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# STEIN EMBEDDING THEOREM FOR $\mathbb{B}$-MANIFOLDS

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*Abstract* An analogue of the Stein embedding theorem for $C^\infty$ manifolds endowed with two equidimensional supplementary foliations is proved.

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## 1. Introduction

The aim of this paper is to provide an intrinsic characterization of the smooth manifolds endowed with two equidimensional supplementary foliations which admit a closed embedding into an affine space $\mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N$, equipped with its natural pair of foliations corresponding to

$$\left\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_N} \right\rangle \quad \text{and} \quad \left\langle \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_N} \right\rangle.$$

Such a class of manifolds appears in a natural way in different geometric settings, and they are designated by distinct names in the specialized literature as paracomplex manifolds, hyperbolic complex manifolds, etc. (see, among others, the survey paper [**2**] and the references mentioned therein, and for the corresponding homogeneous and symmetric spaces see also [**1**, **4**, **9**, **10**, **12**]). In studying these manifolds, we obtain an interesting relationship between $\mathbb{B}$-holomorphy and foliations, $\mathbb{B}$ being the quadratic algebra of double numbers (see § 2). For the theory of $A$-holomorphy and $A$-analyticity on a finite-dimensional commutative $\mathbb{R}$-algebra $A$, we refer the reader to [**5**, **11**, **15**, **16**, **19**, **20**]. Such notions allow one to define the category of $A$-manifolds of class $C^r$, $r \in \mathbb{N}$ or $r \in \{\infty, \omega\}$, which are $C^r$ manifolds endowed with a subsheaf of $A$-algebras of the sheaf of germs of

$A$-valued smooth functions (see [**14**] for a general exposition on the topic, and also [**6**] for the particular case $A = \mathbb{B}$).

On a manifold $M$, the structure given by two equidimensional supplementary distributions $T^-(M)$ and $T^+(M)$ defines a sheaf of $\mathbb{B}$-algebras, denoted by $\mathcal{B}_M^\infty$ and called the sheaf of $J$-holomorphic $\mathbb{B}$-valued functions on $M$ (see § 2.4), which in the integrable case (i.e. when $T^-(M)$, $T^+(M)$ are involutive) completely determines both distributions. Hence a manifold equipped with two equidimensional supplementary foliations can be understood as a sort of complex manifold over $\mathbb{B}$.

Quite surprisingly, the original topological embedding problem can then be reformulated in terms of the ringed manifold $(M, \mathcal{B}_M^\infty)$ by imposing three conditions, which—at least formally—exactly coincide with Stein's conditions for a complex manifold. The geometric meaning of such conditions is, however, rather different, as they refer to the topology of the underlying foliations; especially, the notion of convex holomorphy needs to be suitably translated to $\mathbb{B}$-manifolds.

Accordingly, our main result (Theorem 4.1) is stated as an analogue of Stein's theorem—intrinsically characterizing closed analytic submanifolds of $\mathbb{C}^{2n+1}$ (see, for example, [**8**, Theorem 5.3.9] and [**7**, Theorem VII.C.10])—for the category of such doubly foliated manifolds. Nevertheless, its proof is completely different to that of the complex case, because the basic tools of the standard proof (such as [**8**, Lemmas 5.3.4, 5.3.5]) do not work at all in the present setting, as the ground ring $\mathbb{B}$ is not a field. In fact, the version of Stein's theorem for $\mathbb{B}$-manifolds $M$ presented below can basically be understood as a criterion for the quotients $M/T^-(M)$, $M/T^+(M)$ to exist (see § 3), and this criterion is expressed in terms of the intrinsic properties of the ring of $\mathbb{B}$-holomorphic functions on the $\mathbb{B}$-manifold $M$, even in the case of non-compact leaves.

## 2. Definitions and preliminaries

In this section we give the definition of $\mathbb{B}$-manifolds and several results (without proof) concerning their rings of functions: $\mathbb{B}$-differentiable, $\mathbb{B}$-holomorphic and $\mathbb{B}$-analytic functions. Let $\mathbb{B} = \mathbb{R}[x]/(x^2 - 1) = \{z = x + \mathrm{j}y : x, y \in \mathbb{R},\ \mathrm{j}^2 = 1\}$ be the algebra of double numbers (see [**2**]). We denote by $\bar{z} = x - \mathrm{j}y$ the conjugate of the element $z = x + \mathrm{j}y \in \mathbb{B}$, and by $|z|$ its Euclidean norm.

