mathcal{R} e$. It follows that $e \in E(R_s)$. Finally, $et = ift = it = s$.

(ii) $\Rightarrow$ (iii). Let $f' \in E(L_t)$. By Theorem 2.3.4(2) of [**2**], choose $t' \in V(t) \cap R_{f'}$. Then $t't = f'$ and $tt' \in E(R_t)$. Thus, by (ii) there exists $e \in E(R_s)$ such that $e \leqslant tt'$ and $s = et$. Put $e' = t'et$, then $s = et = t(t'et) = te'$. It is easy to check that $e' \leqslant f'$. Also, $te' = tt'et = et = s$ and $t's = t'et = e'$, so that $s \mathcal{L} e'$. Hence the result.

(iii) $\Rightarrow$ (iv). Let $f' \in E(L_t)$, $e' \in E(L_s)$, where $e' \leqslant f'$ and $s = te'$. Since $f' \in E(L_t)$ there exists $t' \in V(t)$ such that $f' = t't$. Thus $s = te' = te'f' = (te't')t$. But

$$(te't')^2 = te'(t't)e't' = te'f'e't' = te't',$$

and so $te't'$ is an idempotent.

(iv) $\Rightarrow$ (i). Let $s = et = tf$ where e and f are idempotents. From $s = tf$ we have that $R_s \leqslant R_t$. Let $s' \in V(s)$. From $s = et$ we obtain $es = s$ and so $ess' = ss'$. Put $i = ss'e$. Then it is easy to check that $i^2 = i$, $s = it$ and $i \mathcal{R} s$. □

We shall now derive some properties of the natural partial order on semigroups where the idempotents in every local submonoid commute.

Proposition 2.3. *Let S be a semigroup in which the idempotents in every local submonoid commute.*

(i) $|S(e, f)| \leqslant 1$ *for all $e, f \in E(S)$.*

(ii) *If $x, y, u, v \in \operatorname{Reg}(S)$ and $x \leqslant u$, $y \leqslant v$ and $xy, uv \in \operatorname{Reg}(S)$, then $xy \leqslant uv$.*

(iii) *If $x, y \in \operatorname{Reg}(S)$ and e is an idempotent such that $xe = x$ and $ey = y$, then xy is regular.*

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Proof. (i) Let $h, k \in S(e, f)$. We show that $h = k$. We have that

$$fhe = h, \quad ehf = ef \qquad \text{and} \qquad fke = k, \quad ekf = ef.$$

It is easy to check that eh, ek, hf and kf are all idempotents. Furthermore,

$$eh, ek \in E(eSe) \qquad \text{and} \qquad hf, kf \in E(fSf).$$

Thus $ehek = ekeh$ and $hfkf = kfhf$, since the idempotents in every local submonoid commute. Hence $ehk = ekh$ and $hkf = khf$. But

$$ehk = ehfke = efke = ek.$$

By the same token, $ekh = eh$. Thus $ek = eh$. Similarly, $hf = kf$. Now

$$k = fke = fkek = fkeh = kh$$

and

$$h = fhe = hfhe = kfhe = kh.$$

Thus $k = h$.

(ii) Let $u' \in V(u)$ and $v' \in V(v)$. By Proposition 2.2 (ii) and (iii), there exist idempotents e and f such that

$$e \leqslant u'u, \quad e \,\mathcal{L}\, x, \quad x = ue \qquad \text{and} \qquad f \leqslant vv', \quad f \,\mathcal{R}\, y, \quad y = fv.$$

Thus $xy = uefv$. By assumption, xy is regular. But $ey \,\mathcal{L}\, xy$ and so ey is regular, and $ey \,\mathcal{R}\, ef$ and so ef is regular. Hence $S(e, f)$ is non-empty. Let $h \in S(e, f)$. Then $fhe = h$ and $ehf = ef$, and so $xy = uefv = uehfv = u(eh)(hf)v$. Observe that $he = h$ and $hu'u = h$ and so

$$u'uh \,\mathcal{L}\, h, \quad u'uh \leqslant u'u \qquad \text{and} \qquad eh \,\mathcal{L}\, h, \quad eh \leqslant e \leqslant u'u.$$

