THE BULLETIN OF SYMBOLIC LOGIC
Volume 29, Number 3, September 2023

EXTENDED FRAMES AND SEPARATIONS OF LOGICAL PRINCIPLES

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Abstract. We aim at developing a systematic method of separating omniscience principles by constructing Kripke models for intuitionistic predicate logic **IQC** and first-order arithmetic **HA** from a Kripke model for intuitionistic propositional logic **IPC**. To this end, we introduce the notion of an extended frame, and show that each IPC-Kripke model generates an extended frame. By using the extended frame generated by an IPC-Kripke model, we give a separation theorem of a schema from a set of schemata in **IQC** and a separation theorem of a sentence from a set of schemata in **HA**. We see several examples which give us separations among omniscience principles.

§1. Introduction. Omniscience principles have been playing an important role in neutral (Bishop's) constructive mathematics [3–5, 7]. Those are principles which are derivable in classical logic but underivable in intuitionistic logic, and are used to construct a *weak counterexample* which shows that a statement is constructively underivable by proving that the statement implies an omniscience principle, in contrast with a counterexample which shows that a statement is *false*. Also omniscience principles have been a driving force of *constructive reverse mathematics* [18] where we are interested in which (omniscience) principle is necessary and sufficient to prove a (constructively underivable) theorem (see also [30, 34] for classical reverse mathematics).

Then, necessarily, separations among omniscience principles have become crucial. Akama et al. [1] showed some separations in intuitionistic first-order arithmetic **HA** using, case by case, an extension of **HA**, the Kleene realizability, the monotone modified realizability, and the Lifschitz realizability (see also [24]). *Would there be any uniform technique for separating omniscience principles?* In this paper, we present a possible direction of finding such a uniform technique.

Received June 7, 2020.
2020 Mathematics Subject Classification. Primary 03F50, 03B55, 03C30.
Key words and phrases. extended frame, Kripke model, logical principle.

1079-8986/23/2903-0001
DOI :10.1017/bsl.2023.29

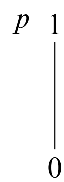

FIGURE 1. Kripke model refuting DNE.

The omniscience principle, called *Markov's principle* (MP), is of the form:

$$\forall x(A \vee \neg A) \rightarrow (\neg\neg\exists x A \rightarrow \exists x A),$$

and it has a special case ($\mathrm{MP_{PR}}$):

$$\neg\neg\exists x A \rightarrow \exists x A \quad (A \text{ primitive recursive})$$

(see [33, Chapter 4, Section 5]). Note that $\mathrm{MP_{PR}}$ is a substitution instance of *double negation elimination* (DNE):

$$\neg\neg p \rightarrow p$$

in intuitionistic propositional logic **IPC** by a Σ_1-formula $\exists x A$, and DNE is refuted in the IPC-Kripke model given by Figure 1. The *weak limited principle of omniscience* (WLPO) is an omniscience principle of the form:

$$\forall x(A \vee \neg A) \rightarrow (\neg\exists x A \vee \neg\neg\exists x A),$$

and its special case ($\mathrm{WLPO_{PR}}$) is the following:

$$\neg\exists x A \vee \neg\neg\exists x A \quad (A \text{ primitive recursive}).$$

Note that $\mathrm{WLPO_{PR}}$ is a substitution instance of the *weak principle of the excluded middle* (WPEM):

$$\neg p \vee \neg\neg p$$

in **IPC** by a Σ_1-formula $\exists x A$, and WPEM is valid in the above IPC-Kripke model (even valid on the frame). Therefore the IPC-Kripke model may be used to separate $\mathrm{MP_{PR}}$ from $\mathrm{WLPO_{PR}}$.

The special cases of many omniscience principles, such as the *limited principle of omniscience* (LPO), the *lesser limited principle of omniscience* (LLPO), the *disjunctive Markov principle* ($\mathrm{MP}^{\vee}$), and Δ_1-PEM, are substitution instances of propositional formulae, such as the *principle of the excluded middle* (PEM): $p \vee \neg p$, *De Morgan's law* (DML): $\neg(p \wedge q) \rightarrow \neg p \vee \neg q$, *weak De Morgan's law* (WDML): $\neg(\neg p \wedge \neg q) \rightarrow \neg\neg p \vee \neg\neg q$, and the *restricted principle of the excluded middle* (RPEM): $(p \leftrightarrow \neg q) \rightarrow p \vee \neg p$,

$$\begin{tikzcd}
 & \mathrm{LPO}_{\mathrm{PR}} \arrow[dl] \arrow[dr] & \\
\mathrm{MP}_{\mathrm{PR}} \arrow[ddrr] & & \mathrm{WLPO}_{\mathrm{PR}} \arrow[d] \\
 & & \mathrm{LLPO}_{\mathrm{PR}} \arrow[d] \\
 & & \mathrm{MP}^{\vee}_{\mathrm{PR}} \arrow[d] \\
 & & \Delta_1\text{-}\mathrm{PEM}_{\mathrm{PR}}
\end{tikzcd}$$

FIGURE 2. Derivabilities between omniscience principles.

respectively, by Σ_1-formulae. Figure 2 shows implications among those special cases. For LPO and LLPO, see [33, Chapter 4, Section 3.4] where those are called $\exists$-PEM and SEP, respectively; for $\mathrm{MP}^{\vee}$, see [17, 28] where it is called LLPE; for Δ_1-PEM, see [9] where it is called III_a, and [25] where it is called Δ^0_1-LEM.

Of course, there are exceptions (see [16, 17, 23]). However, since IPC-Kripke models are simple and easy to handle, a method of separation based on an IPC-Kripke model would give us a good uniform technique for separating many omniscience principles.

In this paper, we aim at developing a systematic method of separating omniscience principles by constructing Kripke models for intuitionistic predicate logic **IQC** and **HA** from an IPC-Kripke model. A similar approach was adopted by de Jongh and Smoryński to show underivability of substitution instances of propositional formulae in **HA** (see [32, Chapter V, Section 3], and also [35]). Here we are interested in not only underivability, but also separation between substitution instances of propositional formulae in **IQC** and **HA**. In Section 2, we introduce the notion of an extended frame which will play a crucial role in the following sections. We show that each IPC-Kripke model generates an extended frame and show a separation theorem (Theorem 2.7). We give several examples of the extended frame generated by an IPC-Kripke model. In Section 3, we introduce the notion of a schema, and, by using the extended frame generated by an IPC-Kripke model, give a separation theorem (Theorem 3.15) of a schema from a set of schemata in **IQC**. We then apply the separation theorem to the examples in the previous section. In Section 4, we apply the results in the previous section to **HA**, and show a separation theorem (Theorem 4.17) of a sentence from a set of schemata. We then see the examples which give us separations among omniscience principles. We conclude the paper with discussing a possible relativization of the results and lifting Theorem 4.17 up to Σ_n-level.

To quickly grasp the story of the paper, follow Definition 2.1, Remark 2.2, Definition 2.3, Example 2.4, Remark 2.6, Theorem 2.7, and Examples 2.10–2.14 in Section 2; Definition 3.7, Definition 3.8, Definition 3.11, Theorem 3.15, and Example 3.16 in Section 3; Definition 4.15, Theorem 4.17, Example 4.18, Theorem 4.23, and Example 4.24 in Section 4.

We use classical logic and set theory at the meta-level.

§2. Extended frames. In this section, we introduce the notion of an extended frame which will play a crucial role in the following sections. We show that each IPC-Kripke model generates an extended frame and show a separation theorem (Theorem 2.7). We give several examples (Examples 2.10–2.14) of the extended frame generated by an IPC-Kripke model.

We use the standard language $\mathcal{L}(\mathbf{IPC})$ of intuitionistic propositional logic **IPC** containing the (countable) set $\mathcal{V}$ of propositional variables, and $\wedge, \vee, \rightarrow$, and $\perp$ as primitive logical operators. *Prime formulae* are atomic formulae (propositional variables) or $\perp$, and we introduce the abbreviations $\neg\varphi \equiv \varphi \rightarrow \perp$ and $\varphi \leftrightarrow \psi \equiv (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$. The set $\mathrm{Vars}(\varphi)$ of propositional variables in a formula φ is defined as usual: $\mathrm{Vars}(\perp) = \emptyset$; $\mathrm{Vars}(p) = \{p\}$; $\mathrm{Vars}(\varphi \circ \psi) = \mathrm{Vars}(\varphi) \cup \mathrm{Vars}(\psi)$ for $\circ \in \{\wedge, \vee, \rightarrow\}$. We sometimes write $\varphi[p_1, \dots, p_n]$ for a formula φ with $\mathrm{Vars}(\varphi) = \{p_1, \dots, p_n\}$. For a formula φ, a sequence $\vec{p} \equiv p_1, \dots, p_n$ of distinct variables and a sequence $\vec{\chi} \equiv \chi_1, \dots, \chi_n$ of formulae, the (simultaneous) substitution $\varphi[\vec{p}/\vec{\chi}]$ is defined as usual: $\perp[\vec{p}/\vec{\chi}] \equiv \perp$; $q[\vec{p}/\vec{\chi}] \equiv \chi_m$ if $q \equiv p_m$ for some $m \in \{1, \dots, n\}$, q otherwise; $(\varphi \circ \psi)[\vec{p}/\vec{\chi}] \equiv \varphi[\vec{p}/\vec{\chi}] \circ \psi[\vec{p}/\vec{\chi}]$ for $\circ \in \{\wedge, \vee, \rightarrow\}$. For a formula $\varphi[p_1, \dots, p_n]$, we write $\varphi[\chi_1, \dots, \chi_n]$ for $\varphi[\vec{p}/\vec{\chi}]$. In the following, we use $\vdash_{\mathbf{IPC}}$ for deducibility in **IPC**, and sometimes write **IPC** for the set of theorems of **IPC**.

Let $(K, \leq)$ be a partially ordered set. Then a subset S of K is *upward closed* if $k \in S$ and $k \leq k'$ imply $k' \in S$ for all $k, k' \in K$, and we write $\boldsymbol{\Omega}_K$ for the class of upward closed subsets of K. For each $k \in K$, we write $\uparrow k$ for the upward closed subset $\{k' \in K \mid k \leq k'\}$, and for each subset S of K, we write $\uparrow S$ for the upward closed subset $\bigcup_{k \in S} \uparrow k$.

DEFINITION 2.1. A *frame* is a nonempty partially ordered set $(K, \leq)$. A *valuation* $\Vdash$ on a frame $(K, \leq)$ is a binary relation between K and $\mathcal{V}$ such that

$$k \Vdash p \text{ and } k \leq k' \Rightarrow k' \Vdash p$$

https://doi.org/10.1017/bsl.2023.29 Published online by Cambridge University Press

for all $k, k' \in K$ and $p \in \mathcal{V}$, that is, $\{k \in K \mid k \Vdash p\} \in \Omega_K$ for all $p \in \mathcal{V}$. The valuation $\Vdash$ is then extended to logically compound formulae of $\mathcal{L}(\mathbf{IPC})$ by the following clauses.

1. $k \nVdash \bot$.
2. $k \Vdash \varphi \wedge \psi \Leftrightarrow k \Vdash \varphi$ and $k \Vdash \psi$.
3. $k \Vdash \varphi \vee \psi \Leftrightarrow k \Vdash \varphi$ or $k \Vdash \psi$.
4. $k \Vdash \varphi \to \psi \Leftrightarrow k' \Vdash \varphi$ implies $k' \Vdash \psi$ for all $k' \geq k$.

Note that $\{k \in K \mid k \Vdash \varphi\} \in \Omega_K$ for all formula φ. An *IPC-Kripke model* is a triple $\mathcal{K} = (K, \leq, \Vdash)$, where $(K, \leq)$ is a frame, and $\Vdash$ is a valuation on it. A formula φ is *valid* in $\mathcal{K}$ if $k \Vdash \varphi$ for all $k \in K$, and we then write $\mathcal{K} \Vdash \varphi$ (see [33, 2.5.2–2.5.4]). A formula φ is *valid* on the frame $(K, \leq)$ if $\mathcal{K} \Vdash \varphi$ for all IPC-Kripke model $\mathcal{K} = (K, \leq, \Vdash)$, that is, for all valuation $\Vdash$ on $(K, \leq)$, and we then write $(K, \leq) \models \varphi$.

Remark 2.2. The set

$$L(K, \leq) = \{\varphi \mid (K, \leq) \models \varphi\}$$

of formulae forms an *intermediate propositional logic* (or simply, a *logic*) in the following sense: $\mathbf{IPC} \subseteq L(K, \leq) \subseteq \mathbf{CPC}$, where $\mathbf{CPC}$ is (the set of theorems of) classical propositional logic; if $\varphi \to \psi, \varphi \in L(K, \leq)$, then $\psi \in L(K, \leq)$; if $\varphi \in L(K, \leq)$, then $\varphi[\vec{p}/\vec{\chi}] \in L(K, \leq)$ for all sequence $\vec{p}$ of distinct propositional variables and sequence $\vec{\chi}$ of formulae.

Definition 2.3. An *extended frame* $\mathcal{E} = ((K, \leq), f, (I, \leq_I))$ is a triple of frames $(K, \leq)$ and $(I, \leq_I)$, and a monotone mapping f between them, that is, $k \leq k'$ implies $f(k) \leq_I f(k')$ for all $k, k' \in K$. Each IPC-Kripke model $\mathcal{I} = (I, \leq_I, \Vdash)$ induces an IPC-Kripke model $\mathcal{K}_{\mathcal{E},\mathcal{I}} = (K, \leq, \Vdash_{\mathcal{E},\mathcal{I}})$ by defining

$$k \Vdash_{\mathcal{E},\mathcal{I}} p \Leftrightarrow f(k) \Vdash p$$

for each $k \in K$ and $p \in \mathcal{V}$. A formula φ is *valid* on $\mathcal{E}$ if $\mathcal{K}_{\mathcal{E},\mathcal{I}} \Vdash_{\mathcal{E},\mathcal{I}} \varphi$ for all IPC-Kripke model $\mathcal{I} = (I, \leq_I, \Vdash)$, that is, for all valuation $\Vdash$ on $(I, \leq_I)$; we then write $\mathcal{E} \models \varphi$.

A trivial example of an extended frame is $((K, \leq), \mathrm{id}_K, (K, \leq))$ for a frame $(K, \leq)$, and a simple, but non-trivial example is given in Figure 3.

Example 2.4. Let $\mathcal{K} = (K, \leq, \Vdash)$ be an IPC-Kripke model, and define a set $\Phi_{\mathcal{K}}$ of upward closed subsets of K by

$$\Phi_{\mathcal{K}} = \{\{k \in K \mid k \Vdash p\} \mid p \in \mathcal{V}\}.$$

Define binary relations $\preceq_{\mathcal{K}}$ and $\sim_{\mathcal{K}}$ on K by

$$k \preceq_{\mathcal{K}} k' \Leftrightarrow \forall U \in \Phi_{\mathcal{K}}(k \in U \Rightarrow k' \in U),$$
$$k \sim_{\mathcal{K}} k' \Leftrightarrow k \preceq_{\mathcal{K}} k' \wedge k' \preceq_{\mathcal{K}} k.$$

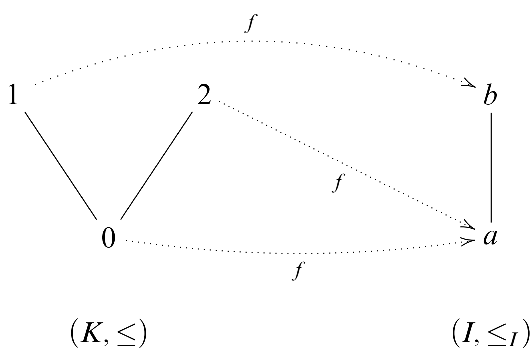

FIGURE 3. An example of extended frame.

Then $\preceq_{\mathcal{K}}$ is a preorder and $\sim_{\mathcal{K}}$ is an equivalence relation on K. Let

$$I_{\mathcal{K}} = K/\sim_{\mathcal{K}}, \qquad [k]_{\mathcal{K}} \leq_{\mathcal{K}} [k']_{\mathcal{K}} \Leftrightarrow k \preceq_{\mathcal{K}} k', \qquad f_{\mathcal{K}}(k) = [k]_{\mathcal{K}},$$

where $[k]_{\mathcal{K}}$ is the equivalence class of k with respect to $\sim_{\mathcal{K}}$. Then

$$\mathcal{E}_{\mathcal{K}} = ((K, \leq), f_{\mathcal{K}}, (I_{\mathcal{K}}, \leq_{\mathcal{K}}))$$

is an extended frame, and we call it the extended frame *generated by* the IPC-Kripke model $\mathcal{K}$.

EXAMPLE 2.5. Let $\mathcal{K} = (K, \leq, \Vdash)$ be an IPC-Kripke model, and define a valuation $\Vdash_{\mathcal{K}}$ on the frame $(I_{\mathcal{K}}, \leq_{\mathcal{K}})$, introduced in Example 2.4, by

$$[k]_{\mathcal{K}} \Vdash_{\mathcal{K}} p \Leftrightarrow k \Vdash p$$

for each $[k]_{\mathcal{K}} \in I_{\mathcal{K}}$ and $p \in \mathcal{V}$. Then $\mathcal{I}_{\mathcal{K}} = (I_{\mathcal{K}}, \leq_{\mathcal{K}}, \Vdash_{\mathcal{K}})$ is an IPC-Kripke model, and induces an IPC-Kripke model $\mathcal{K}_{\mathcal{E}_{\mathcal{K}}, \mathcal{I}_{\mathcal{K}}} = (K, \leq, \Vdash_{\mathcal{E}_{\mathcal{K}}, \mathcal{I}_{\mathcal{K}}})$. Since

$$k \Vdash p \Leftrightarrow [k]_{\mathcal{K}} \Vdash_{\mathcal{K}} p \Leftrightarrow k \Vdash_{\mathcal{E}_{\mathcal{K}}, \mathcal{I}_{\mathcal{K}}} p$$

for all $k \in K$ and $p \in \mathcal{V}$, we have

$$k \Vdash \varphi \Leftrightarrow k \Vdash_{\mathcal{E}_{\mathcal{K}}, \mathcal{I}_{\mathcal{K}}} \varphi$$

for all $k \in K$ and formula φ of $\mathcal{L}(\mathbf{IPC})$.

REMARK 2.6. Let $\mathcal{E} = ((K, \leq), f, (I, \leq_I))$ be an extended frame. Then the set

$$T(\mathcal{E}) = \{\varphi \mid \mathcal{E} \models \varphi\}$$

does not form a logic in general, but forms a *theory* in the following sense: $\mathbf{IPC} \subseteq T(\mathcal{E})$; if $\varphi \to \psi, \varphi \in T(\mathcal{E})$, then $\psi \in T(\mathcal{E})$.

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In fact, consider the extended frame $\mathcal{E} = ((K, \leq), f, (I, \leq_I))$ given in Figure 3. Then $(p \to q) \vee (q \to p) \in T(\mathcal{E})$. However,

$$0 \nVdash_{\mathcal{E},\mathcal{I}} (p \to \neg r) \vee (\neg r \to p)$$

for the IPC-Kripke model $\mathcal{I} = (I, \leq_I, \Vdash)$ with $a \nVdash p$, $a \nVdash r$, $b \Vdash p$, and $b \Vdash r$. Therefore, $((p \to q) \vee (q \to p))[q/\neg r] \notin T(\mathcal{E})$. See Example 5.2 for the details.