### 2.1. $\mathbb{B}$-differentiability and $\mathbb{B}$-holomorphy

**Definition 2.1.** Let $U \subseteq \mathbb{B}$ be an open subset. A function $F : U \to \mathbb{B}$ is said to be $\mathbb{B}$-*differentiable* if for every $z_0 \in U$ the following limit exists in $\mathbb{B}$:

$$\lim_{\substack{z \to z_0 \\ z - z_0 \in \mathbb{B}^*}} \frac{F(z) - F(z_0)}{z - z_0},$$

where $\mathbb{B}^* = \{z \in \mathbb{B} : x^2 \neq y^2\}$ is the group of invertible elements. More generally, a function $F : U \to \mathbb{B}$, defined on an open subset $U \subseteq \mathbb{B}^n$, is said to be $\mathbb{B}$-*differentiable* if the function $z \mapsto F(z_1, \dots, z_{i-1}, z, z_{i+1}, \dots, z_n)$, is $\mathbb{B}$-differentiable on its domain for every $i = 1, \dots, n$. The function $F : U \to \mathbb{B}$ is said to be $\mathbb{B}$-*holomorphic* if it is of class

$C^\infty$ and $\mathbb{B}$-differentiable. We denote by $\mathcal{B}^\infty(U)$ the algebra of $\mathbb{B}$-holomorphic functions on $U$.

We have the following proposition.

**Proposition 2.2.** *A basis of the module of $\mathbb{B}$-derivations of the ring $C^\infty(U,\mathbb{B}) = C^\infty(U)\otimes_{\mathbb{R}}\mathbb{B}$ is the following:*

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \mathrm{j}\frac{\partial}{\partial y_i}\right), \quad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \mathrm{j}\frac{\partial}{\partial y_i}\right), \quad 1 \leqslant i \leqslant n,$$

*where $z_i = x_i + \mathrm{j}y_i$. This is the basis dual to $(\mathrm{d}z_i, \mathrm{d}\bar{z}_i)$, $1 \leqslant i \leqslant n$.*

**Proposition 2.3.** *A function $F \in C^\infty(U,\mathbb{B})$ is $\mathbb{B}$-holomorphic if and only if, for every $1 \leqslant i \leqslant n$, we have $\partial F/\partial \bar{z}_i = 0$; or, equivalently, $\mathrm{d}F = \sum_{i=1}^n (\partial F/\partial z_i)\mathrm{d}z_i$.*

## 2.2. $\mathbb{B}$-analyticity

**Definition 2.4.** A function $F : U \to \mathbb{B}$ on an open subset $U \subseteq \mathbb{B}^n$ is said to be $\mathbb{B}$-*analytic* if for every $z^0 = (z_1^0, \dots, z_n^0) \in U$ there exists a series

$$\sum_{\alpha\in\mathbb{N}^n} b_\alpha z_1^{\alpha_1}\cdots z_n^{\alpha_n} \in \mathbb{B}[[z_1,\dots,z_n]]$$

that is absolutely convergent in a polydisk $|z_i| \leqslant r_i$, $1 \leqslant i \leqslant n$, such that

$$F(z) = \sum_{\alpha\in\mathbb{N}^n} b_\alpha (z_1 - z_1^0)_1^\alpha \cdots (z_n - z_n^0)^{\alpha_n},$$

for every $z = (z_1,\dots,z_n) \in U$, $|z_i - z_i^0| < r_i$, $1 \leqslant i \leqslant n$. We denote by $\mathcal{A}(U)$ the algebra of $\mathbb{B}$-analytic functions on $U$.

We have the following proposition (see [**5**, Proposition 2.5]).