Thus $u'uh \,\mathcal{L}\, eh$ and $u'uh, eh \leqslant u'u$. But $E(u'uSu'u)$ is a commutative semigroup, and so $u'uh = eh$. Similarly, $hvv' = hf$. Hence

$$xy = u(eh)(hf)v = u(u'uh)(hvv')v = uhv.$$

Now

$$hv = (hvv')(vv')v = (vv')(hvv')v = v(v'hv).$$

Thus

$$xy = uhv = uv(v'hv)$$

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where $v'hv$ is an idempotent. Similarly,

$$xy = uhv = (uhu')uv,$$

where uhu' is an idempotent. Hence $xy \leqslant uv$ by Proposition 2.2 (iv).

(iii) Let $x' \in V(x)$. Then $x'xe = x'x$. Thus $ex'x$ is an idempotent and $x'x \,\mathcal{L}\, ex'x \leqslant e$. Hence $x \,\mathcal{L}\, ex'x$. By standard regular semigroup theory there is $x'' \in V(x)$ such that $x''x = ex'x$. Thus we have proved that if $xe = x$, then there is $x' \in V(x)$ such that $x'x \leqslant e$. Similarly, if $ey = y$, then there exists $y' \in V(y)$ such that $yy' \leqslant e$. With these choices of inverses we calculate

$$xy(y'x')xy = x(yy')(x'x)y = x(x'x)(yy')y = xy,$$

since $x'x, yy' \leqslant e$ and so they commute. Thus xy is regular. □

Property (ii) above generalizes a theorem of Nambooripad [**11**] which states that a regular semigroup is locally inverse if and only if the natural partial order is compatible with the multiplication. Property (iii) above will be used repeatedly in what follows to show that certain products of regular elements are again regular.

The following lemma, and its left–right dual, will be needed in § 4.

Lemma 2.4. *Let $x, y \in \mathrm{Reg}(S)$ such that $x \leqslant y$ and $ey = y$ for some idempotent e. Then there exists an idempotent $f \leqslant e$ such that $x = fy$.*

Proof. Let $y' \in V(y)$. Then $eyy' = yy'$. It is easy to check that $yy'e \in E(S)$, $yy'e \leqslant e$ and $yy'e \,\mathcal{R}\, y$. Thus, by Proposition 2.2 (ii), there exists an idempotent $f \leqslant yy'e$ such that $x = fy$. Clearly, $f \leqslant e$. □

3. An associated semigroup

Let S be a semigroup with local units with locally commuting idempotents. We may associate a category with S as follows. Put

$$C(S) = \{(e, x, f) \in E(S) \times S \times E(S) \colon exf = x\}$$

with product given by $(e, x, f)(f, y, j) = (e, xy, j)$ and undefined in all other cases.

Our aim is to convert $C(S)$ into a semigroup with local units and with a normal band of idempotents. To do this, we need to introduce a major assumption on the structure of the semigroup S.

A function $p \colon E(S) \times E(S) \to S$, where we write $p_{u,v} = p(u, v)$, is called a *McAlister sandwich function* if it satisfies the following three conditions.

(M1) $p_{u,v} \in uSv$ and $p_{u,u} = u$.

(M2) $p_{u,v} \in V(p_{v,u})$.

(M3) $p_{u,v}p_{v,f} \leqslant p_{u,f}$.

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To see that condition (M3) makes sense, we have to show that the product $p_{u,v}p_{v,f}$ is always regular. To this end, observe that by condition (M2), both $p_{u,v}$ and $p_{v,f}$ are regular; by condition (M1), we have that $p_{u,v}v = p_{u,v}$ and $vp_{v,f} = p_{v,f}$; thus the regularity of $p_{u,v}p_{v,f}$ follows from Proposition 2.3 (iii). By induction, the above argument implies that any product of the form

$$p_{a,b}p_{b,c}p_{c,d}\cdots$$

is regular.