THEOREM 2.7. *Let $\mathcal{K} = (K, \leq, \Vdash)$ be an IPC-Kripke model, and let φ be a formula of $\mathcal{L}(\mathbf{IPC})$. If $\mathcal{K} \nVdash \varphi$, then*

$$T(\mathcal{E}_{\mathcal{K}}) \nvdash_{\mathbf{IPC}} \varphi.$$

PROOF. Let $\mathcal{K}_{\mathcal{E}_{\mathcal{K}},\mathcal{I}_{\mathcal{K}}} = (K, \leq, \Vdash_{\mathcal{E}_{\mathcal{K}},\mathcal{I}_{\mathcal{K}}})$ be the IPC-Kripke model induced by the IPC-Kripke model $\mathcal{I}_{\mathcal{K}} = (I_{\mathcal{K}}, \leq_{\mathcal{K}}, \Vdash_{\mathcal{K}})$ in Example 2.5. Then $\mathcal{K}_{\mathcal{E}_{\mathcal{K}},\mathcal{I}_{\mathcal{K}}} \nVdash \varphi$, whenever $\mathcal{K} \nVdash \varphi$. On the other hand, we have $\mathcal{K}_{\mathcal{E}_{\mathcal{K}},\mathcal{I}_{\mathcal{K}}} \Vdash \chi$ for all $\chi \in T(\mathcal{E}_{\mathcal{K}})$. By the soundness theorem [33, Chapter 2, Section 5.10], we have $T(\mathcal{E}_{\mathcal{K}}) \nvdash_{\mathbf{IPC}} \varphi$. $\dashv$

REMARK 2.8. Let $\mathcal{K} = (K, \leq, \Vdash)$ be an IPC-Kripke model, and let φ be a formula of $\mathcal{L}(\mathbf{IPC})$. If $\mathcal{K} \nVdash \varphi$, then $k_0 \nVdash \varphi$ for some $k_0 \in K$, and, by considering the *truncated* model $\mathcal{K}' = (K', \leq', \Vdash')$, where $K' = \uparrow k_0$, $\leq' = \leq \cap (K' \times K')$ and $\Vdash' = \Vdash \cap (K' \times \mathcal{V})$, we may assume that $\mathcal{K}$ is an IPC-Kripke model with a root (see [33, Chapter 2, Section 5.4]). Furthermore, if $\mathcal{K}' = (K', \leq', \Vdash')$ is an IPC-Kripke model with root $k_0 \in K'$ and $k_0 \nVdash \varphi$, then there exists a *finite* model $\mathcal{K}'' = (K'', \leq'', \Vdash'')$ such that $K'' \subseteq K'$, $\leq'' = \leq' \cap (K'' \times K'')$ and $k \Vdash'' \psi \Leftrightarrow k \Vdash' \psi$ for all $k \in K''$ and subformula ψ of φ (see [33, Chapter 2, Section 6.11]). Therefore we may assume that $\mathcal{K}$ is a finite IPC-Kripke model with a root.

Note that if $\mathcal{K}$ is a finite IPC-Kripke model with a root, then $(I_{\mathcal{K}}, \leq_{\mathcal{K}})$ is a finite partially ordered set with a root.

DEFINITION 2.9. An extended frame $\mathcal{E} = ((K, \leq), f, (I, \leq_I))$ is *locally directed* if $f^{-1}(\uparrow i) \cap \uparrow k$ is directed for all $i \in I$ and $k \in K$, that is, for each $i \in I$ and $k \in K$, if $l, l' \in f^{-1}(\uparrow i) \cap \uparrow k$, then there exists $l'' \in f^{-1}(\uparrow i) \cap \uparrow k$ such that $l'' \leq l$ and $l'' \leq l'$.

In the following, we shall give several examples of extended frame generated by an IPC-Kripke model (see [8] for more examples). Before that, we quickly review the relation between some intermediate propositional logics and frames.

For a logic L, let $L + \varphi_0 + \cdots + \varphi_{n-1}$ denote the logic obtained from L by adding axiom schemata $\varphi_0, \ldots, \varphi_{n-1}$, and let

$$L^{\mathrm{lin}} = L + (\varphi \to \psi) \vee (\psi \to \varphi).$$

For $n \geq 1$, the slice $\mathcal{S}_n$ consists of logics L such that $L^{\mathrm{lin}} = L(J_n, \leq_n)$, and the slice $\mathcal{S}_\omega$ consists of logics L such that $L^{\mathrm{lin}} = \bigcap_{n<\omega} L(J_n, \leq_n)$, where $(J_n, \leq_n)$

is the linear frame with n elements. Then $\mathcal{S}_1 = \{\mathbf{CPC}\}$ and $\mathbf{IPC} \in \mathcal{S}_\omega$, and any logic belongs to exactly one $\mathcal{S}_n$ $(n \leq \omega)$ (see [14, Part I, 4.1] and [15]).

Let $\rho_1, \rho_2, \dots$ and $\zeta_1, \zeta_2, \dots$ be sequences of axiom schemata defined by

$$\rho_1 \equiv ((\varphi_1 \to \varphi_0) \to \varphi_1) \to \varphi_1,$$
$$\rho_{n+1} \equiv ((\varphi_{n+1} \to \rho_n) \to \varphi_{n+1}) \to \varphi_{n+1},$$
$$\zeta_n \equiv \bigvee_{0 \leq i < j \leq n} (\varphi_i \leftrightarrow \varphi_j)$$

(see [14] where ρ_n and ζ_n are written by P_n and X_n, respectively). For each $k \geq 1$, let $\mathbf{T}_k = \mathbf{IPC} + \rho_2 + \zeta_k$, and let $\mathbf{M}_k = \mathbf{T}_{2^k+1}$ and $\mathbf{M}_\omega = \mathbf{IPC} + \rho_2$. Then $\{\mathbf{M}_1, \mathbf{M}_2, \dots, \mathbf{M}_n, \dots \mathbf{M}_\omega\}$ is the decreasing enumeration of the second slice $\mathcal{S}_2$ (see [15, Theorem 1.6]).

Note that the axiom schema ρ_n says that the height[1] of a frame is at most n. For a rooted frame, the axiom schema ζ_n says that the number of upward closed subsets of the frame is at most n; hence, for a rooted frame with the height 2, the axiom schema ζ_{2^k+1} says that the number of maximal elements of the frame is at most k (see [12, 29] for other axiomatizations).

In contrast with $\mathcal{S}_2$, we know little about the internal structure of $\mathcal{S}_n$ $(n \geq 3)$.

Let PEM, DNE, WPEM, DML, WDML, and RPEM be the following formulae:

$$\mathrm{PEM}[p] \equiv p \vee \neg p.$$
$$\mathrm{DNE}[p] \equiv \neg\neg p \to p.$$
$$\mathrm{WPEM}[p] \equiv \neg p \vee \neg\neg p.$$
$$\mathrm{DML}[p, q] \equiv \neg(p \wedge q) \to \neg p \vee \neg q.$$
$$\mathrm{WDML}[p, q] \equiv \neg(\neg p \wedge \neg q) \to \neg\neg p \vee \neg\neg q.$$
$$\mathrm{RPEM}[p, q] \equiv (p \leftrightarrow \neg q) \to p \vee \neg p.$$

Then we have the following examples of Theorem 2.7.

EXAMPLE 2.10. Let $\mathcal{E}_{\mathcal{K}_1} = ((K_1, \leq_1), f_{\mathcal{K}_1}, (I_{\mathcal{K}_1}, \leq_{\mathcal{K}_1}))$ be the extended frame generated by the IPC-Kripke model $\mathcal{K}_1 = (K_1, \leq_1, \Vdash_1)$ given in Figure 4. Then $\mathcal{K}_1 \nVdash_1 \mathrm{DNE}[p]$, and hence

$$T(\mathcal{E}_{\mathcal{K}_1}) \nvdash_{\mathbf{IPC}} \mathrm{DNE}[p].$$

The theory $T(\mathcal{E}_{\mathcal{K}_1})$ is the logic $L(K_1, \leq_1)$ axiomatized by the axiom schemata ρ_2 and ζ_3.

[1] The height of a frame $(K, \leq)$ is the maximal length n of finite ascending chains in $(K, \leq)$, if it exists; ω, otherwise (see [29]).

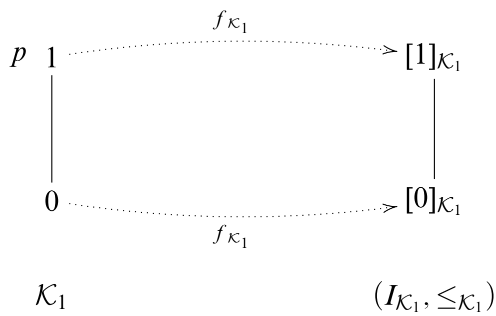

FIGURE 4. The Kripke model and the extended frame in Example 2.10.

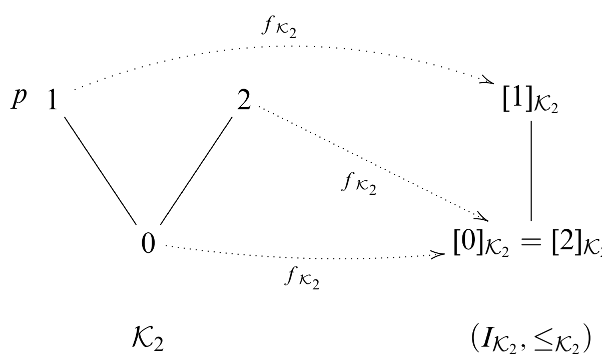

FIGURE 5. The Kripke model and the extended frame in Example 2.11.

Furthermore, $T(\mathcal{E}_{\mathcal{K}_1})$ contains WPEM$[p]$ for all propositional variable p. To see this, consider an IPC-Kripke model $\mathcal{I} = (I_{\mathcal{K}_1}, \leq_{\mathcal{K}_1}, \Vdash')$. If $[1]_{\mathcal{K}_1} \Vdash' p$, then $0 \Vdash_{\mathcal{E}_{\mathcal{K}_1},\mathcal{I}} \neg\neg p$; or else if $[1]_{\mathcal{K}_1} \nVdash' p$, then $0 \Vdash_{\mathcal{E}_{\mathcal{K}_1},\mathcal{I}} \neg p$.

Note that $\mathcal{E}_{\mathcal{K}_1}$ is locally directed.

EXAMPLE 2.11. Let $\mathcal{E}_{\mathcal{K}_2} = ((K_2, \leq_2), f_{\mathcal{K}_2}, (I_{\mathcal{K}_2}, \leq_{\mathcal{K}_2}))$ be the extended frame generated by the IPC-Kripke model $\mathcal{K}_2 = (K_2, \leq_2, \Vdash_2)$ given in Figure 5. Then $\mathcal{K}_2 \nVdash_2$ WPEM$[p]$, and hence

$$T(\mathcal{E}_{\mathcal{K}_2}) \nvdash_{\mathbf{IPC}} \text{WPEM}[p].$$

The theory $T(\mathcal{E}_{\mathcal{K}_2})$ contains the logic $L(K_2, \leq_2)$ axiomatized by the axiom schemata ρ_2 and ζ_5.

Furthermore, $T(\mathcal{E}_{\mathcal{K}_2})$ contains DML$[q, r]$ and DNE$[q]$ for all q and r. To see this, consider an IPC-Kripke model $\mathcal{I} = (I_{\mathcal{K}_2}, \leq_{\mathcal{K}_2}, \Vdash')$. If $0 \Vdash_{\mathcal{E}_{\mathcal{K}_2},\mathcal{I}} \neg(q \wedge r)$, then $[0]_{\mathcal{K}_2} \Vdash' q$ implies $0 \Vdash_{\mathcal{E}_{\mathcal{K}_2},\mathcal{I}} \neg r$, $[0]_{\mathcal{K}_2} \Vdash' r$ implies $0 \Vdash_{\mathcal{E}_{\mathcal{K}_2},\mathcal{I}} \neg q$, and $[0]_{\mathcal{K}_2} \nVdash' q$ and $[0]_{\mathcal{K}_2} \nVdash' r$ imply $0 \Vdash_{\mathcal{E}_{\mathcal{K}_2},\mathcal{I}} \neg q \vee \neg r$, since $[1]_{\mathcal{K}_2} \nVdash' q \wedge r$.

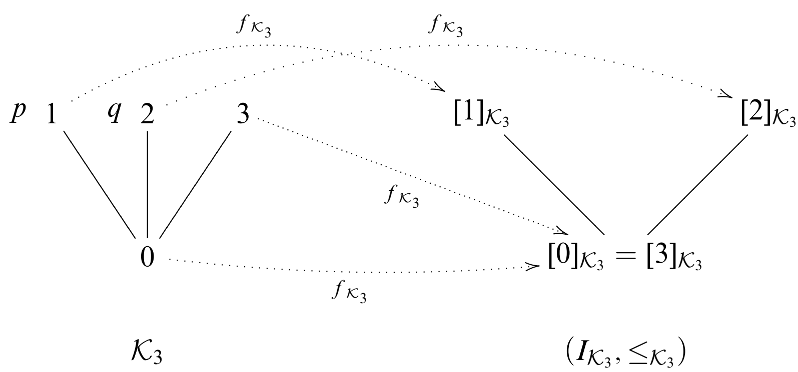

FIGURE 6. The Kripke model and the extended frame in Example 2.12.

If $0 \Vdash_{\mathcal{E}_{\mathcal{K}_2},\mathcal{I}} \neg\neg q$, then $2 \Vdash_{\mathcal{E}_{\mathcal{K}_2},\mathcal{I}} q$, that is, $[2]_{\mathcal{K}_2} = [0]_{\mathcal{K}_2} \Vdash' q$, and hence $0 \Vdash_{\mathcal{E}_{\mathcal{K}_2},\mathcal{I}} q$.

Note that $\mathcal{E}_{\mathcal{K}_2}$ is locally directed.

EXAMPLE 2.12. Let $\mathcal{E}_{\mathcal{K}_3} = ((K_3, \leq_3), f_{\mathcal{K}_3}, (I_{\mathcal{K}_3}, \leq_{\mathcal{K}_3}))$ be the extended frame generated by the IPC-Kripke model $\mathcal{K}_3 = (K_3, \leq_3, \Vdash_3)$ given in Figure 6. Then $\mathcal{K}_3 \nVdash_3 \mathrm{DML}[p, q]$, and hence

$$T(\mathcal{E}_{\mathcal{K}_3}) \nvdash_{\mathbf{IPC}} \mathrm{DML}[p, q].$$

The theory $T(\mathcal{E}_{\mathcal{K}_3})$ contains the logic $L(K_3, \leq_3)$ axiomatized by the axiom schemata ρ_2 and ζ_9.

Furthermore, $T(\mathcal{E}_{\mathcal{K}_3})$ contains DNE[r] for all r. To see this, consider an IPC-Kripke model $\mathcal{I} = (I_{\mathcal{K}_3}, \leq_{\mathcal{K}_3}, \Vdash')$. If $0 \Vdash_{\mathcal{E}_{\mathcal{K}_3},\mathcal{I}} \neg\neg r$, then $3 \Vdash_{\mathcal{E}_{\mathcal{K}_3},\mathcal{I}} r$, that is, $[3]_{\mathcal{K}_3} = [0]_{\mathcal{K}_3} \Vdash' r$, and hence $0 \Vdash_{\mathcal{E}_{\mathcal{K}_3},\mathcal{I}} r$.

Note that $\mathcal{E}_{\mathcal{K}_3}$ is locally directed.

EXAMPLE 2.13. Let $\mathcal{E}_{\mathcal{K}_4} = ((K_4, \leq_4), f_{\mathcal{K}_4}, (I_{\mathcal{K}_4}, \leq_{\mathcal{K}_4}))$ be the extended frame generated by the IPC-Kripke model $\mathcal{K}_4 = (K_4, \leq_4, \Vdash_4)$ given in Figure 7. Then $\mathcal{K}_4 \nVdash_4 \mathrm{WDML}[p, q]$, and hence

$$T(\mathcal{E}_{\mathcal{K}_4}) \nvdash_{\mathbf{IPC}} \mathrm{WDML}[p, q].$$

The theory $T(\mathcal{E}_{\mathcal{K}_4})$ contains the logic $L(K_4, \leq_4)$.

Furthermore $T(\mathcal{E}_{\mathcal{K}_4})$ contains RPEM[r, s] for all r and s. To see this, consider an IPC-Kripke model $\mathcal{I} = (I_{\mathcal{K}_4}, \leq_{\mathcal{K}_4}, \Vdash')$. Suppose that $i \Vdash_{\mathcal{E}_{\mathcal{K}_4},\mathcal{I}} r \leftrightarrow \neg s$. We show that $i \Vdash_{\mathcal{E}_{\mathcal{K}_4},\mathcal{I}} r \vee \neg r$. Suppose otherwise. Then i is not maximal. Without loss of generality, we may assume that $i \leq_4 1$. If $3 \Vdash_{\mathcal{E}_{\mathcal{K}_4},\mathcal{I}} r$, then $3 \Vdash_{\mathcal{E}_{\mathcal{K}_4},\mathcal{I}} \neg s$, and hence $1 \Vdash_{\mathcal{E}_{\mathcal{K}_4},\mathcal{I}} \neg s$, and hence $1 \Vdash_{\mathcal{E}_{\mathcal{K}_4},\mathcal{I}} r$, which implies $0 \Vdash_{\mathcal{E}_{\mathcal{K}_4},\mathcal{I}} r$ by the structure of $\mathcal{E}_{\mathcal{K}_4}$. This is a contradiction. Thus $3 \nVdash_{\mathcal{E}_{\mathcal{K}_4},\mathcal{I}} r$. Then $1 \Vdash_{\mathcal{E}_{\mathcal{K}_4},\mathcal{I}} \neg r$, and hence $i = 0$. Then we also have $4 \nVdash_{\mathcal{E}_{\mathcal{K}_4},\mathcal{I}} r$

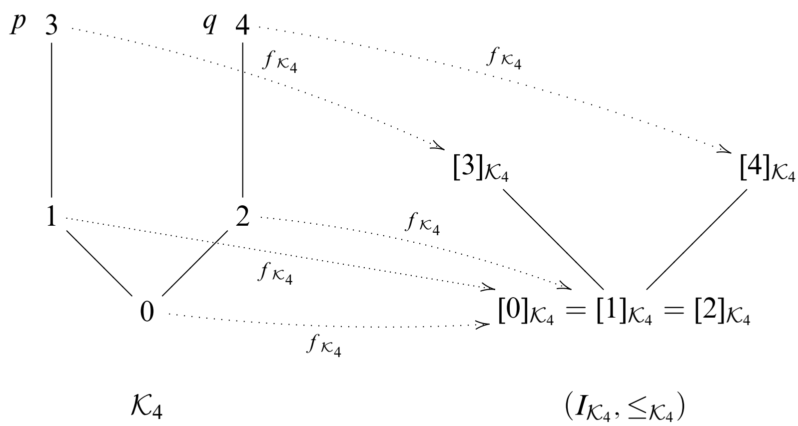

FIGURE 7. The Kripke model and the extended frame in Example 2.13.

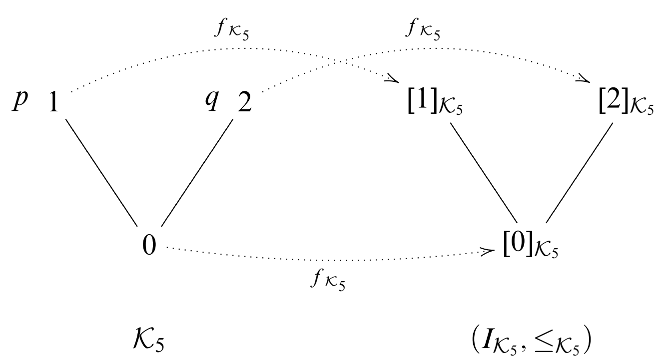

FIGURE 8. The Kripke model and the extended frame in Example 2.14.

as above. Then $0 \Vdash_{\mathcal{E}_{\mathcal{K}_4}, \mathcal{I}} \neg r$ follows. This is a contradiction. Thus we have shown $i \Vdash_{\mathcal{E}_{\mathcal{K}_4}, \mathcal{I}} r \vee \neg r$.

Note that $\mathcal{E}_{\mathcal{K}_4}$ is locally directed.

EXAMPLE 2.14. Let $\mathcal{E}_{\mathcal{K}_5} = ((K_2, \leq_2), f_{\mathcal{K}_5}, (I_{\mathcal{K}_5}, \leq_{\mathcal{K}_5}))$ be the extended frame generated by the IPC-Kripke model $\mathcal{K}_5 = (K_2, \leq_2, \Vdash_5)$ given in Figure 8. Then $\mathcal{K}_5 \nVdash_5$ RPEM$[p, q]$, and hence

$$T(\mathcal{E}_{\mathcal{K}_5}) \nvdash_{\mathbf{IPC}} \mathrm{RPEM}[p, q].$$

The theory $T(\mathcal{E}_{\mathcal{K}_5})$ is the logic $L(K_2, \leq_2)$.

Note that $\mathcal{E}_{\mathcal{K}_5}$ is locally directed.

§3. Separations by extended frames in IQC. In this section, we introduce the notion of a schema, and by using the extended frame generated by an IPC-Kripke model, give a separation theorem (Theorem 3.15) of a schema from a set of schemata in **IQC**. We then apply the separation theorem to the examples in the previous section (Example 3.16).