**Proposition 2.5.** *A $\mathbb{B}$-holomorphic function $F : U \to \mathbb{B}$ on an open subset $U \subseteq \mathbb{B}^n$ is $\mathbb{B}$-analytic if and only if it is of class $C^\omega$; that is, $\mathcal{A}(U) = \mathcal{B}^\infty(U) \cap C^\omega(U,\mathbb{B})$.*

## 2.3. $\mathbb{B}$-manifolds

**Definition 2.6.** An *almost-$\mathbb{B}$-manifold* of class $C^\infty$ (respectively, $C^\omega$) is a $C^\infty$ (respectively, $C^\omega$) manifold and a tensor field $J$ on $M$ of class $C^\infty$ (respectively, $C^\omega$) and type $(1,1)$ such that: (1) $J^2 = \mathrm{id}$, (2) the subbundles of $J$-eigenvectors, $T^-(M)$ and $T^+(M)$, of eigenvalues $-1$ and $+1$, respectively, have the same rank. An almost-$\mathbb{B}$-manifold is said to be a $\mathbb{B}$-*manifold* if the distributions defined by $T^-(M)$ and $T^+(M)$ are involutive. A map $f : (M,J) \to (M',J')$ between almost-$\mathbb{B}$-manifolds is said to be $J$-*holomorphic* (cf. [**10**]) if $f_* \circ J_x = J'_{f(x)} \circ f_*$ for all $x \in M$.

Almost-$\mathbb{B}$-manifolds (respectively, $\mathbb{B}$-manifolds) are usually called almost paracomplex (respectively, paracomplex) manifolds. The terminology in the present paper is due to the fact that we would like to emphasize that they are manifolds modelled over free modules over the ring $\mathbb{B}$ (see [**6**]).

**Example 2.7.** We define two $\mathbb{B}$-manifold structures on an open subset $U \subseteq \mathbb{B}^n$. The *first canonical* $\mathbb{B}$*-structure* is

$$J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \quad J\left(\frac{\partial}{\partial y_i}\right) = \frac{\partial}{\partial x_i}, \qquad 1 \leqslant i \leqslant n;$$

the *second canonical* $\mathbb{B}$*-structure* is (see [**10**])

$$I\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i}, \quad I\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial y_i}, \quad 1 \leqslant i \leqslant n.$$

They are distinct but isomorphic, as the map $\Psi : \mathbb{B}^n = \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{B}^n$, $\Psi(u, v) = \frac{1}{2}(u + v) + \frac{1}{2}\mathrm{j}(u - v)$, transforms $I$ into $J$; that is, $\Psi_* \circ I = J \circ \Psi_*$.

Note that in Theorem 3.5 below the first canonical $\mathbb{B}$-structure above $J$ is used, whereas in Theorem 3.9 below the second one $I$ appears.

### 2.4. $\mathbb{B}$-manifolds as ringed spaces

**Proposition 2.8.** *Let $(M, J)$ be a smooth almost-$\mathbb{B}$-manifold. A map $F$ from $(M, J)$ to $(\mathbb{B}, J)$ is $J$-holomorphic if and only if there exist functions $f, g \in C^\infty(M)$ such that (1) $F = (1 + \mathrm{j})f + (1 - \mathrm{j})g$ and (2) $f$ and $g$ are first integrals of $T^-(M)$ and $T^+(M)$, respectively; that is, $X^-(f) = 0$, $X^+(g) = 0$, for all $X^- \in \Gamma(T^-(M))$, $X^+ \in \Gamma(T^+(M))$. We denote by $\mathcal{B}^\infty_M$ the sheaf of germs of $J$-holomorphic $\mathbb{B}$-valued functions on $M$.*

This notion is consistent with § 2.1 by virtue of the following result.

**Proposition 2.9.** *A $2n$-dimensional almost-$\mathbb{B}$-manifold $(M, J)$ of class $C^\infty$ is a $\mathbb{B}$-manifold if and only if its sheaf of germs of $J$-holomorphic functions is locally isomorphic to the sheaf of germs of $\mathbb{B}$-holomorphic functions on $\mathbb{B}^n$; that is, if and only if $M$ admits an open covering $\{U_\alpha\}_{\alpha \in A}$ such that there exists an isomorphism $\varphi_\alpha : U_\alpha \to \mathbb{B}^n$ of almost-$\mathbb{B}$-manifolds between $U_\alpha$ and an open subset of $\mathbb{B}^n$, for each $\alpha$.*

Therefore, we can conclude the following proposition.