All regular locally inverse semigroups have McAlister sandwich functions by Lemma 2.2 of [**7**].

Proposition 3.1. *Let S be a semigroup with local units with locally commuting idempotents equipped with a McAlister sandwich function. Define a semigroup multiplication on $C(S)$ by*

$$(e,x,f)\cdot(i,y,j) = (e, xp_{f,i}y, j).$$

Then the idempotents of $(C(S),\cdot)$ form a normal band.

Proof. It is evident that $(C(S),\cdot)$ is a semigroup. We begin by locating the idempotents. Observe that $(e,x,f)^2 = (e,x,f)$ if and only if $xp_{f,e}x = x$. Thus, in particular, x is regular.

Suppose that (e,x,f) is an idempotent. By condition (M2), $p_{f,e}p_{e,f}p_{f,e} = p_{f,e}$. Thus $x = xp_{f,e}p_{e,f}p_{f,e}x$. Now $xp_{f,e}, p_{f,e}x \in E(S)$, and $xp_{f,e} \leqslant e$ and $p_{f,e}x \leqslant f$ using condition (M1). The product $xp_{f,e}p_{e,f}$ is regular by Proposition 2.3 (iii) using condition (M1). Thus, by Proposition 2.3 (ii), we have that $xp_{f,e}p_{e,f} \leqslant ep_{e,f} = p_{e,f}$. By Proposition 2.3 (iii) and condition (M1) the product $xp_{f,e}p_{e,f}p_{f,e}x$ is regular and so $xp_{f,e}p_{e,f}p_{f,e}x \leqslant p_{e,f}f \leqslant p_{e,f}$ by Proposition 2.3 (ii). Hence $x \leqslant p_{e,f}$.

Conversely, suppose that x is regular and $x \leqslant p_{e,f}$. Then $x = f'p_{e,f} = p_{e,f}e'$ for some idempotents $e', f' \in S$ by Proposition 2.2 (iv). Thus

$$xp_{f,e}x = f'p_{e,f}p_{f,e}p_{e,f}e' = f'p_{e,f}e' = x.$$

We have therefore proved that

$$E(C(S),\cdot) = \{(e,x,f) \in C(S) \colon x \in \mathrm{Reg}(S) \text{ and } x \leqslant p_{e,f}\}.$$

We now show that the idempotents form a band. Let (e,x,f) and (k,y,l) be idempotents. Then by the result above

$$x \leqslant p_{e,f} \quad \text{and} \quad y \leqslant p_{k,l}.$$

By definition $(e,x,f)\cdot(k,y,l) = (e, xp_{f,k}y, l)$. By Proposition 2.3 (iii) and condition (M1), the product $xp_{f,k}$ is regular, as is $p_{e,f}p_{f,k}$. Thus, by Proposition 2.3 (ii), $xp_{f,k} \leqslant p_{e,f}p_{f,k}$. Similarly, $xp_{f,k}y$ and $p_{e,f}p_{f,k}p_{k,l}$ are both regular, and so by Proposition 2.3 (ii) $xp_{f,k}y \leqslant p_{e,f}p_{f,k}p_{k,l}$. But by two applications of condition (M3), we have that $p_{e,f}p_{f,k}p_{k,l} \leqslant p_{e,l}$. Hence $(e, xp_{f,k}y, l)$ is an idempotent.

Finally, to show that the band is normal, we check that the idempotents in the local submonoids commute (using the fact that a band is normal precisely when it is locally inverse; see [**2**, p. 141, Exercise 18]). Let (e, z, f) be an idempotent and let $(e, x, f), (e, y, f) \leqslant (e, z, f)$ be idempotents. We prove that $(e, x, f) \cdot (e, y, f) = (e, y, f) \cdot (e, x, f)$. By definition,

$$(e, x, f) \cdot (e, y, f) = (e, xp_{f,e}y, f) \quad \text{and} \quad (e, y, f) \cdot (e, x, f) = (e, yp_{f,e}x, f).$$