We use the standard language $\mathcal{L}(\mathbf{IQC})$ of intuitionistic first-order predicate logic **IQC** containing the propositional connectives and $\forall, \exists$ as primitive logical operators. The sets $\mathrm{FV}(t)$ and $\mathrm{FV}(A)$ of free variables in a term t and a formula A, respectively, of $\mathcal{L}(\mathbf{IQC})$ are defined as usual: $\mathrm{FV}(x) = \{x\}$; $\mathrm{FV}(c) = \emptyset$; $\mathrm{FV}(f(t_1, \dots, t_n)) = \mathrm{FV}(t_1) \cup \dots \cup \mathrm{FV}(t_n)$; $\mathrm{FV}(\bot) = \emptyset$; $\mathrm{FV}(R(t_1, \dots, t_n)) = \mathrm{FV}(t_1) \cup \dots \cup \mathrm{FV}(t_n)$; $\mathrm{FV}(B \circ C) = \mathrm{FV}(B) \cup \mathrm{FV}(C)$ for $\circ \in \{\wedge, \vee, \to\}$; $\mathrm{FV}(\forall x B) = \mathrm{FV}(\exists x B) = \mathrm{FV}(B) \setminus \{x\}$; we set $\mathrm{FV}(A_1, \dots, A_n) = \mathrm{FV}(A_1) \cup \dots \cup \mathrm{FV}(A_n)$ for a sequence $A_1, \dots, A_n$ of formulae. In the following, we use $\vdash_{\mathbf{IQC}}$ for deducibility in **IQC**.

DEFINITION 3.1. An *IQC-Kripke model* is a tuple $\mathcal{I} = (I, \leq_I, M, \eta, \Vdash)$, where $(I, \leq_I)$ is a frame, $M = \{M_i\}_{i \in I}$ is a family of nonempty sets, η is a family $\{\eta_{ii'} \in M_i \to M_{i'} \mid i \leq_I i'\}$ of *restrictions* such that:

- η_{ii} is the identity on M_i,
- $\eta_{i'i''} \circ \eta_{ii'} = \eta_{ii''}$ for $i \leq_I i' \leq_I i''$,

and $\Vdash$ is a relation, called a *valuation*, from I to the set of atomic formulae of the language extended by adding a constant symbols a for each element $a \in \bigcup\{M_i \mid i \in I\}$ such that:

- $i \Vdash R(a_1, \dots, a_n) \Rightarrow a_m \in M_i$ for $m \in \{1, \dots, n\}$,
- $i \Vdash R(a_1, \dots, a_n)$ and $i \leq i' \Rightarrow i' \Vdash R(\eta_{ii'}(a_1), \dots, \eta_{ii'}(a_n))$

for all $i, i' \in I$. An n-ary function f is interpreted in $\mathcal{I}$ by a family $f = \{f_i \in M_i^n \to M_i \mid i \in I\}$ commuting with the restrictions for $i' \geq_I i$

$$\eta_{ii'}(f_i(a_1, \dots, a_n)) = f_{i'}(\eta_{ii'}(a_1), \dots, \eta_{ii'}(a_n)).$$

The valuation $\Vdash$ is then extended to logically compound formulae of $\mathcal{L}(\mathbf{IQC})$ by the clauses in the previous section for the propositional connectives and the following clauses:

1. $i \Vdash \forall x A \Leftrightarrow i' \Vdash A[x/a]$ for all $i' \geq i$ and $a \in M_{i'}$.
2. $i \Vdash \exists x A \Leftrightarrow i \Vdash A[x/a]$ for some $a \in M_i$.

Note that $\{i \in I \mid i \Vdash A\} \in \Omega_I$ for all sentence A of $\mathcal{L}(\mathbf{IQC})$. A formula A with $\mathrm{FV}(A) \subseteq \{\vec{x}\}$ is *valid* in $\mathcal{I}$ if $i \Vdash A[\vec{x}/\vec{a}]$ for all $i \in I$ and $\vec{a} \in M_i$, and we then write $\mathcal{I} \Vdash A$ (see [33, Chapter 2, Section 5.12]).

REMARK 3.2. Let $\mathcal{I} = (I, \leq_I, M, \eta, \Vdash)$ be an IQC-Kripke model with a family of interpretations $f = \{f_i \in M_i^n \to M_i \mid i \in I\}$ for all n-ary function f. For each $i \in I$, define a (first-order) *structure* $\mathcal{M}_i = (M_i, \vec{R}_i, \vec{f}_i)$ by

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$$R_i = \{(a_1, \dots, a_m) \mid i \Vdash R(a_1, \dots, a_m)\}, \qquad f_i = f_i.$$

Then $\{\mathcal{M}_i \mid i \in I\}$ is a family of structures satisfying

$$(a_1, \dots, a_m) \in R_i \Rightarrow (\eta_{ii'}(a_1), \dots, \eta_{ii'}(a_m)) \in R_{i'}$$
$$\eta_{ii'}(f_i(a_1, \dots, a_n)) = f_{i'}(\eta_{ii'}(a_1), \dots, \eta_{ii'}(a_n))$$

(see [6, Chapter 2, Section 2.2–2.4]). Conversely, a family $\{\mathcal{M}_i \mid i \in I\}$ of structures together with a family η of restrictions satisfying the above conditions gives an IQC-Kripke model $(I, \leq_I, M, \eta, \Vdash)$ defined by

$$i \Vdash R(a_1, \dots, a_m) \Leftrightarrow \mathcal{M}_i \models_c R(a_1, \dots, a_m)$$

for each $i \in I$, where $\models_c$ denotes the classical interpretation in the structure; we sometimes simply write $M_i \models_c A$ for $\mathcal{M}_i \models_c A$.

DEFINITION 3.3. Let $\mathcal{E} = ((K, \leq), f, (I, \leq_I))$ be an extended frame. Then each IQC-Kripke model $\mathcal{I} = (I, \leq_I, M, \eta, \Vdash)$ induces an IQC-Kripke model $\mathcal{K}_{\mathcal{E},\mathcal{I}} = (K, \leq, D, \varepsilon, \Vdash_{\mathcal{E},\mathcal{I}})$ by defining for each $k, k' \in K$ with $k \leq k'$,

$$D_k = M_{f(k)},$$
$$\varepsilon_{kk'} = \eta_{f(k)f(k')},$$
$$k \Vdash_{\mathcal{E},\mathcal{I}} R(a_1, \dots, a_n) \Leftrightarrow f(k) \Vdash R(a_1, \dots, a_n)$$

for prime formula $R(a_1, \dots, a_n)$ with $a_1, \dots, a_n \in D_k$.

We introduce the notion of a schema following [26], [33, Chapter 2, Section 3.13], and [19, 20].

DEFINITION 3.4. We introduce certain predicate symbols $v_1, v_2, v_3, \dots$ (being outside of our standard language), called *place holders*, to deal with schemata as syntactic objects similar to formulae. *Schemata* are inductively defined by:

1. a prime formula is a schema;
2. if v is an n-ary place holder and $t_1, \dots, t_n$ are terms, then $v(t_1, \dots, t_n)$ is a schema;
3. if α and β are schemata, then $\alpha \circ \beta$ is a schema for $\circ \in \{\wedge, \vee, \rightarrow\}$;
4. if α is a schema, then $Qx\alpha$ is a schema for $Q \in \{\forall, \exists\}$.

Formulae are schemata without place holders.

For example, the *induction schema* is given by a schema

$$v(0) \wedge \forall x(v(x) \rightarrow v(Sx)) \rightarrow \forall x v(x),$$

where v is a unary place holder.

DEFINITION 3.5. Let α be a schema, and let $B_1, \dots, B_k$ be formulae. Let $v_1, \dots, v_k$ be place holders, and let $\vec{x}_1, \dots, \vec{x}_k$ be sequences of variables with lengths of the arities of $v_1, \dots, v_k$, respectively. Then a schema

$$\alpha[v_1/\lambda\vec{x}_1.B_1, \dots, v_k/\lambda\vec{x}_k.B_k]$$

is defined by:

1. $P[v_1/\lambda\vec{x}_1.B_1, \dots, v_k/\lambda\vec{x}_k.B_k] \equiv P$ for P prime;
2. $v(t_1, \dots, t_n)[v_1/\lambda\vec{x}_1.B_1, \dots, v_k/\lambda\vec{x}_k.B_k] \equiv B_i[y_1/t_1, \dots, y_n/t_n]$ if $v \equiv v_i$ and $\vec{x}_i \equiv y_1, \dots, y_n$, and $v(t_1, \dots, t_n)$ otherwise;
3. $(\alpha \circ \beta)[v_1/\lambda\vec{x}_1.B_1, \dots, v_k/\lambda\vec{x}_k.B_k]$
   $\equiv \alpha[v_1/\lambda\vec{x}_1.B_1, \dots, v_k/\lambda\vec{x}_k.B_k] \circ \beta[v_1/\lambda\vec{x}_1.B_1, \dots, v_k/\lambda\vec{x}_k.B_k]$
   for $\circ \in \{\wedge, \vee, \rightarrow\}$;
4. $(Qx\alpha)[v_1/\lambda\vec{x}_1.B_1, \dots, v_k/\lambda\vec{x}_k.B_k]$
   $\equiv Qx(\alpha[v_1/\lambda\vec{x}_1.B_1, \dots, v_k/\lambda\vec{x}_k.B_k])$
   for $Q \in \{\forall, \exists\}$.

We simply write $\alpha[v_1/B_1, \dots, v_k/B_k]$ or even $\alpha[B_1, \dots, B_k]$ for

$$\alpha[v_1/\lambda\vec{x}_1.B_1, \dots, v_k/\lambda\vec{x}_k.B_k]$$

whenever possible, if it does not cause confusion. An *instance* of a schema α with place holders exactly $\vec{v}$ is a formula $\alpha[\vec{v}/\vec{B}]$ or $\alpha[\vec{B}]$.

REMARK 3.6. In using the substitution notations $\alpha[\vec{v}/\vec{B}]$, we shall assume that $\lambda\vec{x}.\vec{B}$ are *free for* $\vec{v}$ *in* α, or we assume that a suitable renaming of bound variables is carried out. Here $\lambda\vec{x}.B$ is free for v in P for prime P; $\lambda\vec{x}.B$ is free for v in $v'(t_1, ..., t_n)$; $\lambda\vec{x}.B$ is free for v in $\alpha \circ \beta$ if $\lambda\vec{x}.B$ is free for v in α and β, where $\circ \in \{\wedge, \vee, \rightarrow\}$; $\lambda\vec{x}.B$ is free for v in $Qy\alpha$ if $y \in \mathrm{FV}(B) \setminus \{\vec{x}\}$ and $\lambda\vec{x}.B$ is free for v in α, where $Q \in \{\forall, \exists\}$.

DEFINITION 3.7. We extend the deducibility relation $\vdash$ and the forcing relation $\Vdash$ to schemata as follows. Let Γ and Δ be a set of schemata, let α be a schema, and let C be a formula. Then $\Gamma \vdash C$ if

$$\{B \mid B \text{ is an instance of a schema in } \Gamma\} \vdash C;$$

$\Gamma \vdash \alpha$ if $\Gamma \vdash B$ for all instance B of α; $\Gamma \vdash \Delta$ if $\Gamma \vdash \alpha$ for all $\alpha \in \Delta$. Similarly, for IQC-Kripke model $\mathcal{K}$, $\mathcal{K} \Vdash \alpha$ if $\mathcal{K} \Vdash B$ for all instance B of α; $\mathcal{K} \Vdash \Delta$ if $\mathcal{K} \Vdash \alpha$ for all $\alpha \in \Delta$.

DEFINITION 3.8. Each formula $\varphi[p_1, \dots, p_n]$ of $\mathcal{L}(\mathbf{IPC})$ may be naturally regarded as a schema

$$\varphi^* \equiv \varphi[v_1^0, \dots, v_n^0],$$

where $v_1^0, \dots, v_n^0$ are nullary place holders.

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LEMMA 3.9. *Let* $(K, \leq, D, \varepsilon, \Vdash)$ *be an IQC-Kripke model, and let* $(K, \leq, \Vdash')$ *be an IPC-Kripke model. Let* $\varphi[p_1, \dots, p_n]$ *be a formula of* $\mathcal{L}(\mathbf{IPC})$*, and let* $A_1, \dots, A_n$ *be formulae of* $\mathcal{L}(\mathbf{IQC})$ *with* $\mathrm{FV}(A_1, \dots, A_n) \subseteq \{\vec{x}\}$*. For each* $k \in K$ *and* $\vec{a} \in D_k$*, if*

$$k' \Vdash' p_m \Leftrightarrow k' \Vdash A_m[\vec{x}/\varepsilon_{kk'}(\vec{a})]$$

for all $k' \geq k$ *and* $m \in \{1, \dots, n\}$*, then*

$$k' \Vdash' \varphi \Leftrightarrow k' \Vdash (\varphi[A_1, \dots, A_n])[\vec{x}/\varepsilon_{kk'}(\vec{a})]$$

for all $k' \geq k$.

PROOF. Straightforward by induction on the complexity of φ. ⊣

PROPOSITION 3.10. *Let* $\mathcal{E} = ((K, \leq), f, (I, \leq_I))$ *be an extended frame. If* $\varphi \in L(K, \leq)$*, then*

$$\mathcal{K}_{\mathcal{E},\mathcal{I}} \Vdash_{\mathcal{E},\mathcal{I}} \varphi^*$$

for all IQC-Kripke model $\mathcal{I} = (I, \leq_I, M, \eta, \Vdash)$.

PROOF. Suppose that $\varphi \in L(K, \leq)$, and consider an IQC-Kripke model $\mathcal{I} = (I, \leq_I, M, \eta, \Vdash)$ and an instance

$$\varphi[A_1, \dots, A_n]$$

of the schema φ^*. Let $\mathcal{K}_{\mathcal{E},\mathcal{I}} = (K, \leq, D, \varepsilon, \Vdash_{\mathcal{E},\mathcal{I}})$ be the induced IQC-Kripke model. Then for each $k \in K$ and $\vec{a} \in D_k$, defining a valuation $\Vdash'$ on $(K, \leq)$ by

$$k' \Vdash' p_m \Leftrightarrow k' \Vdash_{\mathcal{E},\mathcal{I}} A_m[\vec{x}/\varepsilon_{kk'}(\vec{a})]$$

for each $k' \geq k$ and $m \in \{1, \dots, n\}$, by Lemma 3.9, we have

$$k \Vdash' \varphi \Leftrightarrow k \Vdash_{\mathcal{E},\mathcal{I}} (\varphi[A_1, \dots, A_n])[\vec{x}/\vec{a}].$$

Since φ is valid on $(K, \leq)$, we have

$$k \Vdash_{\mathcal{E},\mathcal{I}} (\varphi[A_1, \dots, A_n])[\vec{x}/\vec{a}].$$

⊣

DEFINITION 3.11. For each formula $\varphi[p_1, \dots, p_n]$ of $\mathcal{L}(\mathbf{IPC})$, define a schema Σ-φ by

$$\Sigma\text{-}\varphi \equiv \forall\vec{x}(v_1(\vec{x}) \vee \neg v_1(\vec{x})) \wedge \cdots \wedge \forall\vec{x}(v_n(\vec{x}) \vee \neg v_n(\vec{x})) \\ \to \varphi[\exists\vec{x}\, v_1(\vec{x}), \dots, \exists\vec{x}\, v_n(\vec{x})],$$

where $v_1, \dots, v_n$ are place holders with the arity of the length of $\vec{x}$.

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PROPOSITION 3.12. *Let $\mathcal{E}_{\mathcal{K}} = ((K, \leq), f_{\mathcal{K}}, (I_{\mathcal{K}}, \leq_{\mathcal{K}}))$ be the extended frame generated by an IPC-Kripke model $\mathcal{K} = (K, \leq, \Vdash)$, and let $\varphi[p_1, \dots, p_n]$ be a formula of $\mathcal{L}(\mathbf{IPC})$. If $\mathcal{K} \nVdash \varphi$, then*

$$\mathcal{K}_{\mathcal{E}_{\mathcal{K}}, \mathcal{I}} \nVdash_{\mathcal{E}_{\mathcal{K}}, \mathcal{I}} \Sigma\text{-}\varphi$$

for some IQC-Kripke model $\mathcal{I} = (I_{\mathcal{K}}, \leq_{\mathcal{K}}, M, \eta, \Vdash')$.

PROOF. Recall the construction in Example 2.4 of the extended frame $\mathcal{E}_{\mathcal{K}}$. Suppose that $\mathcal{K} \nVdash \varphi$. Let $R_1(\vec{x}), \dots, R_n(\vec{x})$ be predicate symbols, and let $U_q = \{k \in K \mid k \Vdash q\}$ for each $q \in \mathcal{V}$. Then define an IQC-Kripke model $\mathcal{I} = (I_{\mathcal{K}}, \leq_{\mathcal{K}}, M, \eta, \Vdash')$ by

$$\begin{aligned} M_{[k]_{\mathcal{K}}} &= \{U_q \mid k \Vdash q\} \cup \{K\}, \\ \eta_{[k]_{\mathcal{K}}[k']_{\mathcal{K}}} &= \text{the inclusion map } M_{[k]_{\mathcal{K}}} \to M_{[k']_{\mathcal{K}}}, \\ [k]_{\mathcal{K}} \Vdash' R_m(U_{q_1}, \dots, U_{q_{n'}}) &\Leftrightarrow U_{q_1} \cap \cdots \cap U_{q_{n'}} \subseteq U_{p_m} \end{aligned}$$

for each $[k]_{\mathcal{K}}, [k']_{\mathcal{K}} \in I_{\mathcal{K}}$ with $[k]_{\mathcal{K}} \leq_{\mathcal{K}} [k']_{\mathcal{K}}$, $U_{q_1}, \dots, U_{q_{n'}} \in M_{[k]_{\mathcal{K}}}$ and $m \in \{1, \dots, n\}$. Let $\mathcal{K}_{\mathcal{E}_{\mathcal{K}}, \mathcal{I}} = (K, \leq, D, \varepsilon, \Vdash_{\mathcal{E}_{\mathcal{K}}, \mathcal{I}})$ be the induced IQC-Kripke model, and assume that

$$k \Vdash_{\mathcal{E}_{\mathcal{K}}, \mathcal{I}} \forall \vec{x}(R_1(\vec{x}) \vee \neg R_1(\vec{x})) \wedge \cdots \wedge \forall \vec{x}(R_n(\vec{x}) \vee \neg R_n(\vec{x})) \\ \to \varphi[\exists \vec{x} R_1(\vec{x}), \dots, \exists \vec{x} R_n(\vec{x})]$$

for all $k \in K$. Then given a $k \in K$, if $[k]_{\mathcal{K}} \Vdash' R_m(U_{q_1}, \dots, U_{q_{n'}})$ for some $U_{q_1}, \dots, U_{q_{n'}} \in D_k = M_{[k]_{\mathcal{K}}}$, then $k \in U_{q_1} \cap \cdots \cap U_{q_{n'}} \subseteq U_{p_m}$, and hence $k \Vdash p_m$; if $k \Vdash p_m$, then $U_{p_m} \in M_{[k]_{\mathcal{K}}} = D_k$, and hence $[k]_{\mathcal{K}} \Vdash' R_m(U_{p_m}, \dots, U_{p_m})$. Therefore

$$k \Vdash p_m \Leftrightarrow k \Vdash_{\mathcal{E}_{\mathcal{K}}, \mathcal{I}} \exists \vec{x} R_m(\vec{x}) \tag{$\dagger$}$$

for all $k \in K$ and $m \in \{1, \dots, n\}$. For each $k \in K$, $U_{q_1}, \dots, U_{q_{n'}} \in D_k$ and $m \in \{1, \dots, n\}$, either

$$[k]_{\mathcal{K}} \Vdash' R_m(U_{q_1}, \dots, U_{q_{n'}}) \text{ or } [k]_{\mathcal{K}} \nVdash' R_m(U_{q_1}, \dots, U_{q_{n'}}).$$

In the latter case, if $[k']_{\mathcal{K}} \Vdash' R_m(U_{q_1}, \dots, U_{q_{n'}})$ for some $k' \geq k$, then $U_{q_1} \cap \cdots \cap U_{q_{n'}} \subseteq U_{p_m}$, and hence $[k]_{\mathcal{K}} \Vdash' R_m(U_{q_1}, \dots, U_{q_{n'}})$, a contradiction. Therefore $k \Vdash_{\mathcal{E}_{\mathcal{K}}, \mathcal{I}} \neg R_m(U_{q_1}, \dots, U_{q_{n'}})$. Thus,

$$k \Vdash_{\mathcal{E}_{\mathcal{K}}, \mathcal{I}} \forall \vec{x}(R_m(\vec{x}) \vee \neg R_m(\vec{x}))$$

for all $k \in K$ and $m \in \{1, \dots, n\}$, and so

$$k \Vdash_{\mathcal{E}_{\mathcal{K}}, \mathcal{I}} \varphi[\exists \vec{x} R_1(\vec{x}), \dots, \exists \vec{x} R_n(\vec{x})]$$

for all $k \in K$. By ($\dagger$) and Lemma 3.9, we have $k \Vdash \varphi$ for all $k \in K$, a contradiction. ⊣

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LEMMA 3.13. *Let* $\mathcal{E} = ((K, \leq), f, (I, \leq_I))$ *be a locally directed extended frame, and let* $\mathcal{K}_{\mathcal{E},\mathcal{I}} = (K, \leq, D, \varepsilon, \Vdash_{\mathcal{E},\mathcal{I}})$ *be the induced IQC-Kripke model by an IQC-Kripke model* $\mathcal{I} = (I, \leq_I, M, \eta, \Vdash)$. *Then for each formula* A *of* $\mathcal{L}(\mathbf{IQC})$ *with* $\mathrm{FV}(\exists\vec{x}A) \subseteq \{\vec{y}\}$, $k \in K$, *and* $\vec{b} \in D_k$, *there exists an upward closed subset* U *of* I *such that if*

$$k \Vdash_{\mathcal{E},\mathcal{I}} (\forall\vec{x}(A \vee \neg A))[\vec{y}/\vec{b}],$$

then

$$f(k') \in U \Leftrightarrow k' \Vdash_{\mathcal{E},\mathcal{I}} (\exists\vec{x}A)[\vec{y}/\varepsilon_{kk'}(\vec{b})]$$

for all $k' \geq k$.