**Proposition 2.10.** *A differentiable map $f : M \to M'$ between smooth $\mathbb{B}$-manifolds is $J$-holomorphic if and only if for every open subset $U' \subseteq M'$ and every $F' \in \mathcal{B}^\infty(U')$, the composite map $F' \circ f$ belongs to $\mathcal{B}^\infty(f^{-1}U')$.*

**Example 2.11.** Let $X$, $Y$ be two $C^\infty$ differentiable manifolds of the same dimension $n$. In $M = X \times Y$ there exists a unique $\mathbb{B}$-manifold structure such that for each pair of open subsets $U \subset X$, $V \subset Y$, the ring of $\mathbb{B}$-holomorphic functions on $M$ defined on $U \times V$ is $\{(1+\mathrm{j})f \circ p^- + (1-\mathrm{j})g \circ p^+ : f \in C^\infty(U),\ g \in C^\infty(V)\}$, where $p^- : M = X \times Y \to X$ and $p^+ : M = X \times Y \to Y$ are the canonical projections.

### 2.5. $\mathbb{B}$-complexification

If $(M, J)$ is a $2n$-dimensional $\mathbb{B}$-manifold of class $C^\infty$, for every $x \in M$ we set $T^{\mathbb{B}}_x(M) = T_x(M) \otimes_{\mathbb{R}} \mathbb{B}$, and extend $J$ to $T^{\mathbb{B}}_x(M)$ by simply setting $J^{\mathbb{B}}(X + \mathrm{j}Y) =$

$JX + \mathrm{j}JY$. Dually, we set $T_x^{*\mathbb{B}}(M) = T_x^*(M) \otimes_{\mathbb{R}} \mathbb{B}$, and extend $J^*$ to $T_x^{*\mathbb{B}}(M)$, accordingly. We define

$$T_x^{1,0}(M) = \{Z \in T_x^{\mathbb{B}}(M) : J^{\mathbb{B}}(Z) = \mathrm{j}Z\},$$
$$T_x^{0,1}(M) = \{Z \in T_x^{\mathbb{B}}(M) : J^{\mathbb{B}}(Z) = -\mathrm{j}Z\}.$$

We then have the following.

(1) $T_x^{\mathbb{B}}(M) = T_x^{1,0}(M) \oplus T_x^{0,1}(M)$.

(2) If $(z_i)_{1\leqslant i\leqslant n}$ is a $\mathbb{B}$-coordinate system on an open neighbourhood $U$ of $x \in M$, then $((\partial/\partial z_i)_x)_{1\leqslant i\leqslant n}$ (respectively, $((\partial/\partial \bar{z}_i)_x)_{1\leqslant i\leqslant n}$) is a basis of $T_x^{1,0}$ (respectively, $T_x^{0,1}$) (see Propositions 2.2 and 2.9).

(3) $T_x^{*\mathbb{B}}(M) = T_x^{*1,0}(M) \oplus T_x^{*0,1}(M)$.

(4) If $(z_i)_{1\leqslant i\leqslant n}$ is as in item (2), then $(\mathrm{d}_x z_i)_{1\leqslant i\leqslant n}$ (respectively, $(\mathrm{d}_x \bar{z}_i)_{1\leqslant i\leqslant n}$) is a basis of $T_x^{*1,0}(M)$ (respectively, $T_x^{*0,1}(M)$).

(5) If $F \in \mathcal{B}^\infty(M)$, then $\mathrm{d}_x F \in T_x^{*1,0}(M)$ (cf. Proposition 2.3).

## 3. Stein $\mathbb{B}$-manifolds

**Theorem 3.1.** *Let $(M, J)$ be a $C^\infty$ $\mathbb{B}$-manifold such that the quotient manifolds $M^- = M/T^-(M)$, $M^+ = M/T^+(M)$ exist in the category of $C^\infty$ manifolds. For every $x \in M$, let $\mathcal{F}_x^-$ (respectively, $\mathcal{F}_x^+$) be the leaf of $T^-(M)$ (respectively, of $T^+(M)$) through $x$, and let us endow $M^- \times M^+$ with the $\mathbb{B}$-structure given in Example 2.11. We have the following.*

(1) *The map $\varphi : M \to M^- \times M^+$ defined by $\varphi(x) = (\mathcal{F}_x^-, \mathcal{F}_x^+)$ is an open immersion of $\mathbb{B}$-manifolds.*

(2) *If either every $\mathcal{F}_x^+$ is compact and $M^-$ is simply connected or every $\mathcal{F}_x^-$ is compact and $M^+$ is simply connected, then $\varphi$ is an open embedding.*