Now $y = yp_{f,e}z$ and so

$$xp_{f,e}y = xp_{f,e}yp_{f,e}z.$$

But $xp_{f,e}$ and $yp_{f,e}$ are idempotents, and using condition (M1) we have $xp_{f,e}, yp_{f,e} \leqslant e$. Thus $xp_{f,e}y = yp_{f,e}xp_{f,e}z$. But $x = xp_{f,e}z$ thus $xp_{f,e}y = yp_{f,e}x$. Hence $(e, x, f)(e, y, f) = (e, y, f)(e, x, f)$. □

We shall denote the semigroup $(C(S), \cdot)$ by $C(S)^{\bullet}$.

4. A semigroup with commuting idempotents

The semigroup $C(S)^{\bullet}$ has a normal band of idempotents. In this section, we shall show that we can define a congruence δ on this semigroup in such a way that $C(S)^{\bullet}/\delta$ has commuting idempotents; in addition, the natural homomorphism $\delta^{\natural}$ will be a strict local isomorphism.

In the case of regular semigroups, this result follows from the existence of a minimum inverse congruence on any regular orthodox semigroup, and Proposition 1.4 of [**7**]. Recall that on an orthodox regular semigroup T the minimum inverse congruence γ on T can be defined by

$$(a, b) \in \gamma \Leftrightarrow V(a) \cap V(b) \neq \emptyset.$$

As a first step, we characterize γ on the semigroup $C(S)^{\bullet}$ in the case where S is regular (and therefore locally inverse).

Proposition 4.1. *Let S be a regular locally inverse semigroup, and let (e, x, f), $(i, y, j) \in C(S)^{\bullet}$. Then $(e, x, f)\, \gamma\, (i, y, j)$ if and only if $x = p_{e,i}yp_{j,f}$ and $y = p_{i,e}xp_{f,j}$.*

Proof. Suppose first that $(e, x, f)\, \gamma\, (i, y, j)$. Let $(a, b, c) \in V(e, x, f) \cap V(i, y, j)$. Then

$$x = xp_{f,a}bp_{c,e}x \quad \text{and} \quad y = yp_{j,a}bp_{c,i}y$$

and

$$b = bp_{c,e}xp_{f,a}b \quad \text{and} \quad b = bp_{c,i}yp_{j,a}b.$$

Also, because an element multiplied by an inverse is an idempotent, and using the characterization of idempotents in Proposition 3.1, we have that

$$xp_{f,a}b \leqslant p_{e,c}, \quad bp_{c,e}x \leqslant p_{a,f}, \quad yp_{j,a}b \leqslant p_{i,c} \quad \text{and} \quad bp_{c,i}y \leqslant p_{a,j}.$$

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By assumption,

$$bp_{c,e}xp_{f,a}b = bp_{c,i}yp_{j,a}b.$$

Thus

$$xp_{f,a}(bp_{c,e}xp_{f,a}b)p_{c,e}x = xp_{f,a}(bp_{c,i}yp_{j,a}b)p_{c,e}x.$$

Now

$$xp_{f,a}(bp_{c,e}xp_{f,a}b)p_{c,e}x = xp_{f,a}bp_{c,e}x = x.$$

Thus

$$x = (xp_{f,a}b)p_{c,i}yp_{j,a}(bp_{c,e}x).$$

It follows that

$$x \leqslant p_{e,c}p_{c,i}yp_{j,a}p_{a,f} \leqslant p_{e,i}yp_{j,f}.$$

Now

$$p_{e,i}yp_{j,f} = (p_{e,i}yp_{j,a})b(p_{c,i}yp_{j,f}) = (p_{e,i}yp_{j,a})bp_{c,e}xp_{f,a}b(p_{c,i}yp_{j,f}),$$

which is equal to

$$p_{e,i}(yp_{j,a}b)p_{c,e}xp_{f,a}(bp_{c,i}y)p_{j,f} \leqslant (p_{e,i}p_{i,c}p_{c,e})x(p_{f,a}p_{a,j}p_{j,f}) \leqslant p_{e,e}xp_{f,f},$$

which is just x. Hence $x = p_{e,i}yp_{j,f}$. We may similarly show that $y = p_{i,e}xp_{f,j}$.