PROOF. Consider formula A of $\mathcal{L}(\mathbf{IQC})$ and $k \in K$. We may assume without loss of generality that $\mathrm{FV}(\exists\vec{x}A) = \emptyset$. Define an upward closed subset U of I by

$$U = \bigcup\{\uparrow f(l) \mid k \leq l \wedge l \Vdash_{\mathcal{E},\mathcal{I}} \exists\vec{x}A\},$$

and suppose that $k \Vdash_{\mathcal{E},\mathcal{I}} \forall\vec{x}(A \vee \neg A)$. Then given $k' \geq k$, we show that

$$f(k') \in U \Leftrightarrow k' \Vdash_{\mathcal{E},\mathcal{I}} \exists\vec{x}A.$$

Assume that $f(k') \in U$. Then there exists $l \geq k$ such that $f(l) \leq_I f(k')$ and $l \Vdash_{\mathcal{E},\mathcal{I}} A[\vec{x}/\vec{a}]$ for some $\vec{a} \in D_l$. Since $k', l \in f^{-1}(\uparrow f(l)) \cap \uparrow k$, there exists $l' \in f^{-1}(\uparrow f(l)) \cap \uparrow k$ such that $l' \leq k'$ and $l' \leq l$. Note that $f(l') = f(l)$ and $\vec{a} \in D_l = M_{f(l)} = D_{l'}$. Since $k \Vdash_{\mathcal{E},\mathcal{I}} \forall\vec{x}(A \vee \neg A)$ and $k \leq l'$, either

$$l' \Vdash_{\mathcal{E},\mathcal{I}} A[\vec{x}/\vec{a}] \quad \text{or} \quad l' \Vdash_{\mathcal{E},\mathcal{I}} \neg A[\vec{x}/\vec{a}].$$

However, since $l \Vdash_{\mathcal{E},\mathcal{I}} A[\vec{x}/\vec{a}]$ and $l \geq l'$, the former must be the case. Therefore $l' \Vdash_{\mathcal{E},\mathcal{I}} \exists\vec{x}A$, and, since $l' \leq k'$, we have $k' \Vdash_{\mathcal{E},\mathcal{I}} \exists\vec{x}A$. Conversely, if $k' \Vdash_{\mathcal{E},\mathcal{I}} \exists\vec{x}A$, then trivially $f(k') \in U$. ⊣

PROPOSITION 3.14. *Let* $\mathcal{E} = ((K, \leq), f, (I, \leq_I))$ *be a locally directed extended frame. If* $\varphi \in T(\mathcal{E})$, *then*

$$\mathcal{K}_{\mathcal{E},\mathcal{I}} \Vdash_{\mathcal{E},\mathcal{I}} \Sigma\text{-}\varphi$$

for all IQC-Kripke model $\mathcal{I} = (I, \leq_I, M, \eta, \Vdash)$.

PROOF. Suppose that $\varphi \in T(\mathcal{E})$, and consider an IQC-Kripke model $\mathcal{I} = (I, \leq_I, M, \eta, \Vdash)$ and an instance

$$\forall\vec{x}(A_1 \vee \neg A_1) \wedge \cdots \wedge \forall\vec{x}(A_n \vee \neg A_n) \to \varphi[\exists\vec{x}A_1, \dots, \exists\vec{x}A_n]$$

of the schema Σ-φ. Let $k \in K$. We may assume without loss of generality that $\mathrm{FV}(\exists\vec{x}A_1, \dots, \exists\vec{x}A_n) = \emptyset$. Given a $k' \geq k$, by Lemma 3.13, there exist

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upward closed subsets $U_1, \dots, U_n$ of I such that for each $m \in \{1, \dots, n\}$, if $k' \Vdash_{\mathcal{E},\mathcal{I}} \forall \vec{x}(A_m \vee \neg A_m)$, then

$$f(k'') \in U_m \Leftrightarrow k'' \Vdash_{\mathcal{E},\mathcal{I}} \exists \vec{x} A_m$$

for all $k'' \geq k'$. Let $\mathcal{I}' = (I, \leq_I, \Vdash')$ be an IPC-Kripke model defined by

$$i \Vdash' p_m \Leftrightarrow i \in U_m$$

for each $i \in I$ and $m \in \{1, \dots, n\}$. Assume that $k' \Vdash_{\mathcal{E},\mathcal{I}} \forall \vec{x}(A_m \vee \neg A_m)$ for all $m \in \{1, \dots, n\}$. Then

$$k'' \Vdash_{\mathcal{E},\mathcal{I}'} p_m \Leftrightarrow k'' \Vdash_{\mathcal{E},\mathcal{I}} \exists \vec{x} A_m$$

for all $k'' \geq k'$ and $m \in \{1, \dots, n\}$. By Lemma 3.9, we have

$$k' \Vdash_{\mathcal{E},\mathcal{I}'} \varphi \Leftrightarrow k' \Vdash_{\mathcal{E},\mathcal{I}} \varphi[\exists \vec{x} A_1, \dots, \exists \vec{x} A_n].$$

Therefore, since φ is valid on $\mathcal{E}$, we have

$$k' \Vdash_{\mathcal{E},\mathcal{I}} \varphi[\exists \vec{x} A_1, \dots, \exists \vec{x} A_n].$$

Thus $\quad k \Vdash_{\mathcal{E},\mathcal{I}} \forall \vec{x}(A_1 \vee \neg A_1) \wedge \dots \wedge \forall \vec{x}(A_n \vee \neg A_n) \to \varphi[\exists \vec{x} A_1, \dots, \exists \vec{x} A_n].$ ⊣

For a set S of formulae of $\mathcal{L}(\mathbf{IPC})$, define sets S^* and Σ-S of schemata by $S^* = \{\varphi^* \mid \varphi \in S\}$ and Σ-$S = \{\Sigma\text{-}\varphi \mid \varphi \in S\}$.

Now, we arrive at a separation theorem for **IQC**.

Theorem 3.15. *Let $\mathcal{K} = (K, \leq, \Vdash)$ be an IPC-Kripke model, and let φ be a formula of $\mathcal{L}(\mathbf{IPC})$. If $\mathcal{K} \nVdash \varphi$ and $\mathcal{E}_{\mathcal{K}}$ is locally directed, then*

$$L(K, \leq)^* + \Sigma\text{-}T(\mathcal{E}_{\mathcal{K}}) \nvdash_{\mathbf{IQC}} \Sigma\text{-}\varphi.$$

Proof. Straightforward by Proposition 3.10, Proposition 3.12, Proposition 3.14, and the soundness theorem [33, Chapter 2, Section 5.10]. ⊣

Example 3.16. By applying Theorem 3.15 to Example 2.10, we have

$$L(K_1, \leq_1)^* \nvdash_{\mathbf{IQC}} \Sigma\text{-DNE},$$

especially $\text{WPEM}^* \nvdash_{\mathbf{IQC}} \Sigma$-DNE; to Example 2.11, we have

$$L(K_2, \leq_2)^* + \Sigma\text{-}T(\mathcal{E}_{\mathcal{K}_2}) \nvdash_{\mathbf{IQC}} \Sigma\text{-WPEM},$$

especially $\Sigma\text{-DML} + \Sigma\text{-DNE} \nvdash_{\mathbf{IQC}} \Sigma$-WPEM; to Example 2.12, we have

$$L(K_3, \leq_3)^* + \Sigma\text{-}T(\mathcal{E}_{\mathcal{K}_3}) \nvdash_{\mathbf{IQC}} \Sigma\text{-DML},$$

especially $\Sigma\text{-DNE} \nvdash_{\mathbf{IQC}} \Sigma$-DML; to Example 2.13, we have

$$L(K_4, \leq_4)^* + \Sigma\text{-}T(\mathcal{E}_{\mathcal{K}_4}) \nvdash_{\mathbf{IQC}} \Sigma\text{-WDML},$$

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especially Σ-RPEM $\nvdash_{\mathbf{IQC}}$ Σ-WDML; to Example 2.14, we have

$$L(K_2, \leq_2)^* \nvdash_{\mathbf{IQC}} \Sigma\text{-RPEM}.$$

Note that Σ-DNE, Σ-WPEM, Σ-DML, Σ-WDML, and Σ-RPEM correspond to MP, WLPO, LLPO, $\mathrm{MP}^{\vee}$, and Δ_1-PEM, respectively.

§4. Separations by extended frames in HA. In this section, we apply the results in the previous section to **HA**, and show a separation theorem (Theorem 4.17) of a sentence from a set of schemata. We then see the examples which give us separations among omniscience principles (Example 4.18).

We use the standard language $\mathcal{L}_1 = \mathcal{L}(\mathbf{HA})$ of intuitionistic first-order arithmetic **HA** containing the constant 0, the unary function symbol S, the binary function symbols $+$ and $\times$, and the binary predicate $=$ (equality). The axioms and rules are those of **IQC** with equality, the axioms

$$Sx = Sy \to x = y, \qquad \neg 0 = Sx, \qquad \neg 0 = x \to \exists y(x = Sy)$$

and

$$x + 0 = x, \quad x + Sy = S(x + y), \quad x \times 0 = 0, \quad x \times Sy = (x \times y) + x,$$

and the induction axiom schema

$$A(0) \wedge \forall x(A(x) \to A(Sx)) \to \forall x A(x)$$

(see [27, Chapter 1]).

A Σ_0-formula (and Π_0-*formula*) is a formula built up from prime formulae by the propositional connectives $\wedge$, $\vee$, and $\to$, and the bounded quantifiers $\forall x \leq t$ and $\exists x \leq t$; A Σ_{n+1}-*formula* is a Π_n-formula or a formula of the form $\exists x A$, where A is a Π_n-formula, and a Π_{n+1}-*formula* is a Σ_n-formula or a formula of the form $\forall x A$, where A is a Σ_n-formula. A Σ_n-*sentence* (respectively, Π_n-*sentence*) is a Σ_n-formula (respectively, Π_n-formula) without free variables.

In the following, we use $\vdash$ and $\vdash_c$ for deducibilities of intuitionistic and classical first-order predicate logic **IQC** and **CQC**, respectively. Recall that $\models_c$ denotes the classical interpretation in a (first-order) structure.

Definition 4.1. Let **HA** also denote the (recursive) set of $\mathcal{L}_1$-sentences consisting of (the universal closures of) axioms and instances of the axiom schema of **HA**. Then $\mathbf{HA} \vdash_c A$ means that a formula A is derivable in classical first-order arithmetic **PA**. Let $\mathrm{Th}(\omega)$ denote the set of $\mathcal{L}_1$-sentences which are true in ω, that is, $\mathrm{Th}(\omega) = \{A \mid \omega \models_c A\}$. For a set T of $\mathcal{L}_1$-sentences, a Σ_1-*representation* of T is a Σ_1-formula $\tau(x)$ such that

$$\mathbf{HA} \vdash_c \forall x[(\tau(x) \to \text{“}x \text{ is a sentence”}) \wedge (\tau_{\mathbf{HA}}(x) \to \tau(x))], \text{ and}$$
$$A \in T \text{ if and only if } \mathbf{HA} \vdash_c \tau(\ulcorner A \urcorner) (\text{ or equivalently } \omega \models_c \tau(\ulcorner A \urcorner))$$

for all $\mathcal{L}_1$-sentence A, where $\ulcorner A \urcorner$ is a Gödel number of A and $\tau_{\mathbf{HA}}(x)$ denotes

$$\text{“}x \text{ is (a Gödel number of) a sentence”}$$
$$\wedge \text{ “}x \text{ is an axiom or an instance of the axiom schema of } \mathbf{HA}.\text{”}$$

A set T of $\mathcal{L}_1$-sentences is *well-behaved* if T has a Σ_1-representation and $\mathbf{HA} \subseteq T \subseteq \mathrm{Th}(\omega)$.

REMARK 4.2. It is easy to see that if T has a Σ_1-representation, then it is recursively enumerable. Conversely, if T ($\supseteq \mathbf{HA}$) is recursively enumerable, then there exists a Σ_1-formula which represents T. For each recursively enumerable set T ($\supseteq \mathbf{HA}$), we fix a Σ_1-representation (say, by choosing the one with the smallest Gödel number), and identify T with its Σ_1-representation. With this identification, one may refer to the consistency statement $\mathrm{Con}(T) \equiv \neg\text{“}T \vdash_c \perp''$ of T within $\mathbf{HA}$.

DEFINITION 4.3. Let M be an $\mathcal{L}_1$-structure. Then a subset S of M^n is *definable in* M if there exists a formula B with $\mathrm{FV}(B) = \{x_1, \dots, x_n\}$ such that

$$S = \{(a_1, \dots, a_n) \in M^n \mid M \models_c B[x_1, \dots, x_n/a_1, \dots, a_n]\};$$

a function is *definable in* M if its graph is a definable subset in M. An $\mathcal{L}_1$-structure $\hat{M}$ is a *definable structure in* M if the universe of $\hat{M}$ is a definable subset in M, and the interpretations of $+$ and $\times$ in $\hat{M}$ are definable functions in M (see [13, 22] for the basic notions of model theory of arithmetic).

REMARK 4.4. Let M be an $\mathcal{L}_1$-structure such that $M \models_c \mathbf{HA}$, and let $\hat{M}$ be a definable $\mathcal{L}_1$-structure in M. Then the interpretation of closed $\mathcal{L}_1$-terms in $\hat{M}$, that is, a function σ_0 from the codes (in M) of closed $\mathcal{L}_1$-terms into $\hat{M} \subseteq M$, is definable in M. This σ_0 yields the *canonical embedding* $\theta : M \to \hat{M}$, defined by $\theta(m) = \sigma_0(\bar{m})$ where $\bar{m}$ is the m-th numeral coded in M. Moreover, this θ is a Σ_0-*elementary* embedding, that is, $M \models_c B[\vec{x}/\vec{a}] \Leftrightarrow \hat{M} \models_c B[\vec{x}/\theta(\vec{a})]$ for all Σ_0-formula B and $\vec{a} \in M$ (see [31, Lemma 6.12 and Lemma 3.8]).

THEOREM 4.5 (Arithmetized completeness theorem). *Let T be a well-behaved set of $\mathcal{L}_1$-sentences, and let M be an $\mathcal{L}_1$-structure such that*

$$M \models_c \mathbf{HA} + \mathrm{Con}(T).$$

Then there exist a definable $\mathcal{L}_1$-structure $\hat{M}$ in M and a Σ_0-elementary embedding $\theta : M \to \hat{M}$ such that $\hat{M} \models_c T$. Moreover, for each formula B, there exists a formula $[B]$ such that $\mathrm{FV}([B]) = \mathrm{FV}(B)$ *and*

$$M \models_c [B][\vec{x}/\vec{a}] \Leftrightarrow \hat{M} \models_c B[\vec{x}/\theta(\vec{a})]$$

for all $\vec{a} \in M$, where $\vec{x}$ are the free variables of B.

This is the semantic form of the arithmetized completeness theorem [31, Theorem 6.10] together with the Σ_0-elementary canonical embedding obtained in Remark 4.4. This theorem is also referred to as the interpretability theorem (see, e.g., [2, 21]). We will digest the proof of the relativized form of this theorem in Appendix B.

LEMMA 4.6 (Fixed point lemma [27, Chapter 1, Lemma 1(b)]). *Let $k \geq 1$. For any Π_k-formula $\gamma(w, n)$, there exists a Π_k-formula $\xi \equiv \xi(v)$ such that*

$$\mathbf{HA} \vdash_c \forall n(\xi(n) \leftrightarrow \gamma(\ulcorner \xi \urcorner, n)).$$

LEMMA 4.7 [27, Chapter 2, Exercise 11]. *Let T be a well-behaved set of $\mathcal{L}_1$-sentences. Then there exists a Π_1-formula $K(v)$ such that $T \nvdash_c K(m)$ for all natural number m, and $\mathbf{HA} \vdash_c K(m) \vee K(l)$ for all natural numbers m and l with $m \neq l$.*

PROOF. We follow the hint in [27]. Let $\gamma(w, n)$ be a formula which denotes

$$\begin{aligned}&\text{“}w \text{ is a formula such that } \mathrm{FV}(w) \subseteq \{v\}\text{”}\\ &\to \forall y[\text{“}y \text{ is a proof of } T \vdash_c w[v/\bar{n}]\text{”}\\ &\to \exists z \exists u(\langle u, z\rangle < \langle n, y\rangle \wedge \text{“}z \text{ is a proof of } T \vdash_c w[v/\bar{u}]\text{”})],\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual pairing function. Then apply the fixed point lemma and obtain a Π_1-formula $K \equiv K(v)$ such that $\mathbf{HA} \vdash_c \forall n(K(n) \leftrightarrow \gamma(\ulcorner K \urcorner, n))$. It is easy to check that this $K(v)$ satisfies the desired conditions. ⊣

LEMMA 4.8. *Let $\{T_1, \dots, T_n\}$ be a nonempty and finite set of well-behaved sets of $\mathcal{L}_1$-sentences. Then there exist a well-behaved set T of $\mathcal{L}_1$-sentences with $\bigcup_{j=1}^n T_j \subseteq T$ and a set $\{A_1, \dots, A_n\}$ of Σ_1-sentences such that:*

1. $T \vdash_c \mathrm{Con}(T_j + A_j)$,
2. $T \vdash_c \neg A_j$,
3. *if $j \neq j'$, then $\mathbf{HA} \vdash_c \neg(A_j \wedge A_{j'})$,*

for all $j, j' \in \{1, \dots, n\}$.

PROOF. Let $\{T_1, \dots, T_n\}$ be a nonempty and finite set of well-behaved sets of $\mathcal{L}_1$-sentences, and let $T' = \bigcup_{j=1}^n T_j$. Then T' is a well-behaved set of $\mathcal{L}_1$-sentences. By Lemma 4.7, there exists a Σ_1-formula $K(x)$ such that $T' \nvdash_c \neg K(m)$ for all natural number m, and $\mathbf{HA} \vdash_c \neg(K(m) \wedge K(l))$ for all natural numbers m and l with $m \neq l$. Note that for each natural number m, since $T' + K(m) \nvdash_c \perp$, we have $\mathrm{Con}(T' + K(m)) \in \mathrm{Th}(\omega)$. Since

$$\mathbf{HA} \vdash_c K(m) \to \text{“}\mathbf{HA} \vdash_c K(m),\text{”}$$

by Σ_1-completeness of the provability predicate [27, Chapter 1, Fact 9], we have

$$\mathbf{HA} \vdash_c K(m) \to \text{“}\mathbf{HA} + K(l) \vdash_c K(m),\text{”}$$

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and hence $\mathbf{HA} \vdash_c K(m) \to$ "$\mathbf{HA} + K(l) \vdash_c \perp$" for all natural numbers m and l with $m \neq l$. Therefore,

$$\mathbf{HA} \vdash_c K(m) \to \neg\mathrm{Con}(\mathbf{HA} + K(l)),$$

and so $\mathbf{HA} \vdash_c \mathrm{Con}(\mathbf{HA} + K(l)) \to \neg K(m)$. Let $A_j \equiv K(j)$ for each $j \in \{1, \dots, n\}$, and let

$$T = T' + \{\mathrm{Con}(T' + K(0)), \mathrm{Con}(T' + A_1), \dots, \mathrm{Con}(T' + A_n)\}.$$

Then $\mathbf{HA} \vdash_c \neg(A_j \wedge A_{j'})$ for all $j, j' \in \{1, \dots, n\}$ with $j \neq j'$, and T is a well-behaved set of $\mathcal{L}_1$-sentences. For each $j \in \{1, \dots, n\}$, since $\mathbf{HA} \vdash_c \mathrm{Con}(T' + A_j) \to \mathrm{Con}(T_j + A_j)$ and $\mathbf{HA} \vdash_c \mathrm{Con}(T' + K(0)) \to \mathrm{Con}(\mathbf{HA} + K(0))$, we have $T \vdash_c \mathrm{Con}(T_j + A_j)$ and $T \vdash_c \neg A_j$. ⊣

In the following, we call an IQC-Kripke model for the language $\mathcal{L}_1$ an $\mathcal{L}_1$-*Kripke model*. The construction in the proof of the following lemma is essentially the same as Smoryński's construction for the refinement of de Jongh's theorem (see [32, Chapter V, Section 2–3]).