(3) *If for every couple $x \neq y \in M$, there is $F \in \mathcal{B}^\infty(M)$ such that $F(x) \neq F(y)$, then $\varphi$ is an open embedding.*

**Proof.** (1) Let $p^- : M \to M^-$ and $p^+ : M \to M^+$ be the quotient mappings. As is well known, $p^+$ and $p^-$ are submersions of $C^\infty$ manifolds [**13**, I.§ 5.Theorem X] such that $\operatorname{Ker} p_*^- = T^-(M)$ and $\operatorname{Ker} p_*^+ = T^+(M)$. We have $\varphi = (p^-, p^+)$. Hence $\varphi_* X = 0$ for $X \in T(M)$ implies $p_*^-(X) = 0$ and $p_*^+(X) = 0$, and consequently $X = 0$. It thus follows that $\varphi$ is an immersion and since $\dim M = \dim(M^- \times M^+)$, we conclude that $\varphi$ is an open immersion.

(2) Assume that every leaf $\mathcal{F}_x^+$ is compact and $M^-$ is simply connected. Let $\pi : \mathcal{F}_x^+ \to M^-$ be the restriction to the leaf of the quotient mapping $p^- : M \to M^-$. Since $\operatorname{Ker} p_*^- = T^-(M)$, $\pi$ is a local diffeomorphism, and since $\mathcal{F}_x^+$ is connected and compact,

it follows that $\pi$ is surjective; hence $\pi$ is a covering map and by virtue of the hypothesis, $\pi$ is a global diffeomorphism. Accordingly, the relations $\mathcal{F}_x^- = \mathcal{F}_y^-$ (i.e. $y \in \mathcal{F}_x^-$) and $\mathcal{F}_x^+ = \mathcal{F}_y^+$ (i.e. $\pi(x) = \pi(y)$), imply $x = y$. Therefore, $\varphi$ is injective.

(3) We only need to prove that either $\mathcal{F}_x^- \neq \mathcal{F}_y^-$ or $\mathcal{F}_x^- = \mathcal{F}_y^-$. Consider the function $F = (1+\mathrm{j})f + (1-\mathrm{j})g$, where $f$ and $g$ are first integrals of $T^-(M)$ and $T^+(M)$, respectively. Assume $\mathcal{F}_x^- = \mathcal{F}_y^-$ (or, equivalently, $y \in \mathcal{F}_x^-$). Then $g(x) = g(y)$. Similarly, $\mathcal{F}_x^+ \neq \mathcal{F}_y^+$ implies $f(x) = f(y)$, and the result follows from our hypothesis. $\square$

**Remark 3.2.** The quotient manifolds $M^-$ and $M^+$ may exist, however $\varphi$ may not be injective. For example, take $M = \{(z, w) \in \mathbb{C}^2 : w \neq 0\}$, endowed with the $\mathbb{B}$-structure defined by the fibres of the projections $p^- : M \to \mathbb{C}$ and $p^+ : M \to \mathbb{C}$ defined by $p^-(z, w) = z$ and $p^+(z, w) = z + w^2$. Then $M^-$ and $M^+$ exist and both coincide with $\mathbb{C}$. Nevertheless, $\mathcal{F}^-_{(z,w)} \cap \mathcal{F}^+_{(z,w)} = \{(z, w), (z, -w)\}$ for every $(z, w) \in M$. In the general case, $\mathcal{F}_x^- \cap \mathcal{F}_x^+$ is a discrete subset of $M$, since $\mathcal{F}_x^-$ and $\mathcal{F}_x^+$ cut transversally at each point $x \in \mathcal{F}_x^- \cap \mathcal{F}_x^+$, and $T_x(\mathcal{F}_x^-) \cap T_x(\mathcal{F}_x^+) = \{0\}$. We set $\nu(M) = \sup_{x \in M} \#(\mathcal{F}_x^- \cap \mathcal{F}_x^+)$ (this is a positive integer or $\infty$). Assume $M/T^-(M)$ and $M/T^+(M)$ exist in the category of $C^\infty$-manifolds. Then a necessary and sufficient condition for $M$ to be an open subset of the product $\mathbb{B}$-structure defined in Theorem 3.1 is $\nu(M) = 1$.