To prove the converse, suppose that $x = p_{e,i}yp_{j,f}$ and $y = p_{i,e}xp_{f,j}$. We shall show that $V(e,x,f) \cap V(i,y,j) \neq \emptyset$. Observe that

$$y = p_{i,e}xp_{f,j} = (p_{i,e}p_{e,i})y(p_{j,f}p_{f,j}),$$

and, similarly,

$$x = (p_{e,i}p_{i,e})x(p_{f,j}p_{j,f}).$$

Now $x \in eSf$ implies that there is $x' \in V(x) \cap fSe$. Thus $(f,x',e) \in C(S)$. Next observe that

$$y(p_{j,f}x'p_{e,i})y = (p_{i,e}xp_{f,j})p_{j,f}x'p_{e,i}(p_{i,e}xp_{f,j}) = p_{i,e}xx'xp_{f,j} = y$$

and

$$(p_{j,f}x'p_{e,i})y(p_{j,f}x'p_{e,i}) = p_{j,f}x'p_{e,i}p_{i,e}xp_{f,j}p_{j,f}x'p_{e,i} = p_{j,f}x'p_{e,i}.$$

Thus $p_{j,f}x'p_{e,i} \in V(y)$. It is now easy to check that

$$(f,x',e) \in V(e,x,f) \cap V(i,y,j).$$

Thus $V(e,x,f) \cap V(i,y,j) \neq \emptyset$. □

Now let S be a semigroup with local units with locally commuting idempotents, equipped with a McAlister sandwich function. Motivated by Proposition 4.1, define the relation δ on the semigroup $C(S)^{\bullet}$ by

$$(e,x,f)\,\delta\,(i,y,j) \Leftrightarrow x = p_{e,i}yp_{j,f} \quad \text{and} \quad y = p_{i,e}xp_{f,j}.$$

Theorem 4.2. *The relation δ is a congruence on the semigroup $C(S)^{\bullet}$, and the idempotents in the quotient semigroup $C(S)^{\bullet}/\delta$ commute. Furthermore, $\delta^{\natural}$ is a strict local isomorphism.*

Proof. The proof is long and we have to be careful to manipulate regular elements correctly.

(1) δ is an equivalence relation

Both reflexivity and symmetry are straightforward to check. We prove transitivity explicitly. Let

$$(e, x, f)\,\delta\,(i, y, j) \quad \text{and} \quad (i, y, j)\,\delta\,(k, z, l).$$

We prove that

$$(e, x, f)\,\delta\,(k, z, l).$$

By definition,

$$x = p_{e,i}yp_{j,f} \quad \text{and} \quad y = p_{i,e}xp_{f,j}$$

and

$$y = p_{i,k}zp_{l,j} \quad \text{and} \quad z = p_{k,i}yp_{j,l}.$$

Now

$$x = p_{e,i}yp_{j,f} = p_{e,i}(p_{i,k}zp_{l,j})p_{j,f}.$$

By condition (M3), we have that

$$p_{e,i}p_{i,k} \leqslant p_{e,k} \quad \text{and} \quad p_{l,j}p_{j,f} \leqslant p_{l,f}.$$

Thus, by Lemma 2.4 and its dual, there are idempotents $\alpha \leqslant e$ and $\beta \leqslant f$ such that

$$p_{e,i}p_{i,k} = \alpha p_{e,k} \quad \text{and} \quad p_{l,j}p_{j,f} = p_{l,f}\beta.$$

Hence

$$x = \alpha p_{e,k}zp_{l,f}\beta = \alpha p_{e,k}(p_{k,i}yp_{j,l})p_{l,f}\beta = \alpha p_{e,k}p_{k,i}(p_{i,e}xp_{f,j})p_{j,l}p_{l,f}\beta.$$