Lemma 4.9. *Let T_0 be a well-behaved set of $\mathcal{L}_1$-sentences, and let $(I, \leq_I)$ be a finite tree with the root $i_0 \in I$. Then there exists a well-behaved set T of $\mathcal{L}_1$-sentences with $T_0 \subseteq T$ such that for each Σ_1-sentence C_0 and $\mathcal{L}_1$-structure M_0, if*

$$M_0 \models_c T + C_0,$$

then there exist an $\mathcal{L}_1$-Kripke model $\mathcal{I} = (I, \leq_I, \{M_i \mid i \in I\}, \eta, \Vdash)$ and a family $\{C_i \mid i \in I\}$ of Σ_1-sentences with $M_{i_0} = M_0$ and $C_{i_0} \equiv C_0$ satisfying that for each $i \in I$,

1. $M_i \models_c T_0 + C_i$;
2. *$i \leq_I i'$ implies $\mathbf{HA} \vdash_c C_{i'} \to C_i$ for all $i' \in I$;*
3. *$M_{i'} \models_c C_i$ implies $i \leq_I i'$ for all $i' \in I$;*
4. *for each $i' \in I$ with $i \leq_I i'$ and formula B, there exists an $\mathcal{L}_1$-formula $[B]_i^{i'}$ such that $\mathrm{FV}([B]_i^{i'}) = \mathrm{FV}(B)$ and*

$$M_i \models_c [B]_i^{i'}[\vec{x}/\vec{a}] \Leftrightarrow M_{i'} \models_c B[\vec{x}/\eta_{ii'}(\vec{a})]$$

for all $\vec{a} \in M_i$, where $\vec{x}$ are the free variables of B;

5. *if B is a Σ_0-formula, then*

$$M_i \models_c B[\vec{x}/\vec{a}] \Leftrightarrow M_{i'} \models_c B[\vec{x}/\eta_{ii'}(\vec{a})]$$

for all $i' \in I$ with $i \leq_I i'$ and $\vec{a} \in M_i$, where $\vec{x}$ are the free variables of B.

Proof. We proceed by induction on the complexity of $(I, \leq_I)$.

Basis: If i_0 is a leaf, then note that $I = \{i_0\}$, and set $T = T_0$. For each Σ_1-sentence C_0 and $\mathcal{L}_1$-structure M_0 with

$$M_0 \models_c T + C_0,$$

setting $M_{i_0} = M_0$, define an $\mathcal{L}_1$-Kripke model $\mathcal{I} = (I, \leq_I, M, \eta, \Vdash)$ by $M = \{M_{i_0}\}, \eta = \{\mathrm{id}_{M_{i_0}}\}$ and $i_0 \Vdash P \Leftrightarrow M_{i_0} \models_c P$ for each atomic P, and set $C_{i_0} \equiv C_0$. Then it is straightforward to see that $\mathcal{I}$ and $\{C_{i_0}\}$ satisfy (1)–(5) for all $i \in I$.

Induction step: Let J be the (nonempty and finite) set of direct descendants (children) of i_0, and note that $i_0 = i \vee \exists! j \in J (j \leq_I i)$ for all $i \in I$. Then, since $(\uparrow j, \leq_I)$ is a finite tree with the root j for all $j \in J$, there exist, by the induction hypothesis, the set $\{T_j \mid j \in J\}$ of well-behaved sets including T_0 such that for each $j \in J$, Σ_1-sentence C_j and $\mathcal{L}_1$-structure M_j, if $M_j \models_c T_j + C_j$, then there exist an $\mathcal{L}_1$-Kripke model $(\uparrow j, \leq_I, M^j, \eta^j, \Vdash^j)$ and a family $\{C_i^j \mid i \in \uparrow j\}$ of Σ_1-sentences satisfying that (1)–(5) for all $i \in \uparrow j$. By Lemma 4.8, there exist a well-behaved set T of $\mathcal{L}_1$-sentences with $\bigcup_{j \in J} T_j \subseteq T$ and a set $\{A_j \mid j \in J\}$ of Σ_1-sentences such that $T \vdash_c \mathrm{Con}(T_j + A_j)$, $T \vdash_c \neg A_j$, and if $j \neq j'$, then $\mathbf{HA} \vdash_c \neg(A_j \wedge A_{j'})$ for all $j, j' \in J$. Consider a Σ_1-sentence C_0 and an $\mathcal{L}_1$-structure M_0 such that

$$M_0 \models_c T + C_0.$$

Then $M_0 \models_c \mathbf{HA} + \mathrm{Con}(T_j + A_j)$ and $M_0 \models_c \neg A_j$ for all $j \in J$. By the arithmetized completeness theorem, for each $j \in J$ there exist a definable $\mathcal{L}_1$-structure $\hat{M}_j$ in M_0 and a Σ_0-elementary embedding $\theta_j : M_0 \to \hat{M}_j$ such that $\hat{M}_j \models_c T_j + A_j$. Let C_j be a Σ_1-sentences such that $\mathbf{HA} \vdash_c C_j \leftrightarrow C_0 \wedge A_j$. Then, since $\hat{M}_j$ is a Σ_0-elementary extension of M_0, we have $\hat{M}_j \models_c C_0$, and hence

$$\hat{M}_j \models_c T_j + C_j.$$

Therefore there exist an $\mathcal{L}_1$-Kripke model $(\uparrow j, \leq_I, M^j, \eta^j, \Vdash^j)$, where $M^j = \{M_i^j \mid i \in \uparrow j\}$, and a family $\{C_i^j \mid i \in \uparrow j\}$ of Σ_1-sentences satisfying (1)–(5), and $M_j^j = \hat{M}_j$ and $C_j^j \equiv C_j$. Setting $M_{i_0} = M_0$, define an $\mathcal{L}_1$-Kripke model $\mathcal{I} = (I, \leq_I, M, \eta, \Vdash)$ by

$$M = \{M_{i_0}\} \cup \bigcup_{j \in J} M^j, \quad \eta = \{\eta_{i_0 i} \mid i \in I\} \cup \bigcup_{j \in J} \eta^j, \quad \Vdash = \{\Vdash_0\} \cup \bigcup_{j \in J} \Vdash^j,$$

where $\eta_{i_0 i_0} = \mathrm{id}_{M_{i_0}}$ and $\eta_{i_0 i} = \eta_{ji}^j \circ \theta_j$ for $i \in I$ with $j \in J$ and $j \leq_I i$, and $\Vdash_0$ is defined by $i_0 \Vdash_0 P \Leftrightarrow M_{i_0} \models_c P$ for each atomic P. Set $C_i \equiv C_0$ if $i = i_0$; $C_i \equiv C_i^j$ if $i \in I$ with $j \in J$ and $j \leq_I i$. We show that $\mathcal{I}$ and $\{C_i \mid i \in I\}$ satisfy (1)–(5). Let $i \in I$.

(1): Either (i) $i = i_0$; or (ii) $j \leq_I i$ for some $j \in J$.
  (i): We have $M_{i_0} \models_c T_0 + C_{i_0}$.
  (ii): We have $M_i \models_c T_0 + C_i$, by the induction hypothesis.
(2): If $i \leq_I i'$, then either (i) $i = i' = i_0$; (ii) $i = i_0$ and $j \leq_I i'$ for some $j \in J$; or (iii) $j \leq_I i \leq_I i'$ for some $j \in J$.
  (i): It is trivial.
  (ii): Since $\mathbf{HA} \vdash_c C_j \to C_{i_0}$, we have $\mathbf{HA} \vdash_c C_{i'} \to C_i$, by the induction hypothesis.
  (iii): It follows from the induction hypothesis.
(3): Suppose that $M_{i'} \models_c C_i$. Then either (i) $i = i_0$; (ii) $i' = i_0$ and $j \leq_I i$ for some $j \in J$; or (iii) $j \leq_I i$ and $j' \leq_I i'$ for some $j, j' \in J$.
  (i): There is nothing to prove.
  (ii): Since $M_{i_0} \models_c C_i$ and $\mathbf{HA} \vdash_c C_i \to C_j$, we have $M_{i_0} \models_c C_j$, and hence $M_{i_0} \models_c A_j$, a contradiction to $M_{i_0} \models_c \neg A_j$.
  (iii): Since $M_{i'} \models_c C_i$, $M_{i'} \models_c C_{i'}$, $\mathbf{HA} \vdash_c C_i \to C_j$, and $\mathbf{HA} \vdash_c C_{i'} \to C_{j'}$, we have $M_{i'} \models_c C_j$ and $M_{i'} \models_c C_{j'}$, and hence $M_{i'} \models_c A_j \wedge A_{j'}$. Therefore, since $j \neq j'$ implies a contradiction to $M_{i'} \models_c \neg(A_j \wedge A_{j'})$, we have $j = j'$, and so $i \leq i'$, by the induction hypothesis.
(4): Consider $i' \in I$ with $i \leq_I i'$. Then either (i) $i = i' = i_0$; (ii) $i = i_0$ and $j \leq_I i'$ for some $j \in J$; or (iii) $j \leq_I i$ for some $j \in J$.
  (i): It is trivial.
  (ii): Since for each formula B there exists a formula $[B]$ with $\mathrm{FV}(B) = \mathrm{FV}([B]) = \{\vec{x}\}$ such that

$$M_{i_0} \models_c [B][\vec{x}/\vec{a}] \Leftrightarrow \hat{M}_j \models_c B[\vec{x}/\theta_j(\vec{a})]$$

for all $\vec{a} \in M_{i_0}$, we have $\mathrm{FV}(B) = \mathrm{FV}([[B]_j^{i'}])$ and

$$\begin{aligned} M_{i_0} \models_c [[B]_j^{i'}][\vec{x}/\vec{a}] &\Leftrightarrow M_j \models_c [B]_j^{i'}[\vec{x}/\theta_j(\vec{a})] \\ &\Leftrightarrow M_{i'} \models_c B[\vec{x}/\eta_{ji'}^j(\theta_j(\vec{a}))], \end{aligned}$$

for all $\vec{a} \in M_{i_0}$, by the induction hypothesis, and hence

$$M_{i_0} \models_c [[B]_j^{i'}][\vec{x}/\vec{a}] \Leftrightarrow M_{i'} \models_c B[\vec{x}/\eta_{i_0 i'}(\vec{a})]$$

for all $\vec{a} \in M_{i_0}$.
  (iii): It follows from the induction hypothesis.
(5): Consider $i' \in I$ with $i \leq_I i'$. Then either (i) $i = i' = i_0$; (ii) $i = i_0$ and $j \leq_I i'$ for some $j \in J$; or (iii) $j \leq_I i$ for some $j \in J$.
  (i): It is trivial.
  (ii): It suffices to show that if B is a Σ_0-formula, then

$$M_{i_0} \models_c B[\vec{x}/\vec{a}] \Leftrightarrow M_j \models_c B[\vec{x}/\theta_j(\vec{a})]$$

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for all $j \in J$ and $\vec{a} \in M_{i_0}$. This is the case, since M_j is a Σ_0-elementary extension of M_{i_0}.

(iii): It follows from the induction hypothesis. $\dashv$

We need a couple of technical lemmas in the language of $\mathcal{L}(\mathbf{IQC})$ whose proofs are given in Appendix A.

DEFINITION 4.10. Recall the definition of a schema in Definition 3.4. Then we define simultaneously classes $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ of schemata as follows. Let P range over atomic formulae, v over expressions $v_k(t_1 \dots t_n)$ (v_k an n-ary place holder symbol), α and α' over $\mathcal{A}$, β and β' over $\mathcal{B}$, γ and γ' over $\mathcal{C}$, and δ and δ' over $\mathcal{D}$. Then $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are inductively generated by the clauses

$$
\begin{aligned}
&\bot, \alpha \wedge \alpha', \alpha \vee \alpha', \forall x\alpha, \gamma \to \alpha \in \mathcal{A};\\
&\alpha, P, v, \beta \wedge \beta', \beta \vee \beta', \forall x\beta, \exists x\beta, \gamma \to \beta \in \mathcal{B};\\
&\bot, P, v, \gamma \wedge \gamma', \gamma \vee \gamma', \exists x\gamma, \alpha \to \gamma \in \mathcal{C};\\
&\gamma, \delta \wedge \delta', \forall x\delta, \beta \to \delta \in \mathcal{D}.
\end{aligned}
$$

LEMMA 4.11. *Let $\mathcal{E} = ((K, \leq), f, (I, \leq_I))$ be an extended frame such that $(K, \leq)$ is finite, and let $\mathcal{K}_{\mathcal{E},\mathcal{I}} = (K, \leq, D, \varepsilon, \Vdash_{\mathcal{E},\mathcal{I}})$ be the induced IQC-Kripke model by an IQC-Kripke model $\mathcal{I} = (I, \leq_I, M, \eta, \Vdash)$. Assume that for each $i, i' \in I$ with $i \leq i'$ and formula B, there exists a formula $[B]_i^{i'}$ such that $\mathrm{FV}([B]_i^{i'}) = \mathrm{FV}(B)$ and*

$$M_i \models_c [B]_i^{i'}[\vec{x}/\vec{a}] \Leftrightarrow M_{i'} \models_c B[\vec{x}/\eta_{ii'}(\vec{a})]$$

for all $\vec{a} \in M_i$, where $\vec{x}$ are the free variables of B. If $\Gamma \subseteq \mathcal{D}$ and $M_{f(k)} \models_c \Gamma$ for all $k \in K$, then $\mathcal{K}_{\mathcal{E},\mathcal{I}} \Vdash \Gamma$.

DEFINITION 4.12. Let Γ be a class of formulae. Then Γ is *closed under subformulae* if:

1. if $A \circ B \in \Gamma$, then $A \in \Gamma$ and $B \in \Gamma$, where $\circ \in \{\wedge, \vee, \to\}$;
2. if $\forall x A \in \Gamma$ or $\exists x A \in \Gamma$, then $A[x/t] \in \Gamma$ for all term t.

The class $\exists(\Gamma)$ of formulae is inductively generated by the rules:

1. $A \in \exists(\Gamma)$ for $A \in \Gamma$;
2. if $A, B \in \exists(\Gamma)$, then $A \wedge B, A \vee B \in \exists(\Gamma)$;
3. if $A[x/t] \in \exists(\Gamma)$ for all term t, then $\exists x A \in \exists(\Gamma)$.

REMARK 4.13. In the language $\mathcal{L}_1$ of $\mathbf{HA}$, the classes of Σ_n-formulae and Π_n-formulae are closed under subformulae, and each Σ_{n+1}-formula belongs to a class $\exists(\Gamma)$ of formulae, where Γ is the class of Π_n-formulae.

LEMMA 4.14. *Let $\mathcal{E} = ((K, \leq), f, (I, \leq_I))$ be an extended frame, let Γ be a class of formulae closed under subformulae, and let $\mathcal{I} = (I, \leq_I, M, \eta, \Vdash)$ be an*

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IQC-Kripke model such that if $B \in \Gamma$, then

$$M_i \models_c B[\vec{x}/\vec{a}] \Leftrightarrow M_j \models_c B[\vec{x}/\eta_{ij}(\vec{a})]$$

for all $i, j \in I$ with $i \leq_I j$ and $\vec{a} \in M_i$. For the induced IQC-Kripke model $\mathcal{K}_{\mathcal{E},\mathcal{I}} = (K, \leq, D, \varepsilon, \Vdash_{\mathcal{E},\mathcal{I}})$, if $B \in \exists(\Gamma)$, then

$$k \Vdash_{\mathcal{E},\mathcal{I}} B[\vec{x}/\vec{a}] \Leftrightarrow M_{f(k)} \models_c B[\vec{x}/\vec{a}]$$

for all $k \in K$ and $\vec{a} \in D_k = M_{f(k)}$.

Definition 4.15. For each formula $\varphi[p_1, \dots, p_m]$ of $\mathcal{L}(\mathbf{IPC})$ and n, we define a set Σ_n-φ of $\mathcal{L}_1$-formulae by

$$\Sigma_n\text{-}\varphi = \{\varphi[A_1, \dots, A_m] \mid A_1, \dots, A_m \text{ are } \Sigma_n\text{-formulae}\}.$$

Proposition 4.16. *Let $\mathcal{E}_{\mathcal{K}} = ((K, \leq), f_{\mathcal{K}}, (I_{\mathcal{K}}, \leq_{\mathcal{K}}))$ be the extended frame generated by a finite IPC-Kripke model $\mathcal{K} = (K, \leq, \Vdash)$ such that $(I_{\mathcal{K}}, \leq_{\mathcal{K}})$ is a rooted (and finite) tree, and let $\varphi[p_1, \dots, p_n]$ be a formula of $\mathcal{L}(\mathbf{IPC})$. If $\mathcal{K} \nVdash \varphi$, then*

$$\mathcal{K}_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} \Vdash_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} \mathbf{HA} \quad \textit{and} \quad \mathcal{K}_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} \nVdash_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} \Sigma_1\text{-}\varphi$$

for some $\mathcal{L}_1$-Kripke model $\mathcal{I} = (I_{\mathcal{K}}, \leq_{\mathcal{K}}, M, \eta, \Vdash')$.

Proof. Let $i_0 \in I_{\mathcal{K}}$ be the root of the finite tree $(I_{\mathcal{K}}, \leq_{\mathcal{K}})$. Then, by Lemma 4.9 with $T_0 = \mathbf{HA}$, there exists a well-behaved set T of $\mathcal{L}_1$-sentences such that for each Σ_1-sentence C_{i_0} and $\mathcal{L}_1$-structure M_{i_0}, if

$$M_{i_0} \models_c T + C_{i_0},$$

there exist an $\mathcal{L}_1$-Kripke model $(I_{\mathcal{K}}, \leq_{\mathcal{K}}, M, \eta, \Vdash')$ and a family $\{C_i \mid i \in I_{\mathcal{K}}\}$ of Σ_1-sentences satisfying that for each $i \in I$, (1)–(5) of Lemma 4.9. Let M_{i_0} be a model of T, and let $C_{i_0} \equiv 0 = 0$. Then $M_{i_0} \models_c T + C_{i_0}$, and hence there exist an $\mathcal{L}_1$-Kripke model $\mathcal{I} = (I_{\mathcal{K}}, \leq_{\mathcal{K}}, M, \eta, \Vdash')$ and a family $\{C_i \mid i \in I_{\mathcal{K}}\}$ of Σ_1-sentences satisfying that (1)–(5) for all $i \in I_{\mathcal{K}}$.

Let $\mathcal{K}_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} = (K, \leq, D, \varepsilon, \Vdash_{\mathcal{E}_{\mathcal{K}},\mathcal{I}})$ be the induced $\mathcal{L}_1$-Kripke model by $\mathcal{I}$. We show that $\mathcal{K}_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} \Vdash_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} \mathbf{HA}$ and $\mathcal{K}_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} \nVdash_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} \varphi[B_1, \dots, B_n]$ for some Σ_1-sentences $B_1, \dots, B_n$. Since $(K, \leq)$ is finite and the property (4) holds for all $i \in I_{\mathcal{K}}$, by Lemma 4.11, if $\Gamma \subseteq \mathcal{D}$ and $M_{f_{\mathcal{K}}(k)} \models_c \Gamma$ for all $k \in K$, then $\mathcal{K}_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} \Vdash_{\mathcal{E},\mathcal{I}} \Gamma$. Therefore, since the axioms and the axiom schema of $\mathbf{HA}$ belong to the class $\mathcal{D}$ and $M_{f_{\mathcal{K}}(k)} \models_c \mathbf{HA}$ for all $k \in K$, we have $\mathcal{K}_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} \Vdash_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} \mathbf{HA}$.