**Corollary 3.3.** *Under the hypotheses of Theorem 3.1 (1), $M$ admits a compatible structure of a $\mathbb{B}$-analytic manifold.*

**Proof.** From Whitney's Theorem on $C^\omega$ analytization and Proposition 2.5 it follows that $M^- \times M^+$ has a compatible structure of a $\mathbb{B}$-analytic manifold. Then it is not difficult to see that there exists a unique structure of a $\mathbb{B}$-analytic manifold on $M$ such that a $\mathbb{B}$-holomorphic mapping $f : N \to M$ of $\mathbb{B}$-analytic manifolds is $\mathbb{B}$-analytic if and only if the composite map $\varphi \circ f$ ($\varphi$ being as in Theorem 3.1) is $\mathbb{B}$-analytic. $\square$

**Lemma 3.4.** *Let $V$ be a free $\mathbb{B}$-module of rank $N$. Every system of $\mathbb{B}$-linearly independent vectors $(v_1, \dots, v_n)$ can be extended to a basis of $V$.*

**Proof.** We proceed by induction on $n$. If $n = 1$ and $N = 1$, it is immediate that $v_1$ is linearly independent if and only if $v_1$ is invertible. If $n = 1$ and $N \geqslant 2$, then either $v_1$ has an invertible component or it can be written as $v_1 = (1 + \mathrm{j})u_1 + (1 - \mathrm{j})u_2$ in a basis $(u_1, \dots, u_N)$ of $V$. In both cases we can conclude the proof. In the general case the above argument proves that $V/\mathbb{B} \cdot v_1$ is a free $\mathbb{B}$-module of rank $N - 1$, and the cosets $(\bar{v}_2, \dots, \bar{v}_n)$ are linearly independent. The proof is thus finished by simply applying the induction hypothesis. $\square$

**Theorem 3.5.** *If a $2n$-dimensional $\mathbb{B}$-manifold $M$ admits a $\mathbb{B}$-immersion into an affine space $(\mathbb{B}^N, J)$, then for every $x \in M$, global functions $F_1, \dots, F_n \in \mathbb{B}^\infty(M)$ exist such that $(\mathrm{d}_x F_1, \dots, \mathrm{d}_x F_n)$ is a basis of $T_x^{*1,0}(M)$. If $M$ is compact, the converse is also true.*

**Proof.** Let $F_i = P_i + \mathrm{j}Q_i$, $1 \leqslant i \leqslant N$, be the components of a $\mathbb{B}$-holomorphic mapping $F : M \to \mathbb{B}^N$. According to Proposition 2.9, for every $x \in M$, we can consider a system of $\mathbb{B}$-coordinates $z_h = x_h + \mathrm{j} y_h$, $1 \leqslant h \leqslant n$, around $x$. Let $A$ be the $N \times n$ matrix $A = (\partial P_i/\partial x_h(x))$, and let $B$ be the $N \times n$ matrix $B = (\partial P_i/\partial y_h(x))$. The Jacobian of $F$ at $x$ is

$$\Lambda = \begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

since $F$ is $\mathbb{B}$-holomorphic. Hence $F$ is an immersion if and only if $\operatorname{rank} \Lambda = 2n$. Moreover, the matrix of the covectors $\mathrm{d}_x F_1, \dots, \mathrm{d}_x F_N$ in the basis $(\mathrm{d}_x z_1, \dots, \mathrm{d}_x z_n)$ is ${}^{\mathrm{t}}A + \mathrm{j}\,{}^{\mathrm{t}}B$, thus proving that the columns of $A + \mathrm{j}B$ are $\mathbb{B}$-linearly independent. We conclude by applying Lemma 3.4. Conversely, if $M$ is compact, there are a finite open covering $(U_1, \dots, U_r)$ of $M$, and functions $F_1^\alpha, \dots, F_n^\alpha \in \mathcal{B}^\infty(M)$ such that for every $\alpha = 1, \dots, r$, $(F_1^\alpha|_{U_\alpha}, \dots, F_n^\alpha|_{U_\alpha})$ is a $\mathbb{B}$-holomorphic coordinate system. It follows that the $\mathbb{B}$-holomorphic mapping $F : M \to \mathbb{B}^{rn}$, whose components are $F_h^\alpha$, is an immersion. $\square$

**Remark 3.6.** Wazewsky [**17, 18**] obtained a one-dimensional $C^\infty$-foliation $\mathcal{F}^+$ on $\mathbb{R}^2$ such that every global first integral of $\mathcal{F}^+$ is a constant. Let $\mathcal{F}^-$ be the orthogonal foliation with respect to the Euclidean metric. We thus define a $\mathbb{B}$-structure on the plane which does not admit any immersion into $\mathbb{B}^N$, as follows from § 2.4 and Theorem 3.5.