In particular, $\alpha x = x = x\beta$. Now

$$p_{e,k}p_{k,i}p_{i,e} \leqslant p_{e,e} = e,$$

and $\alpha \leqslant e$. But $E(eSe)$ is a semilattice. Thus

$$\alpha(p_{e,k}p_{k,i}p_{i,e}) = (p_{e,k}p_{k,i}p_{i,e})\alpha.$$

Similarly,

$$\beta(p_{f,j}p_{j,l}p_{l,f}) = (p_{f,j}p_{j,l}p_{l,f})\beta.$$

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Hence

$$x = p_{e,k}p_{k,i}p_{i,e}\alpha x\beta p_{f,j}p_{j,l}p_{l,f} = p_{e,k}p_{k,i}(p_{i,e}xp_{f,j})p_{j,l}p_{l,f} = p_{e,k}(p_{k,i}yp_{j,l})p_{l,f},$$

which is equal to $p_{e,k}zp_{l,f}$. We may show, in a similar way, that $z = p_{k,e}xp_{f,l}$. Hence $(e, x, f)\,\delta\,(k, z, l)$, as required.

(2) δ is a congruence

We prove that δ is left compatible with the multiplication. The proof that it is right compatible is similar. Let $(e, x, f)\,\delta\,(i, y, j)$ and let (a, b, c) be arbitrary. Then

$$(a, b, c)(e, x, f) = (a, bp_{c,e}x, f) \quad \text{and} \quad (a, b, c)(i, y, j) = (a, bp_{c,i}y, j).$$

We prove that

$$(a, bp_{c,e}x, f)\,\delta\,(a, bp_{c,i}y, j).$$

To do this, we need to show that

$$bp_{c,e}x = bp_{c,i}yp_{j,f} \quad \text{and} \quad bp_{c,i}y = bp_{c,e}xp_{f,j}.$$

We shall prove the former equality explicitly; the latter equality is established in a similar way.

By assumption,

$$x = p_{e,i}yp_{j,f} \quad \text{and} \quad y = p_{i,e}xp_{f,j}.$$

Now

$$p_{c,e}xp_{f,j} = p_{c,e}(p_{e,i}yp_{j,f})p_{f,j} = p_{c,e}p_{e,i}p_{i,e}xp_{f,j}p_{j,f}p_{f,j}.$$

Now $p_{c,e}p_{e,i} = \gamma p_{c,i}$ for some idempotent $\gamma \leqslant c$, and $p_{c,i}p_{i,e} = \alpha p_{c,e}$ for some idempotent $\alpha \leqslant c$. Thus

$$p_{c,e}xp_{f,j} = \gamma\alpha p_{c,e}xp_{f,j}p_{j,f}p_{f,j}.$$

Since $\alpha, \gamma \in cSc$ we have that $\gamma\alpha = \alpha\gamma$. Hence $\alpha(p_{c,e}xp_{f,j}) = p_{c,e}xp_{f,j}$. We now have that

$$p_{c,i}yp_{j,f} = p_{c,i}p_{i,e}xp_{f,j}p_{j,f} = \alpha(p_{c,e}xp_{f,j})p_{j,f} = p_{c,e}xp_{f,j}p_{j,f}.$$

However

$$x = p_{e,i}yp_{j,f},$$

and so by condition (M2) we have that $xp_{f,j}p_{j,f} = x$. Thus

$$p_{c,i}yp_{j,f} = p_{c,e}x.$$

It follows that

$$bp_{c,e}x = bp_{c,i}yp_{j,f},$$

as required.

(3) Idempotents in $U(S) = C(S)^\bullet/\delta$ commute

First of all we characterize the idempotents in $U(S)$. Suppose that $\delta(e,x,f)$ is an idempotent in $U(S)$. Then $(e, xp_{f,e}x, f)\,\delta\,(e,x,f)$. Thus

$$xp_{f,e}x = p_{e,e}xp_{f,f} = x.$$

Hence (e,x,f) is an idempotent in $C(S)^\bullet$. Thus $\delta(e,x,f)$ is an idempotent in $U(S)$ if and only if (e,x,f) is an idempotent in $C(S)^\bullet$. In particular, x is regular.