For each $m \in \{1, \dots, n\}$, let B_m be a Σ_1-sentence such that

$$\mathbf{HA} \vdash B_m \leftrightarrow \bigvee_{i \in \{[k]_{\mathcal{K}} \mid k \Vdash p_m\}} C_i.$$

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Note that, by (1)–(3), we have $M_{i'} \models_c C_i \Leftrightarrow i \leq_{\mathcal{K}} i'$ for all $i, i' \in I_{\mathcal{K}}$. Then, by (5) and Lemma 4.14, we have

$$\begin{aligned} k \Vdash_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} B_m &\Leftrightarrow \exists k' \in K(k' \Vdash p_m \wedge k \Vdash_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} C_{[k']_{\mathcal{K}}}) \\ &\Leftrightarrow \exists k' \in K(k' \Vdash p_m \wedge M_{[k]_{\mathcal{K}}} \models_c C_{[k']_{\mathcal{K}}}) \\ &\Leftrightarrow \exists k' \in K(k' \Vdash p_m \wedge [k']_{\mathcal{K}} \leq_{\mathcal{K}} [k]_{\mathcal{K}}) \Leftrightarrow k \Vdash p_m \end{aligned}$$

for all $k \in K$ and $m \in \{1, \dots, n\}$. By Lemma 3.9, we have

$$\mathcal{K}_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} \nVdash_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} \varphi[B_1, \dots, B_n]. \qquad \dashv$$

Now, we arrive at a separation theorem for **HA**.

THEOREM 4.17. *Let $\mathcal{K} = (K, \leq, \Vdash)$ be a finite IPC-Kripke model such that $(I_{\mathcal{K}}, \leq_{\mathcal{K}})$ is a rooted (and finite) tree, and let φ be a formula of $\mathcal{L}(\mathbf{IPC})$. If $\mathcal{K} \nVdash \varphi$ and $\mathcal{E}_{\mathcal{K}}$ is locally directed, then*

$$\mathbf{HA} + L(K, \leq)^* + \Sigma\text{-}T(\mathcal{E}_{\mathcal{K}}) \nvdash \Sigma_1\text{-}\varphi.$$

PROOF. Straightforward by Proposition 3.10, Proposition 3.14, Proposition 4.16, and the soundness theorem [33, Chapter 2, Section 5.10]. $\dashv$

EXAMPLE 4.18. By applying Theorem 4.17 to Example 2.10, we have

$$\mathbf{HA} + L(K_1, \leq_1)^* \nvdash \Sigma_1\text{-DNE},$$

especially $\mathbf{HA} + \mathrm{WPEM}^* \nvdash \Sigma_1\text{-DNE}$; to Example 2.11, we have

$$\mathbf{HA} + L(K_2, \leq_2)^* + \Sigma\text{-}T(\mathcal{E}_{\mathcal{K}_2}) \nvdash \Sigma_1\text{-WPEM},$$

especially $\mathbf{HA} + \Sigma\text{-DML} + \Sigma\text{-DNE} \nvdash \Sigma_1\text{-WPEM}$; to Example 2.12, we have

$$\mathbf{HA} + L(K_3, \leq_3)^* + \Sigma\text{-}T(\mathcal{E}_{\mathcal{K}_3}) \nvdash \Sigma_1\text{-DML},$$

especially $\mathbf{HA} + \Sigma\text{-DNE} \nvdash \Sigma_1\text{-DML}$; to Example 2.13, we have

$$\mathbf{HA} + L(K_4, \leq_4)^* + \Sigma\text{-}T(\mathcal{E}_{\mathcal{K}_4}) \nvdash \Sigma_1\text{-WDML},$$

especially $\mathbf{HA} + \Sigma\text{-RPEM} \nvdash \Sigma_1\text{-WDML}$; to Example 2.14, we have

$$\mathbf{HA} + L(K_2, \leq_2)^* \nvdash \Sigma_1\text{-RPEM}.$$

Note that Σ_1-DNE, Σ_1-WPEM, Σ_1-DML, Σ_1-WDML, and Σ_1-RPEM correspond to $\mathrm{MP}_{\mathrm{PR}}$, $\mathrm{WLPO}_{\mathrm{PR}}$, $\mathrm{LLPO}_{\mathrm{PR}}$, $\mathrm{MP}^{\vee}_{\mathrm{PR}}$, and Δ_1-$\mathrm{PEM}_{\mathrm{PR}}$, respectively.

REMARK 4.19. Σ-DNE (respectively, Σ-WPEM, Σ-DML, Σ-WDML, and Σ-RPEM) is stronger than Σ_1-DNE (respectively, Σ_1-WPEM, Σ_1-DML, Σ_1-WDML, and Σ_1-RPEM) in **HA**, and they are equivalent in many-sorted extensions of **HA** with countable choice.

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$$\begin{tikzcd}
 & \mathrm{LPO}_{\mathrm{PR}} \arrow[dl] \arrow[dr] & \\
\mathrm{MP}_{\mathrm{PR}} \arrow[drr, "/" marking] \arrow[ddrr] & & \mathrm{WLPO}_{\mathrm{PR}} \arrow[ll, "/" marking] \arrow[d, shift left] \\
 & & \mathrm{LLPO}_{\mathrm{PR}} \arrow[u, shift left, "/" marking] \arrow[d] \\
 & & \mathrm{MP}^{\vee}_{\mathrm{PR}} \arrow[d, shift left] \\
 & & \Delta_1\text{-}\mathrm{PEM}_{\mathrm{PR}} \arrow[u, shift left, "/" marking]
\end{tikzcd}$$

FIGURE 9. Derivabilities and underivabilities between omniscience principles.

Figure 9 summarizes derivabilities among the omniscience principles in the introduction from which derivability between any pair of principles follows.

We conclude the paper with discussing a possible relativization of the previous results and lifting Theorem 4.17 up to Σ_n-level.

Let $\mathcal{L}_1^Q$ be an extension of $\mathcal{L}_1$ by adding a unary predicate symbol Q. Σ_0^Q- and Σ_1^Q-formulae are similarly defined with extra atomic formulae of the form "$Q(t)$." $\mathbf{HA}^*$ denotes the set of $\mathcal{L}_1^Q$-sentences consisting of axioms and instances of the axiom schema of $\mathbf{HA}$. An $\mathcal{L}_1^Q$-structure is a pair (M, Q^M) where M is an $\mathcal{L}_1$-structure and Q^M is an interpretation of Q on M.

For a set Q_0 of natural numbers, we let

$$\mathbf{HA}^{Q_0} = \mathbf{HA}^* \cup \{Q(\bar{n}) \mid n \in Q_0\} \cup \{\neg Q(\bar{n}) \mid n \notin Q_0\}.$$

Note that $\mathbf{HA}^{Q_0}$ is Σ_1^Q-complete for (ω, Q_0), in other words, $\mathbf{HA}^{Q_0} \vdash_c A$ if and only if $(\omega, Q_0) \models_c A$ for all Σ_1^Q-sentence A. Consider a Σ_1^Q-formula $\tau_{\mathbf{HA}}^Q(x)$ denoting

"x is (a Gödel number of) an $\mathcal{L}_1^Q$-sentence"
$\wedge$ ["x is an axiom or an $\mathcal{L}_1^Q$-instance of the axiom schema of $\mathbf{HA}$"
$\vee \exists y($"x is $Q(\bar{y})$" $\wedge Q(y)) \vee \exists y($"$x$ is $\neg Q(\bar{y})$" $\wedge \neg Q(y))$].

(Here, $\bar{y}$ denotes y-th numeral as in Remark 7.)

Then, $\tau_{\mathbf{HA}}^Q$ (uniformly) represents $\mathbf{HA}^{Q_0}$ over $\mathbf{HA}^{Q_0}$ in the sense that $A \in \mathbf{HA}^{Q_0}$ if and only if $\mathbf{HA}^{Q_0} \vdash_c \tau_{\mathbf{HA}}^Q(\ulcorner A \urcorner)$ for all $\mathcal{L}_1^Q$-sentence A.

Definition 4.20. Let $Q_0 \subseteq \omega$, and let T be a set of $\mathcal{L}_1^Q$-sentences. Then a Σ_1^Q*-representation over* Q_0 of T is a Σ_1^Q-formula $\tau(x)$ such that

$$\mathbf{HA}^* \vdash_c \forall x[(\tau(x) \to \text{“}x \text{ is a sentence”}) \wedge (\tau_{\mathbf{HA}}^Q(x) \to \tau(x))], \text{ and}$$
$$A \in T \text{ if and only if } \mathbf{HA}^{Q_0} \vdash_c \tau(\ulcorner A \urcorner) \text{ (or equivalently } (\omega, Q_0) \models_c \tau(\ulcorner A \urcorner))$$

for all $\mathcal{L}_1^Q$-sentence A. A set of $\mathcal{L}_1^Q$-sentences T is *well-behaved over* Q_0 if T has a Σ_1^Q-representation over Q_0 and $\mathbf{HA}^{Q_0} \subseteq T \subseteq \mathrm{Th}(\omega, Q_0)$. We then identify T ($\supseteq \mathbf{HA}^*$) with its Σ_1^Q-representation. Note that the condition $T \subseteq \mathrm{Th}(\omega, Q_0)$ implies $(\omega, Q_0) \models \mathrm{Con}(T)$.

As same as the usual relativization in computability theory, the discussions and proofs in this section remain valid with the new predicate Q and its interpretation Q_0. In other words, for the extended language $\mathcal{L}_1^Q$, the arithmetized completeness theorem (Theorem 4.5), and its applications, Lemma 4.8, Lemma 4.9, and Proposition 4.16 are all relativizable in the sense that they still hold if we replace $\mathcal{L}_1$, Σ_n, $\mathbf{HA}$, and well-behavedness by $\mathcal{L}_1^Q$, Σ_n^Q, $\mathbf{HA}^{Q_0}$, and well-behavedness over Q_0, respectively. (We will see the relativization of Theorem 4.5 in Appendix B.)

As a typical application of relativization, we consider the case that Q denotes the Σ_n-satisfaction predicate. Then, we have the following generalization of Proposition 4.16.

Definition 4.21. A Σ_n*-satisfaction predicate* is a Σ_n-formula $\mathrm{Sat}_n(e, x)$ such that

$$\mathbf{HA} \vdash_c \forall \vec{x}[A \leftrightarrow \mathrm{Sat}_n(\ulcorner A \urcorner, \langle \vec{x} \rangle)]$$

for all Σ_n-formula A with the free variables $\vec{x} = x_1, \dots, x_n$, where $\langle \vec{x} \rangle = \langle x_1, \dots, x_n \rangle$ (see, e.g., [13, Chapter I, Section 2, 2.55–2.57] for the definition of Σ_n-satisfaction predicates).

Proposition 4.22. *Let* $\mathcal{E}_\mathcal{K} = ((K, \leq), f_\mathcal{K}, (I_\mathcal{K}, \leq_\mathcal{K}))$ *be the extended frame generated by a finite IPC-Kripke model* $\mathcal{K} = (K, \leq, \Vdash)$ *such that* $(I_\mathcal{K}, \leq_\mathcal{K})$ *is a rooted (and finite) tree, and let* $\varphi[p_1, \dots, p_m]$ *be a formula of* $\mathcal{L}(\mathbf{IPC})$*. If* $\mathcal{K} \nVdash \varphi$*, then for each* n

$$\mathcal{K}_{\mathcal{E}_\mathcal{K}, \mathcal{I}} \Vdash_{\mathcal{E}_\mathcal{K}, \mathcal{I}} \mathbf{HA} + \Sigma_n\text{-PEM} \quad \textit{and} \quad \mathcal{K}_{\mathcal{E}_\mathcal{K}, \mathcal{I}} \nVdash_{\mathcal{E}_\mathcal{K}, \mathcal{I}} \Sigma_{n+1}\text{-}\varphi$$

for some $\mathcal{L}_1$*-Kripke model* $\mathcal{I} = (I_\mathcal{K}, \leq_\mathcal{K}, M, \eta, \Vdash')$.

Proof (sketch). Let $Q_0 \subseteq \omega$ be given by

$$Q_0 = \{a \in \omega \mid \omega \models_c \exists e \exists x (a = \langle e, x \rangle \wedge \mathrm{Sat}_n(e, x))\},$$

and define a set T^{Sat_n} of $\mathcal{L}_1^Q$-sentences by

$$T^{\mathrm{Sat}_n} = \mathbf{HA}^{Q_0} \cup \{\forall y [Q(y) \leftrightarrow \exists e \exists x (y = \langle e, x \rangle \wedge \mathrm{Sat}_n(e, x))]\}.$$

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Then T^{Sat_n} is well-behaved over Q_0. We follow the proof of Proposition 4.16. Let $i_0 \in I_{\mathcal{K}}$ be the root of the finite tree $(I_{\mathcal{K}}, \leq_{\mathcal{K}})$. Then, by (the relativization of) Lemma 4.9 with $T_0 = T^{\mathrm{Sat}_n}$, there exists a set T of $\mathcal{L}_1^Q$-sentences well-behaved over Q_0 such that for each Σ_1^Q-sentence C_{i_0} and $\mathcal{L}_1^Q$-structure $(M_{i_0}, Q^{M_{i_0}})$, if

$$(M_{i_0}, Q^{M_{i_0}}) \models_c T + C_{i_0},$$

there exist an $\mathcal{L}_1^Q$-Kripke model $(I_{\mathcal{K}}, \leq_{\mathcal{K}}, \{(M_i, Q^{M_i}) \mid i \in I_{\mathcal{K}}\}, \eta, \Vdash')$ and a family $\{C_i \mid i \in I_{\mathcal{K}}\}$ of Σ_1^Q-sentences satisfying that for each $i \in I$, (1)–(5) of (the relativization of) Lemma 4.9. Let $(M_{i_0}, Q^{M_{i_0}})$ be a model of T, and let $C_{i_0} \equiv 0 = 0$. Then $(M_{i_0}, Q^{M_{i_0}}) \models_c T + C_{i_0}$, and hence there exist an $\mathcal{L}_1^Q$-Kripke model $\mathcal{I} = (I_{\mathcal{K}}, \leq_{\mathcal{K}}, \{(M_i, Q^{M_i}) \mid i \in I_{\mathcal{K}}\}, \eta, \Vdash')$ and a family $\{C_i \mid i \in I_{\mathcal{K}}\}$ of Σ_1^Q-sentences satisfying that (1)–(5) for all $i \in I_{\mathcal{K}}$.

Let $\mathcal{K}_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} = (K, \leq, D, \varepsilon, \Vdash_{\mathcal{E}_{\mathcal{K}},\mathcal{I}})$ be the induced $\mathcal{L}_1^Q$-Kripke model by $\mathcal{I}$. Then, similar to the proof of Proposition 4.16, we have $\mathcal{K}_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} \Vdash_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} \mathbf{HA}$. Moreover, for each Σ_n-formula B with the free variables $\vec{x}$, since

$$M_i \models_c B[\vec{x}/\vec{a}] \Leftrightarrow (M_i, Q^{M_i}) \models_c Q(\langle \ulcorner B \urcorner, \langle \vec{a} \rangle \rangle)$$

for all $i \in I$, by the Σ_0^Q-elementarity of η_{ij}, we have

$$M_i \models_c B[\vec{x}/\vec{a}] \Leftrightarrow M_j \models_c B[\vec{x}/\eta_{ij}(\vec{a})]$$

for all $i, j \in I_{\mathcal{K}}$ with $i \leq_{\mathcal{K}} j$. Therefore for each $k \in K$, Σ_n-formula B with the free variables $\vec{x}$ and $\vec{a} \in M_{f_{\mathcal{K}}(k)}$, by Lemma 4.14, we have

$$\begin{aligned} k \Vdash_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} B[\vec{x}/\vec{a}] &\Leftrightarrow M_{f_{\mathcal{K}}(k)} \models_c B[\vec{x}/\vec{a}] \\ &\Leftrightarrow M_{f_{\mathcal{K}}(k')} \models_c B[\vec{x}/\eta_{f_{\mathcal{K}}(k) f_{\mathcal{K}}(k')}(\vec{a})] \\ &\Leftrightarrow k' \Vdash_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} B[\vec{x}/\eta_{f_{\mathcal{K}}(k) f_{\mathcal{K}}(k')}(\vec{a})] \end{aligned}$$

for all $k' \in K$ with $k \leq k'$. In particular, if $k \nVdash_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} B[\vec{x}/\vec{a}]$, then $k \Vdash_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} \neg B[\vec{x}/\vec{a}]$. Thus $\mathcal{K}_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} \Vdash_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} \Sigma_n$-PEM.

Finally, we show that there exist Σ_{n+1}-sentences $B_1, \dots, B_n$ such that $\mathcal{K}_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} \nVdash_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} \varphi[B_1, \dots, B_n]$. As in the proof of Proposition 4.16, $\mathcal{K}_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} \nVdash_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} \varphi[A_1, \dots, A_n]$ for some Σ_1^Q-sentences $A_1, \dots, A_n$. Hence it suffices to show that for each Σ_1^Q-sentence A there exists a Σ_{n+1}-sentence B such that $\mathcal{K}_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} \Vdash_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} A \leftrightarrow B$. For each Σ_1^Q-sentence A, since $Q(y)$ is equivalent to a Σ_n-formula $\exists e \exists x (y = \langle e, x \rangle \wedge \mathrm{Sat}_n(e, x))$ over T^{Sat_n}, there exists a Σ_{n+1}-sentence B such that $T^{\mathrm{Sat}_n} \vdash_c A \leftrightarrow B$. Therefore for each $k \in K$, since $(M_{f_{\mathcal{K}}(k)}, Q^{M_{f_{\mathcal{K}}(k)}}) \models_c T^{\mathrm{Sat}_n}$, by Lemma 4.14, we have

$$\begin{aligned} k \Vdash_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} A &\Leftrightarrow (M_{f_{\mathcal{K}}(k)}, Q^{M_{f_{\mathcal{K}}(k)}}) \models_c A \\ &\Leftrightarrow (M_{f_{\mathcal{K}}(k)}, Q^{M_{f_{\mathcal{K}}(k)}}) \models_c B \Leftrightarrow k \Vdash_{\mathcal{E}_{\mathcal{K}},\mathcal{I}} B. \end{aligned}$$ ⊣

THEOREM 4.23. *Let $\mathcal{K} = (K, \leq, \Vdash)$ be a finite IPC-Kripke model such that $(I_{\mathcal{K}}, \leq_{\mathcal{K}})$ is a rooted (and finite) tree, and let φ be a formula of $\mathcal{L}(\mathbf{IPC})$. If $\mathcal{K} \nVdash \varphi$ and $\mathcal{E}_{\mathcal{K}}$ is locally directed, then for each n*

$$\mathbf{HA} + \Sigma_n\text{-PEM} + L(K, \leq)^* + \Sigma\text{-}T(\mathcal{E}_{\mathcal{K}}) \nvdash \Sigma_{n+1}\text{-}\varphi.$$

EXAMPLE 4.24. By applying Theorem 4.23 to Example 2.10 with $n \geq 0$, we have

$$\mathbf{HA} + \Sigma_n\text{-PEM} + L(K_1, \leq_1)^* \nvdash \Sigma_{n+1}\text{-DNE},$$

especially $\mathbf{HA} + \Sigma_n\text{-PEM} + \text{WPEM}^* \nvdash \Sigma_{n+1}\text{-DNE}$; to Example 2.11, we have

$$\mathbf{HA} + \Sigma_n\text{-PEM} + L(K_2, \leq_2)^* + \Sigma\text{-}T(\mathcal{E}_{\mathcal{K}_2}) \nvdash \Sigma_{n+1}\text{-WPEM},$$

especially $\mathbf{HA} + \Sigma_n\text{-PEM} + \Sigma\text{-DML} + \Sigma\text{-DNE} \nvdash \Sigma_{n+1}\text{-WPEM}$; to Example 2.12, we have

$$\mathbf{HA} + \Sigma_n\text{-PEM} + L(K_3, \leq_3)^* + \Sigma\text{-}T(\mathcal{E}_{\mathcal{K}_3}) \nvdash \Sigma_{n+1}\text{-DML},$$

especially $\mathbf{HA} + \Sigma_n\text{-PEM} + \Sigma\text{-DNE} \nvdash \Sigma_{n+1}\text{-DML}$; to Example 2.13, we have

$$\mathbf{HA} + \Sigma_n\text{-PEM} + L(K_4, \leq_4)^* + \Sigma\text{-}T(\mathcal{E}_{\mathcal{K}_4}) \nvdash \Sigma_{n+1}\text{-WDML},$$

especially $\mathbf{HA} + \Sigma_n\text{-PEM} + \Sigma\text{-RPEM} \nvdash \Sigma_{n+1}\text{-WDML}$; to Example 2.14, we have

$$\mathbf{HA} + \Sigma_n\text{-PEM} + L(K_2, \leq_2)^* \nvdash \Sigma_{n+1}\text{-RPEM}.$$

Note that Σ-DNE (respectively, Σ-WPEM, Σ-DML, Σ-WDML, and Σ-RPEM) implies Σ_{n+1}-DNE (respectively, Σ_{n+1}-WPEM, Σ_{n+1}-DML, Σ_{n+1}-WDML, and Σ_{n+1}-RPEM) in $\mathbf{HA} + \Sigma_n$-PEM.

§5. Concluding remarks. In this paper, we have developed a general technique to separate omniscience principles over **HA** by constructing a Kripke model of **HA** from an IPC-Kripke model. Example 4.24 shows that all the separation results in [1] obtained by using several different kinds of functional interpretations can be proved uniformly by applying Theorem 4.23 to simple IPC-Kripke models; for some further separation results among weaker principles by using Theorem 4.17, see [8].

On the other hand, our technique is not totally universal. According to the recent study of the hierarchical structure of the logical principles restricted to prenex formulae (including the principles studied in [1, 8, 9]) by Fujiwara and Kurahashi [11], some principles in the $(n + 1)$-th hierarchy are mutually equivalent in the presence of DNE or the *double negation shift* (DNS):

$$\forall x \neg\neg v(x) \to \neg\neg\forall x v(x),$$

in the n-th hierarchy. For example, Σ_{n+1}-DML is equivalent to $(\Pi_{n+1} \vee \Pi_{n+1})$-DNE in the presence of Σ_n-DNE, but it is still open whether Σ_{n+1}-DML implies $(\Pi_{n+1} \vee \Pi_{n+1})$-DNE over **HA** (cf. [11, Figure 3]). Since the relativized theory already contains Σ_n-PEM (which is stronger than Σ_n-DNE and Σ_n-DNS), our Theorem 4.23 does not provide separations of equivalent principles, such as Σ_{n+1}-DML and $(\Pi_{n+1} \vee \Pi_{n+1})$-DNE, in the presence of Σ_n-PEM.

Furthermore, our technique is available only for a separation of logical principles which are obtained from those in propositional logic by substituting propositional variables by predicate formulae, and therefore the fragments of DNS (investigated in [10]) are outside of the range of our technique.

It is also remarkable that separation of two propositional theories by an IPC-Kripke model does not necessarily induce a separation of their Σ_1-substitution instances in **HA**. Consider the following propositional formulae:

$$\begin{aligned}\mathrm{LIN}[p,q] &\equiv (p \to q) \vee (q \to p),\\ \mathrm{LIN}'[p,q] &\equiv (p \to \neg q) \vee (\neg q \to p).\end{aligned}$$

Example 5.1. We show the following.

1. $\mathrm{LIN}' \nvdash_{\mathbf{IPC}} \mathrm{LIN}$;
2. $\Sigma\text{-LIN}' \nvdash_{\mathbf{IQC}} \Sigma\text{-LIN}$;
3. $\mathbf{HA} + \Sigma_1\text{-LIN}' \vdash \Sigma_1\text{-LIN}$.

For (1) and (2), consider an IPC-Kripke model $\mathcal{K}_6 = (K_6, \leq_6, \Vdash_6)$ given in Figure 10. Then $\mathcal{K}_6 \nVdash_6 \mathrm{LIN}[p,q]$. On the other hand, LIN' is valid on the Kripke frame $(K_6, \leq_6)$. Therefore we have $\mathrm{LIN}' \nvdash_{\mathbf{IPC}} \mathrm{LIN}$. Furthermore, since the extended frame $\mathcal{E}_{\mathcal{K}_6}$ generated by $\mathcal{K}_6$ is locally directed (cf. Figure 10), we have

$$L(K_6, \leq_6)^* + \Sigma\text{-}T(\mathcal{E}_{\mathcal{K}_6}) \nvdash_{\mathbf{IQC}} \Sigma\text{-LIN},$$

by Theorem 3.15. Since $T(\mathcal{E}_{\mathcal{K}_6})$ contains $\mathrm{LIN}'[p,q]$ for all p and q, we have $\Sigma\text{-LIN}' \nvdash_{\mathbf{IQC}} \Sigma\text{-LIN}$. Note that the Kripke frame $(I_{\mathcal{K}_6}, \leq_{\mathcal{K}_6})$ generated by $\mathcal{K}_6$ is not a tree, and Theorem 4.17 is not applicable.

For (3), first note that, since an instance

$$(p \to \neg p) \vee (\neg p \to p)$$

of LIN' is equivalent to $\mathrm{WPEM}[p]$, Σ_1-LIN' implies Σ_1-WPEM. Assume that Σ_1-LIN'. Then to show Σ_1-LIN in **HA**, it suffices to prove that

$$(\exists x\,(s(x) = 0) \to \exists x\,(t(x) = 0)) \vee (\exists x\,(t(x) = 0) \to \exists x\,(s(x) = 0)),$$

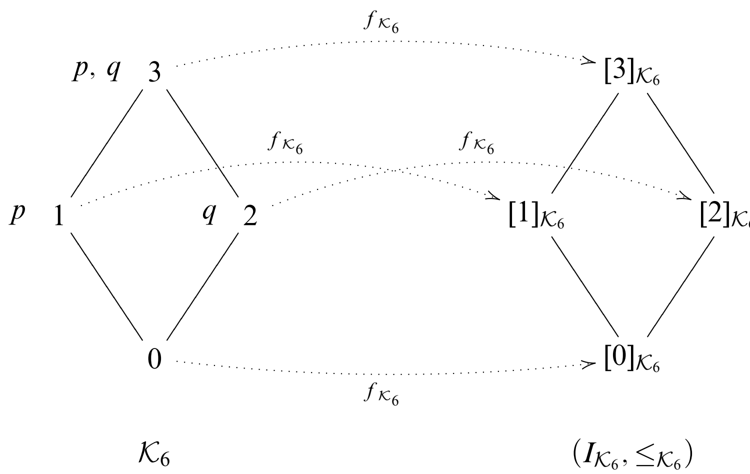

FIGURE 10. The Kripke model and the extended frame in Example 5.1.

where s and t are (primitive recursive) terms of **HA**. Consider terms s and t of **HA**, and let $A(x)$ and $B(x)$ be the following formulae:

$$\begin{aligned} A(x) &\equiv \quad s(x) = 0 \wedge \forall y \leq x\, (t(y) \neq 0), \\ B(x) &\equiv \quad t(x) = 0 \wedge \forall y \leq x\, (s(y) \neq 0). \end{aligned}$$

Then

$$\mathbf{HA} \vdash \neg(\exists x A(x) \wedge \exists x B(x)).$$

Since $\neg\exists x A(x) \vee \neg\neg\exists x A(x)$ and $\neg\exists x B(x) \vee \neg\neg\exists x B(x)$, by Σ_1-WPEM, we have

$$\neg\exists x A(x) \vee \neg\exists x B(x).$$

In the former case, if $s(x) = 0$, then $\neg\forall y \leq x\, (t(y) \neq 0)$, that is,

$$\exists y \leq x\, (t(y) = 0);$$

hence $\exists x\, (s(x) = 0) \rightarrow \exists x\, (t(x) = 0)$. In the latter case, similarly we have $\exists x\, (t(x) = 0) \rightarrow \exists x\, (s(x) = 0)$.

Interestingly, our technique can be applied even to an IPC-Kripke model, which does not separate propositional principles, for separating their substitution instances in **IQC** or **HA**.

EXAMPLE 5.2. We show the following.

1. LIN $\vdash_{\mathbf{IPC}}$ LIN$'$;
2. Σ-LIN $\nvdash_{\mathbf{IQC}}$ Σ-LIN$'$;
3. $\mathbf{HA} + \Sigma_1$-LIN $\nvdash \Sigma_1$-LIN$'$.

Note that, since LIN$'[p, q]$ is an instance of LIN$[p, q]$, (1) is trivial.

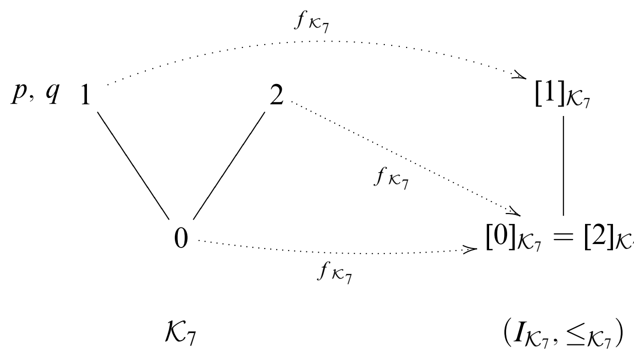

FIGURE 11. The Kripke model and the extended frame in Example 5.2.

For (2), let $\mathcal{E}_{\mathcal{K}_7} = ((K_2, \leq_2), f_{\mathcal{K}_7}, (I_{\mathcal{K}_7}, \leq_{\mathcal{K}_7}))$ be the extended frame generated by the IPC-Kripke model $\mathcal{K}_7 = (K_2, \leq_2, \Vdash_7)$ given in Figure 11, where $(K_2, \leq_2)$ is the Kripke frame in Example 2.11. Then it is easy to see $\mathcal{K}_7 \nVdash_7 \mathrm{LIN}'[p, q]$. Since the extended frame $\mathcal{E}_{\mathcal{K}_7}$ is identical with $\mathcal{E}_{\mathcal{K}_2}$, which is locally directed, we have

$$L(K_2, \leq_2)^* + \Sigma\text{-}T(\mathcal{E}_{\mathcal{K}_7}) \nvdash_{\mathbf{IQC}} \Sigma\text{-LIN}',$$

by Theorem 3.15. To prove that $\Sigma\text{-LIN} \nvdash_{\mathbf{IQC}} \Sigma\text{-LIN}'$, it suffices to show that $T(\mathcal{E}_{\mathcal{K}_7})$ contains $\mathrm{LIN}[p, q]$ for all p and q. Consider an IPC-Kripke model $\mathcal{I} = (I_{\mathcal{K}_7}, \leq_{\mathcal{K}_7}, \Vdash')$.

Then either $0 \Vdash_{\mathcal{E}_{\mathcal{K}_7}, \mathcal{I}} p$, $0 \Vdash_{\mathcal{E}_{\mathcal{K}_7}, \mathcal{I}} q$ or $0 \nVdash_{\mathcal{E}_{\mathcal{K}_7}, \mathcal{I}} p, q$. In the first and second cases, we have $0 \Vdash_{\mathcal{E}_{\mathcal{K}_7}, \mathcal{I}} q \to p$ and $0 \Vdash_{\mathcal{E}_{\mathcal{K}_7}, \mathcal{I}} p \to q$, respectively. In the third case, either $1 \nVdash_{\mathcal{E}_{\mathcal{K}_7}, \mathcal{I}} p$, $1 \nVdash_{\mathcal{E}_{\mathcal{K}_7}, \mathcal{I}} q$ or $1 \Vdash_{\mathcal{E}_{\mathcal{K}_7}, \mathcal{I}} p, q$; in the first and second cases, since $2 \nVdash_{\mathcal{E}_{\mathcal{K}_7}, \mathcal{I}} p, q$, we have $0 \Vdash_{\mathcal{E}_{\mathcal{K}_7}, \mathcal{I}} p \to q$ and $0 \Vdash_{\mathcal{E}_{\mathcal{K}_7}, \mathcal{I}} q \to p$, respectively; in the third case, we have $0 \Vdash_{\mathcal{E}_{\mathcal{K}_7}, \mathcal{I}} p \to q$, $q \to p$. Therefore, in any case, we have $0 \Vdash_{\mathcal{E}_{\mathcal{K}_7}, \mathcal{I}} (p \to q) \vee (q \to p)$.

For (3), since $(I_{\mathcal{K}_7}, \leq_{\mathcal{K}_7})$ is a rooted tree, applying Theorem 4.17, we have $\mathbf{HA} + \Sigma_1\text{-LIN} \nvdash \Sigma_1\text{-LIN}'$.

To conclude the paper, we briefly examine the following assumptions on a Kripke model $\mathcal{K} = (K, \leq, \Vdash)$ in Theorem 4.17 and Theorem 4.23, respectively:

1. The Kripke frame $(I_{\mathcal{K}}, \leq_{\mathcal{K}})$ generated by $\mathcal{K}$ is a rooted finite tree.
2. The extended frame $\mathcal{E}_{\mathcal{K}}$ generated by $\mathcal{K}$ is locally directed.

The first and second assumptions are crucial for our constructions of Kripke models of **HA** in Proposition 4.16 and of IQC-Kripke models in Lemma 3.13, respectively. Although, as we have seen in Examples 2.10–2.14 and 5.2, many useful IPC-Kripke models enjoy these assumptions, we cannot remove the first assumption from Theorem 4.17. In fact, if we omitted it in Theorem 4.17, since the locally directed extended frame $\mathcal{E}_{\mathcal{K}_6}$ in Example 5.1 separates LIN from LIN$'$, we could have

$$\mathbf{HA} + \Sigma_1\text{-LIN}' \nvdash \Sigma_1\text{-LIN},$$

which contradicts Example 5.1(3). Note that the Kripke frame $(I_{\mathcal{K}_6}, \leq_{\mathcal{K}_6})$ generated by $\mathcal{K}_6$ has a confluent point, and is not a tree. On the other hand, we still do not know whether the second assumption is essential or not for Theorem 4.17.

Appendix A. Proofs of Lemmas 4.11 and 4.14.

LEMMA A.1. *Let $\mathcal{E} = ((K, \leq), f, (I, \leq_I))$ be an extended frame such that $(K, \leq)$ is finite, and let $\mathcal{K}_{\mathcal{E},\mathcal{I}} = (K, \leq, D, \varepsilon, \Vdash_{\mathcal{E},\mathcal{I}})$ be the induced IQC-Kripke model by an IQC-Kripke model $\mathcal{I} = (I, \leq_I, M, \eta, \Vdash)$. Assume that for each $i, i' \in I$ with $i \leq i'$ and formula B, there exists a formula $[B]_i^{i'}$ such that $\mathrm{FV}([B]_i^{i'}) = \mathrm{FV}(B)$ and*

$$M_i \models_c [B]_i^{i'}[\vec{x}/\vec{a}] \Leftrightarrow M_{i'} \models_c B[\vec{x}/\eta_{ii'}(\vec{a})]$$

for all $\vec{a} \in M_i$, where $\vec{x}$ are the free variables of B. Then for each $k \in K$ and formula A, there exists a formula $\hat{A}_k$ such that $\mathrm{FV}(\hat{A}_k) = \mathrm{FV}(A)$ and

$$k \Vdash_{\mathcal{E},\mathcal{I}} A[\vec{x}/\vec{a}] \Leftrightarrow D_k \models_c \hat{A}_k[\vec{x}/\vec{a}]$$

for all $\vec{a} \in D_k$, where $\vec{x}$ are the free variables of A.

PROOF. Given a $k \in K$, define $\hat{A}_k$ by induction on the complexity of a formula A as follows:

- $\hat{A}_k \equiv A$ for A prime;
- $\hat{A}_k \equiv \hat{B}_k \wedge \hat{C}_k$ for $A \equiv B \wedge C$;
- $\hat{A}_k \equiv \hat{B}_k \vee \hat{C}_k$ for $A \equiv B \vee C$;
- $\hat{A}_k \equiv \bigwedge_{k' \geq k}([\hat{B}_{k'} \to \hat{C}_{k'}]_{f(k)}^{f(k')})$ for $A \equiv B \to C$;
- $\hat{A}_k \equiv \exists y \hat{B}_k$ for $A \equiv \exists y B$;
- $\hat{A}_k \equiv \bigwedge_{k' \geq k}([\forall y \hat{B}_{k'}]_{f(k)}^{f(k')})$ for $A \equiv \forall y B$.

Then it is straightforward to show that

$$k \Vdash_{\mathcal{E},\mathcal{I}} A[\vec{x}/\vec{a}] \Leftrightarrow D_k \models_c \hat{A}_k[\vec{x}/\vec{a}],$$

for all $k \in K$ and $\vec{a} \in D_k$, by induction on the complexity of A except the cases for $\to$ and $\forall$. For $\to$, we have

$$\begin{aligned} k \Vdash_{\mathcal{E},\mathcal{I}} A[\vec{x}/\vec{a}] &\Leftrightarrow \forall k' \geq k (k' \Vdash_{\mathcal{E},\mathcal{I}} B[\vec{x}/\varepsilon_{kk'}(\vec{a})] \Rightarrow k' \Vdash_{\mathcal{E},\mathcal{I}} C[\vec{x}/\varepsilon_{kk'}(\vec{a})]) \\ &\Leftrightarrow \forall k' \geq k (D_{k'} \models_c \hat{B}_{k'}[\vec{x}/\varepsilon_{kk'}(\vec{a})] \Rightarrow D_{k'} \models_c \hat{C}_{k'}[\vec{x}/\varepsilon_{kk'}(\vec{a})]) \end{aligned}$$

$$
\begin{aligned}
&\Leftrightarrow \forall k' \geq k(D_{k'} \models_c (\hat{B}_{k'} \to \hat{C}_{k'})[\vec{x}/\varepsilon_{kk'}(\vec{a})]) \\
&\Leftrightarrow \forall k' \geq k(M_{f(k')} \models_c (\hat{B}_{k'} \to \hat{C}_{k'})[\vec{x}/\eta_{f(k)f(k')}(\vec{a})]) \\
&\Leftrightarrow \forall k' \geq k(M_{f(k)} \models_c [\hat{B}_{k'} \to \hat{C}_{k'}]^{f(k')}_{f(k)}[\vec{x}/\vec{a}]) \\
&\Leftrightarrow M_{f(k)} \models_c \left(\bigwedge_{k' \geq k} [\hat{B}_{k'} \to \hat{C}_{k'}]^{f(k')}_{f(k)} \right) [\vec{x}/\vec{a}] \\
&\Leftrightarrow D_k \models_c \hat{A}_k[\vec{x}/\vec{a}],
\end{aligned}
$$

for all $\vec{a} \in D_k$. For $\forall$, we have

$$
\begin{aligned}
k \Vdash_{\mathcal{E},\mathcal{I}} A[\vec{x}/\vec{a}] &\Leftrightarrow \forall k' \geq k \forall b \in D_{k'}(k' \Vdash_{\mathcal{E},\mathcal{I}} B[\vec{x}, y/\varepsilon_{kk'}(\vec{a}), b]) \\
&\Leftrightarrow \forall k' \geq k \forall b \in D_{k'}(D_{k'} \models_c \hat{B}_{k'}[\vec{x}, y/\varepsilon_{kk'}(\vec{a}), b]) \\
&\Leftrightarrow \forall k' \geq k(D_{k'} \models_c (\forall y \hat{B}_{k'})[\vec{x}/\varepsilon_{kk'}(\vec{a})]) \\
&\Leftrightarrow \forall k' \geq k(M_{f(k')} \models_c (\forall y \hat{B}_{k'})[\vec{x}/\eta_{f(k)f(k')}(\vec{a})]) \\
&\Leftrightarrow \forall k' \geq k(M_{f(k)} \models_c [\forall y \hat{B}_{k'}]^{f(k')}_{f(k)}[\vec{x}/\vec{a}]) \\
&\Leftrightarrow M_{f(k)} \models_c \left(\bigwedge_{k' \geq k} [\forall y \hat{B}_{k'}]^{f(k')}_{f(k)} \right) [\vec{x}/\vec{a}] \\
&\Leftrightarrow D_k \models_c \hat{A}_k[\vec{x}/\vec{a}],
\end{aligned}
$$

for all $a \in D_k$. $\dashv$

Recall the definition of the classes $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ of schemata in Definition 4.10.

LEMMA A.2. *Let $\mathcal{K} = (K, \leq, D, \Vdash, \varepsilon)$ be an IQC-Kripke model. Assume that for each $k \in K$ and formula A, there exists a formula $\hat{A}_k$ such that* $\mathrm{FV}(\hat{A}_k) = \mathrm{FV}(A)$ *and*

$$k \Vdash A[\vec{x}/\vec{a}] \Leftrightarrow D_k \models_c \hat{A}_k[\vec{x}/\vec{a}]$$

for all $\vec{a} \in D_k$, where $\vec{x}$ are the free variables of A. Then for each formula $A^1, \dots, A^n$, $k \in K$, and $\vec{a} \in D_k$,

1. *if $\alpha \in \mathcal{A}$, then*

$$\exists k' \geq k(k' \Vdash (\alpha[A^1, \dots, A^n])[\vec{x}/\varepsilon_{kk'}(\vec{a})]) \Rightarrow D_k \models_c (\alpha[\hat{A}^1_k, \dots, \hat{A}^n_k])[\vec{x}/\vec{a}],$$

2. *if $\beta \in \mathcal{B}$, then*

$$k \Vdash (\beta[A^1, \dots, A^n])[\vec{x}/\vec{a}] \Rightarrow D_k \models_c (\beta[\hat{A}^1_k, \dots, \hat{A}^n_k])[\vec{x}/\vec{a}],$$

3. *if $\gamma \in \mathcal{C}$, then*

$$D_k \models_c (\gamma[\hat{A}^1_k, \dots, \hat{A}^n_k])[\vec{x}/\vec{a}] \Rightarrow k \Vdash (\gamma[A^1, \dots, A^n])[\vec{x}/\vec{a}],$$

4. *if* $\delta \in \mathcal{D}$, *then*

$$\forall k' \geq k(D_{k'} \models_c (\delta[\hat{A}^1_{k'}, \dots, \hat{A}^n_{k'}])[\vec{x}/\varepsilon_{kk'}(\vec{a})]) \Rightarrow k \Vdash (\delta[A^1, \dots, A^n])[\vec{x}/\vec{a}],$$

where $\vec{x}$ *are the free variables of* $\alpha[A^1, \dots, A^n]$ (*respectively,* $\beta[A^1, \dots, A^n]$, $\gamma[A^1, \dots, A^n]$ *and* $\delta[A^1, \dots, A^n]$).