Let $\delta(e,x,f)$ and $\delta(i,y,j)$ be idempotents in $U(S)$. We shall prove that they commute. We therefore need to show that

$$\delta(e, xp_{f,i}y, j) = \delta(i, yp_{j,e}x, f);$$

that is, we need to prove that

$$xp_{f,i}y = p_{e,i}(yp_{j,e}x)p_{f,j} \quad \text{and} \quad yp_{j,e}x = p_{i,e}(xp_{f,i}y)p_{j,f}.$$

We shall prove the former equality; the proof of the latter equality is similar. Observe also that

$$x \leqslant p_{e,f} \quad \text{and} \quad y \leqslant p_{i,j}$$

from the proof of Proposition 3.1 and the fact that (e,x,f) and (i,y,j) are idempotents in $C(S)^\bullet$.

We have that

$$xp_{f,i}y = (xp_{f,i}y)p_{j,e}(xp_{f,i}y),$$

since $(e, xp_{f,i}y, j)$ is an idempotent by Proposition 3.1. Thus

$$xp_{f,i}y = (xp_{f,i})yp_{j,e}x(p_{f,i}y) \leqslant (p_{e,f}p_{f,i})yp_{j,e}x(p_{f,i}p_{i,j}) \leqslant p_{e,i}(yp_{j,e}x)p_{f,j}$$

using Proposition 2.3 (ii),(iii) and the fact that all elements involved are regular. Similarly,

$$yp_{j,e}x \leqslant p_{i,e}(xp_{f,i}y)p_{j,f}.$$

Hence

$$xp_{f,i}y \leqslant p_{e,i}(yp_{j,e}x)p_{f,j} \leqslant p_{e,i}p_{i,e}(xp_{f,i}y)p_{j,f}p_{f,j} \leqslant p_{e,e}(xp_{f,i}y)p_{j,j} = e(xp_{f,i}y)j,$$

and this is equal to $xp_{f,i}y$. Thus

$$xp_{f,i}y = p_{e,i}(yp_{j,e}x)p_{f,j},$$

as required.

(4) $U(S)$ is a semigroup with local units

Let $\delta(e,x,f)$ be an element of $U(S)$. Observe that (e,e,e) and (f,f,f) are both idempotents of $C(S)^\bullet$, by condition (M1). Thus $\delta(e,e,e)$ and $\delta(f,f,f)$ are both idempotents in $U(S)$, and clearly $\delta(e,x,f)\delta(f,f,f) = \delta(e,x,f)$ and $\delta(e,e,e)\delta(e,x,f) = \delta(e,x,f)$.

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(5) $\delta^\natural$ is a strict local isomorphism

Let (a,b,c) and (d,e,f) be idempotents in $C(S)^\bullet$. Then elements of $(a,b,c)\cdot C(S)\cdot (d,e,f)$ will certainly have the form (a,z,f) for suitable z. Let

$$(a,x,f),(a,y,f)\in(a,b,c)\cdot C(S)\cdot(d,e,f).$$

Then if $\delta(a,x,f)=\delta(a,y,f)$, it follows that $x=p_{a,a}yp_{f,f}=y$. □

5. The covering theorem

Let S be a semigroup with local units having locally commuting idempotents and equipped with a McAlister sandwich function. We may therefore construct the semigroup with local units $U(S)$ that has commuting idempotents. The map $\delta^\natural\colon C(S)^\bullet\to U(S)$ is a strict local isomorphism. Define $q\colon E(S)\times E(S)\to U(S)$ by $q(v,u)=q_{v,u}=\delta(v,vu,u)$. We may therefore form the Rees matrix semigroup $\mathcal{M}=\mathcal{M}(U(S);E(S),E(S);Q)$.

Put $E'=\{(e,\delta(e,e,e),e)\colon e\in E(S)\}$. Then E' is a set of idempotents of $\mathcal{M}$. The semigroup $E'\mathcal{M}E'$ is a subsemigroup of $\mathcal{M}$ and a semigroup with local units.