PROOF. By simultaneous induction on $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$.
Basis. It is straightforward using the assumption.
Induction step. Since the cases for the propositional operators $\wedge$ and $\vee$, and the quantifier $\exists$ are straightforward, we review the propositional operator $\rightarrow$ and the quantifier $\forall$.
Case 1: $\forall y\alpha \in \mathcal{A}$. Suppose that $k' \Vdash ((\forall y\alpha)[A^1, \dots, A^n])[\vec{x}/\varepsilon_{kk'}(\vec{a})]$ for some $k' \geq k$. Then, since $k' \Vdash (\forall y(\alpha[A^1, \dots, A^n]))[\vec{x}/\varepsilon_{kk'}(\vec{a})]$, we have $k' \Vdash (\alpha[A^1, \dots, A^n])[\vec{x}, y/\varepsilon_{kk'}(\vec{a}), c]$ for all $c \in D_{k'}$, and hence

$$k' \Vdash (\alpha[A^1, \dots, A^n])[\vec{x}, y/\varepsilon_{kk'}(\vec{a}), \varepsilon_{kk'}(b)]$$

for all $b \in D_k$. By the induction hypothesis, we have $D_k \models_c (\alpha[\hat{A}^1_k, \dots, \hat{A}^n_k])[\vec{x}, y/\vec{a}, b]$ for all $b \in D_k$, and so $D_k \models_c (\forall y(\alpha[\hat{A}^1_k, \dots, \hat{A}^n_k]))[\vec{x}/\vec{a}]$. Thus $D_k \models_c ((\forall y\alpha)[\hat{A}^1_k, \dots, \hat{A}^n_k])[\vec{x}/\vec{a}]$.
Case 2: $\gamma \rightarrow \alpha \in \mathcal{A}$. Suppose that $k' \Vdash ((\gamma \rightarrow \alpha)[A^1, \dots, A^n])[\vec{x}/\varepsilon_{kk'}(\vec{a})]$ for some $k' \geq k$. Then

$$k' \Vdash (\gamma[A^1, \dots, A^n])[\vec{x}/\varepsilon_{kk'}(\vec{a})] \rightarrow (\alpha[A^1, \dots, A^n])[\vec{x}/\varepsilon_{kk'}(\vec{a})].$$

Assume that $D_k \models_c (\gamma[\hat{A}^1_k, \dots, \hat{A}^n_k])[\vec{x}/\vec{a}]$. By the induction hypothesis, we have $k \Vdash (\gamma[A^1, \dots, A^n])[\vec{x}/\vec{a}]$, and hence $k' \Vdash (\gamma[A^1, \dots, A^n])[\vec{x}/\varepsilon_{kk'}(\vec{a})]$. Therefore $k' \Vdash (\alpha[A^1, \dots, A^n])[\vec{x}/\varepsilon_{kk'}(\vec{a})]$. By the induction hypothesis, we have

$$D_k \models_c (\alpha[\hat{A}^1_k, \dots, \hat{A}^n_k])[\vec{x}/\vec{a}].$$

Thus $D_k \models_c ((\gamma \rightarrow \alpha)[\hat{A}^1_k, \dots, \hat{A}^n_k])[\vec{x}/\vec{a}]$.
Case 3: $\forall y\beta \in \mathcal{B}$. Suppose that $k \Vdash ((\forall y\beta)[A^1, \dots, A^n])[\vec{x}/\vec{a}]$. Then, since $k \Vdash \forall y((\beta[A^1, \dots, A^n])[\vec{x}/\vec{a}])$, we have

$$k \Vdash (\beta[A^1, \dots, A^n])[\vec{x}, y/\vec{a}, b]$$

for all $b \in D_k$. By the induction hypothesis, we have $D_k \models_c (\beta[\hat{A}^1_k, \dots, \hat{A}^n_k])[\vec{x}, y/\vec{a}, b]$ for all $b \in D_k$. Therefore

$$D_k \models_c ((\forall y\beta)[\hat{A}^1_k, \dots, \hat{A}^n_k])[\vec{x}/\vec{a}].$$

Case 4: $\gamma \rightarrow \beta \in \mathcal{B}$. Suppose that $k \Vdash ((\gamma \rightarrow \beta)[A^1, \dots, A^n])[\vec{x}/\vec{a}]$. Then

$$k \Vdash (\gamma[A^1, \dots, A^n])[\vec{x}/\vec{a}] \rightarrow (\beta[A^1, \dots, A^n])[\vec{x}/\vec{a}].$$

Assume that $D_k \models_c (\gamma[\hat{A}^1_k, \dots, \hat{A}^n_k])[\vec{x}/\vec{a}]$. By the induction hypothesis, we have $k \Vdash (\gamma[A^1, \dots, A^n])[\vec{x}/\vec{a}]$, and hence $k \Vdash (\beta[A^1, \dots, A^n])[\vec{x}/\vec{a}]$. By the induction hypothesis, we have $D_k \models_c (\beta[\hat{A}^1_k, \dots, \hat{A}^n_k])[\vec{x}/\vec{a}]$. Thus

$$D_k \models_c ((\gamma \to \beta)[\hat{A}^1_k, \dots, \hat{A}^n_k])[\vec{x}/\vec{a}].$$

Case 5: $\alpha \to \gamma \in \mathcal{C}$. Suppose that $D_k \models_c ((\alpha \to \gamma)[\hat{A}^1_k, \dots, \hat{A}^n_k])[\vec{x}/\vec{a}]$. Then

$$D_k \models_c (\alpha[\hat{A}^1_k, \dots, \hat{A}^n_k])[\vec{x}/\vec{a}] \to (\gamma[\hat{A}^1_k, \dots, \hat{A}^n_k])[\vec{x}/\vec{a}].$$

Consider $k' \geq k$, and assume that $k' \Vdash (\alpha[A^1, \dots, A^n])[\vec{x}/\varepsilon_{kk'}(\vec{a})]$. By the induction hypothesis, we have $D_k \models_c (\alpha[\hat{A}^1_k, \dots, \hat{A}^n_k])[\vec{x}/\vec{a}]$, and hence

$$D_k \models_c (\gamma[\hat{A}^1_k, \dots, \hat{A}^n_k])[\vec{x}/\vec{a}].$$

By the induction hypothesis, we have $k \Vdash (\gamma[A^1, \dots, A^n])[\vec{x}/\vec{a}]$, and so $k' \Vdash (\gamma[A^1, \dots, A^n])[\vec{x}/\varepsilon_{kk'}(\vec{a})]$. Thus

$$k \Vdash ((\alpha \to \gamma)[A^1, \dots, A^n])[\vec{x}/\vec{a}].$$

Case 6: $\forall y\delta \in \mathcal{D}$. Suppose that $D_{k'} \models_c ((\forall y\delta)[\hat{A}^1_{k'}, \dots, \hat{A}^n_{k'}])[\vec{x}/\varepsilon_{kk'}(\vec{a})]$ for all $k' \geq k$. Then $D_{k'} \models_c (\delta[\hat{A}^1_{k'}, \dots, \hat{A}^n_{k'}])[\vec{x}, y/\varepsilon_{kk'}(\vec{a}), c]$ for all $k' \geq k$ and $c \in D_{k'}$. Therefore if $k' \geq k$ and $c \in D_{k'}$, then

$$D_{k''} \models_c (\delta[\hat{A}^1_{k''}, \dots, \hat{A}^n_{k''}])[\vec{x}, y/\varepsilon_{k'k''}(\varepsilon_{kk'}(\vec{a})), \varepsilon_{k'k''}(c)]$$

for all $k'' \geq k'$. By the induction hypothesis, we have $k' \Vdash (\delta[A^1, \dots, A^n])[\vec{x}, y/\varepsilon_{kk'}(\vec{a}), c]$. Thus $k \Vdash ((\forall y\delta)[A^1, \dots, A^n])[\vec{x}/\vec{a}]$.
Case 7: $\beta \to \delta \in \mathcal{D}$. Suppose that $D_{k'} \models_c ((\beta \to \delta)[\hat{A}^1_{k'}, \dots, \hat{A}^n_{k'}])[\vec{x}/\varepsilon_{kk'}(\vec{a})]$ for all $k' \geq k$. Then

$$D_{k'} \models_c (\beta[\hat{A}^1_{k'}, \dots, \hat{A}^n_{k'}])[\vec{x}/\varepsilon_{kk'}(\vec{a})] \to (\delta[\hat{A}^1_{k'}, \dots, \hat{A}^n_{k'}])[\vec{x}/\varepsilon_{kk'}(\vec{a})]$$

for all $k' \geq k$. Consider $k' \geq k$, and assume that

$$k' \Vdash (\beta[A^1, \dots, A^n])[\vec{x}/\varepsilon_{kk'}(\vec{a})].$$

Then $k'' \Vdash (\beta[A^1, \dots, A^n])[\vec{x}/\varepsilon_{k'k''}(\varepsilon_{kk'}(\vec{a}))]$. for all $k'' \geq k'$. By the induction hypothesis, we have $D_{k''} \models_c (\beta[\hat{A}^1_{k''}, \dots, \hat{A}^n_{k''}])[\vec{x}/\varepsilon_{k'k''}(\varepsilon_{kk'}(\vec{a}))]$ for all $k'' \geq k'$. Therefore $D_{k''} \models_c (\delta[\hat{A}^1_{k''}, \dots, \hat{A}^n_{k''}])[\vec{x}/\varepsilon_{k'k''}(\varepsilon_{kk'}(\vec{a}))]$ for all $k'' \geq k'$. By the induction hypothesis, we have $k' \Vdash (\delta[A^1, \dots, A^n])[\vec{x}/\varepsilon_{kk'}(\vec{a})]$. Thus $k \Vdash ((\beta \to \delta)[A^1, \dots, A^n])[\vec{x}/\vec{a}]$. ⊣

PROOF OF LEMMA 4.11. Consider a schema $\delta[v_1, \dots, v_n]$ in Γ. By Lemma A.1, for each $k \in K$ and formulae $A^1, \dots, A^n$, there exist formulae $\hat{A}^1_k, \dots, \hat{A}^n_k$ such that

$$k \Vdash_{\mathcal{E},\mathcal{I}} A^i[\vec{x}/\vec{a}] \Leftrightarrow M_{f(k)} \models_c \hat{A}^i_k[\vec{x}/\vec{a}]$$

https://doi.org/10.1017/bsl.2023.29 Published online by Cambridge University Press

for all $i \in \{1, \dots, n\}$ and $\vec{a} \in M_{f(k)}$. Since

$$M_{f(k)} \models_c (\delta[\hat{A}_k^1, \dots, \hat{A}_k^n)[\vec{x}/\vec{a}]$$

for all $k \in K$ and $\vec{a} \in M_{f(k)}$, by Lemma A.2, we have

$$k \Vdash (\delta[A^1, \dots, A^n])[\vec{x}/\vec{a}]$$

for all $\vec{a} \in M_{f(k)}$. ⊣

Proof of Lemma 4.14. It suffices to show that if $B \in \Gamma$, then

$$k \Vdash_{\mathcal{E},\mathcal{I}} B[\vec{x}/\vec{a}] \Leftrightarrow M_{f(k)} \models_c B[\vec{x}/\vec{a}]$$

for all $i \in I$ and $\vec{a} \in M_{f(k)}$. We proceed by induction on the complexity of B, and review only the cases for $\to$ and $\forall$. If $B \equiv B' \to B''$, by the induction hypothesis and the assumption, we have

$$\begin{aligned}
k \Vdash_{\mathcal{E},\mathcal{I}} B[\vec{x}/\vec{a}] &\Leftrightarrow \forall k' \geq k(k' \Vdash_{\mathcal{E},\mathcal{I}} B'[\vec{x}/\varepsilon_{kk'}(\vec{a})] \Rightarrow k' \Vdash_{\mathcal{E},\mathcal{I}} B''[\vec{x}/\varepsilon_{kk'}(\vec{a})]) \\
&\Leftrightarrow \forall k' \geq k(M_{f(k')} \models_c B'[\vec{x}/\eta_{f(k)f(k')}(\vec{a})] \\
&\qquad\qquad\qquad\qquad \Rightarrow M_{f(k')} \models_c B''[\vec{x}/\eta_{f(k)f(k')}(\vec{a})]) \\
&\Leftrightarrow \forall k' \geq k(M_{f(k')} \models_c B[\vec{x}/\eta_{f(k)f(k')}(\vec{a})]) \\
&\Leftrightarrow M_{f(k)} \models_c B[\vec{x}/\vec{a}].
\end{aligned}$$

If $B \equiv \forall y B'$, again by the induction hypothesis and the assumption, we have

$$\begin{aligned}
k \Vdash_{\mathcal{E},\mathcal{I}} B[\vec{x}/\vec{a}] &\Leftrightarrow \forall k' \geq k \forall b \in D_{k'}(k' \Vdash_{\mathcal{E},\mathcal{I}} B'[\vec{x}, y/\varepsilon_{kk'}(\vec{a}), b]) \\
&\Leftrightarrow \forall k' \geq k \forall b \in M_{f(k')}(M_{f(k')} \models_c B'[\vec{x}, y/\eta_{f(k)f(k')}(\vec{a}), b]) \\
&\Leftrightarrow \forall k' \geq k(M_{f(k')} \models_c B[\vec{x}/\eta_{f(k)f(k')}(\vec{a})]) \\
&\Leftrightarrow M_{f(k)} \models_c B[\vec{x}/\vec{a}].
\end{aligned}$$

⊣

Appendix B. The relativization of the arithmetized completeness theorem. Recall that $\mathcal{L}_1^Q$ is an extension of $\mathcal{L}_1$ by adding a unary predicate symbol Q and an $\mathcal{L}_1^Q$-structure is a pair (M, Q^M) where M is an $\mathcal{L}_1$-structure and Q^M is an interpretation of Q on M. Thanks to $\mathcal{L}_1^Q$-instances of the induction axiom schema in $\mathbf{HA}^*$, most proofs and discussions for $\mathcal{L}_1$-statements and structures within $\mathbf{HA}$ can be carried out analogously for $\mathcal{L}_1^Q$ within $\mathbf{HA}^*$.

Theorem B.1 (Arithmetized completeness theorem—relativized). *Let $Q_0 \subseteq \omega$, let T be a set of $\mathcal{L}_1^Q$-sentences which is well-behaved over Q_0, and let (M, Q^M) be an $\mathcal{L}_1^Q$-structure such that*

$$(M, Q^M) \models_c \mathbf{HA}^{Q_0} + \mathrm{Con}(T).$$

Then there exist a definable $\mathcal{L}_1^Q$-structure $(\hat{M}, Q^{\hat{M}})$ in (M, Q^M) and a Σ_0^Q-elementary embedding $\theta : M \to \hat{M}$ such that $(\hat{M}, Q^{\hat{M}}) \models_c T$. Moreover, for *each formula B, there exists an $\mathcal{L}_1^Q$-formula $[B]$ such that* $\mathrm{FV}([B]) = \mathrm{FV}(B)$ *and*

$$(M, Q^M) \models_c [B][\vec{x}/\vec{a}] \Leftrightarrow (\hat{M}, Q^{\hat{M}}) \models_c B[\vec{x}/\theta(\vec{a})]$$

for all $\vec{a} \in M$, where $\vec{x}$ are the free variables of B.

Note that a special case by letting $Q_0 = \omega$ and forgetting Q is the original arithmetized completeness theorem (Theorem 4.5).

Proof. Let $\tau(x)$ be a Σ_1^Q-representation over Q_0 of T. By arithmetizing the usual argument within $\mathbf{HA}^*$, let $\hat{T}$ be the Henkin extension of T. In other words, obtain $\hat{\tau}(x)$ which represents a set of $\mathcal{L}_1^Q \cup \mathcal{C}$-sentences which extends $\tau(x)$ together with Henkin axioms for $\mathcal{L}_1^Q$ where $\mathcal{C}$ is the set of Henkin constants (coded by numbers). Then $\mathbf{HA}^* \vdash_c \forall x(\tau(x) \to \hat{\tau}(x))$ and $\mathbf{HA}^* \vdash_c \mathrm{Con}(T) \to \mathrm{Con}(\hat{T})$. Apply the syntactic form of the arithmetized completeness theorem [31, Theorem 6.8] and obtain $\chi(x)$ so that

$$\begin{aligned}\mathbf{HA}^* \vdash_c & \forall x[(\chi(x) \to \text{“}x \text{ is a sentence”}) \wedge (\hat{\tau}(x) \to \chi(x))] \\ & \wedge [\mathrm{Con}(\hat{T}) \to \text{“}\chi(x) \text{ defines a complete set of } \mathcal{L}_1^Q \cup \mathcal{C}\text{-sentences”}].\end{aligned}$$

Now we see the semantic form of the arithmetized completeness theorem [31, Theorem 6.10] in our formulation. Let $(M, Q^M) \models_c \mathbf{HA}^{Q_0} + \mathrm{Con}(T)$. Then we have $(M, Q^M) \models_c \mathrm{Con}(\hat{T})$. Hence, by arithmetizing the usual Henkin construction using the complete theory defined by $\chi(x)$, one may obtain a definable $\mathcal{L}_1^Q$-structure $(\hat{M}, Q^{\hat{M}})$ in (M, Q^M) such that for any $\mathcal{L}_1^Q$-sentence A, $(M, Q^M) \models_c \chi(\ulcorner A \urcorner) \Leftrightarrow (\hat{M}, Q^{\hat{M}}) \models_c A$ and thus $(\hat{M}, Q^{\hat{M}}) \models_c \mathbf{HA}^*$. For a given $\mathcal{L}_1^Q$-formula B, one may obtain $[B]$ by formalizing Tarski's truth definition. Moreover, for each standard number $m \in \omega$, we have $m \in Q_0 \Leftrightarrow (M, Q^M) \models_c Q(\bar{m}) \Leftrightarrow (M, Q^M) \models_c \tau(\ulcorner Q(\bar{m}) \urcorner) \Leftrightarrow (M, Q^M) \models_c \chi(\ulcorner Q(\bar{m}) \urcorner) \Leftrightarrow (\hat{M}, Q^{\hat{M}}) \models_c Q(\bar{m})$. Therefore $(\hat{M}, Q^{\hat{M}}) \models_c \mathbf{HA}^{Q_0}$.

Now consider the canonical embedding $\theta : M \to \hat{M}$ as in Remark 4.4, then M can be embedded onto an initial segment of $\hat{M}$ by [31, Lemma 6.12]. Moreover, for each $m \in M$, we have $(M, Q^M) \models_c Q(m) \Leftrightarrow (\hat{M}, Q^{\hat{M}}) \models_c Q(\bar{m}) \Leftrightarrow (\hat{M}, Q^{\hat{M}}) \models_c Q(\theta(m))$, and hence θ is $\mathcal{L}_1^Q$-homeomorphism. Therefore, θ is Σ_0^Q-elementary by [31, Lemma 3.8]. ⊣

See also [13, Chapter I, Section 4, especially 4.27] and [36] for stronger forms of the arithmetized completeness theorem and a modern proof.

Acknowledgment. The authors thank Taishi Kurahashi for his useful comments.

Funding. The authors thank the Japan Society for the Promotion of Science (JSPS), Core-to-Core Program (A. Advanced Research Networks) for supporting the research. The first author was supported by JSPS KAKENHI Grant Numbers JP16H07289, JP18K13450, JP19J01239, and JP20K14354, the second author by JP16K05251, the third author by JP18K03392, the forth author by JP24540120, JP26245026, JP16K05252, JP20K03716, and JP21H00694, the fifth author by JP19K03601, and the first, second, third, and fifth authors by JP21KK0045.

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https://doi.org/10.1017/bsl.2023.29 Published online by Cambridge University Press

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