Lemma 5.1. $E'\mathcal{M}E'=\{(u,\delta(u,x,v),v)\in\mathcal{M}\}$.

Proof. Observe that

$$(u,\delta(u,x,v),v)=(u,\delta(u,u,u),u)(u,\delta(u,x,v),v)(v,\delta(v,v,v),v).$$

Thus $\{(u,\delta(u,x,v),v)\in\mathcal{M}\}$ is contained in $E'\mathcal{M}E'$. On the other hand,

$$(e,\delta(e,e,e),e)(u,\delta(i,x,j),v)(f,\delta(f,f,f),f)=(e,\delta(e,eup_{u,i}xp_{j,v}vf,f),f),$$

which is of the required form. □

Put $\mathcal{EM}=E'\mathcal{M}E'$. Observe that in the regular case, $\mathcal{EM}=\mathcal{RM}$ the set of regular elements of $\mathcal{M}$ (see the proof of Theorem 2.1 at the foot of p. 731 in [**7**]).

Define $\theta\colon\mathcal{EM}\to S$ by $\theta(u,\delta(u,x,v),v)=x$; this is well defined because if $\delta(u,x,v)=\delta(u,y,v)$, then $x=y$ by the last part of the proof of Theorem 4.2.

Proposition 5.2. *The function θ is a surjective strict local isomorphism along which idempotents can be lifted.*

Proof. Let $s\in S$. Then because S has local units, we can find idempotents $e,f\in S$ such that $es=s=sf$. But then $\theta(e,\delta(e,s,f),f)=s$, and so θ is surjective.

To show that θ is a homomorphism, let $(u,\delta(u,x,v),v),(g,\delta(g,y,k),k)\in\mathcal{EM}$. Then

$$(u,\delta(u,x,v),v)(g,\delta(g,y,k),k)=(u,\delta(u,xy,k),k).$$

The result is now clear.

To show that θ is a strict local isomorphism, it is sufficient to check that if

$$\theta(u,\delta(u,x,v),v)=\theta(u,\delta(u,y,v),v),$$

then $(u,\delta(u,x,v),v)=(g,\delta(g,y,k),k)$; but this is immediate from the definition.

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Suppose now that $e \in E(S)$. Then $\theta(e, \delta(e, e, e,), e) = e$ and $(e, \delta(e, e, e,), e) \in E(\mathcal{EM})$. Thus idempotents lift along θ. $\square$

We have proved the following covering theorem.

Theorem 5.3. *Let S be a semigroup with local units having locally commuting idempotents. If S has a McAlister sandwich set, then there exists a semigroup U with local units whose idempotents commute, a square Rees matrix semigroup $\mathcal{M} = \mathcal{M}(U; I, I; Q)$ over U, and a subsemigroup T of $\mathcal{M}$ that has local units, and a surjective homomorphism $\theta \colon T \to S$ that is a strict local isomorphism along which idempotents can be lifted.*

It is natural to ask when a semigroup S has a McAlister sandwich function. It is proved in [**4**] that if S possesses an idempotent e such that every element of $eE(S)$ is regular, then S has a McAlister sandwich function constructed in the same way as the one in McAlister's original paper [**7**]. In particular, if the regular elements of S form a subsemigroup then S has a McAlister sandwich function. In our second paper [**3**], we characterize semigroups with local units which possess a McAlister sandwich function.

McAlister's work on the local structure of regular semigroups [**6**–**8**] is related to Talwar's Morita theory of semigroups developed in [**12**] and [**13**]; however, Talwar is interested in more general semigroups than regular; in particular, in [**12**] he develops Morita theory for semigroups with local units, whereas in [**13**], he develops a Morita theory for the more general class of *factorizable* semigroups: a semigroup S is factorizable if $S = S^2$. In [**5**], Lawson and Márki generalized McAlister's local structure theorem to factorizable semigroups. It is therefore natural to ask if all of McAlister's work can be generalized from regular semigroups not just to semigroups with local units but to semigroups which are factorizable.

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