**Remark 3.7.** Compact examples are easy to show: the torus $S^1 \times S^1$ with the foliations spanned by

$$X_1 = \frac{\partial}{\partial \theta_1} + \alpha_1 \frac{\partial}{\partial \theta_2}, \qquad X_2 = \frac{\partial}{\partial \theta_1} + \alpha_2 \frac{\partial}{\partial \theta_2},$$

where $\alpha_1$, $\alpha_2$ are two distinct irrational numbers.

**Proposition 3.8.** *If $\varphi : M \to M'$ is a $J$-holomorphic map of $\mathbb{B}$-manifolds, then for every $x_0 \in M$ we have $\varphi(\mathcal{F}^-_{x_0}) \subseteq \mathcal{F}^-_{\varphi(x_0)}$, $\varphi(\mathcal{F}^+_{x_0}) \subseteq \mathcal{F}^+_{\varphi(x_0)}$.*

**Proof.** We first prove that there exists an open neighbourhood $U$ of $x_0 \in M$ such that $\varphi(\mathcal{F}^-_{x_0} \cap U) \subseteq \mathcal{F}^-_{\varphi(x_0)}$. From § 2.4 and Proposition 2.9 it follows that an open neighbourhood $V$ of $\varphi(x_0)$ and functions $F_i = (1+\mathrm{j})f_i + (1-\mathrm{j})g_i \in \mathcal{B}^\infty_{M'}(V)$, $1 \leqslant i \leqslant n$, exist such that $\mathcal{F}^-_{\varphi(x_0)} \cap V = \{y \in V : f_i(y) = 0,\ 1 \leqslant i \leqslant n\}$, and from § 2.4 and Proposition 2.10 we know that $f_i(\varphi(x)) = f_i(\varphi(x_0))$ for $x \in \mathcal{F}^-_{x_0} \cap U$, where $U$ is the connected component of $\varphi^{-1}(V)$ through $x_0$. Let $x_1$ be another arbitrary point in $\mathcal{F}^-_{x_0}$, and let $\sigma : [0,1] \to \mathcal{F}^-_{x_0}$ be a continuous arc such that $\sigma(0) = x_0$, $\sigma(1) = x_1$. We denote by $S$ the subset of $t \in [0,1]$ such that $\varphi(\sigma[0,t]) \subset \mathcal{F}^-_{\varphi(x_0)}$. By virtue of the first part of the proof, $S$ contains the origin and it is open, as $t_0 \in S$ implies $\varphi(\sigma(t_0)) \in \mathcal{F}^-_{\varphi(x_0)} = \mathcal{F}^-_{\varphi(\sigma(t_0))}$, and hence for small enough $\varepsilon > 0$ each $\sigma(t)$ with $|t - t_0| < \varepsilon$ lies on a neighbourhood of $\sigma(t_0)$ in the leaf $\mathcal{F}^-_{\sigma(t_0)}$, which is mapped by $\varphi$ into $\mathcal{F}^-_{\varphi(x_0)}$. Moreover, $S$ is also closed. In fact, assume that $t_0 \in \bar{S}$. If $\varphi(\sigma(t_0)) \notin \mathcal{F}^-_{\varphi(x_0)}$, then for every small enough $\varepsilon > 0$, we have $\varphi(\sigma(t)) \in \mathcal{F}^-_{\varphi(\sigma(t_0))}$ for $|t - t_0| < \varepsilon$; but we know that a point $t_0 - \varepsilon$ exists such that $\varphi(\sigma(t_0 - \varepsilon))$ belongs to $\mathcal{F}^-_{\varphi(x_0)}$. As $\mathcal{F}^-_{\varphi(x_0)} \cap \mathcal{F}^-_{\varphi(\sigma(t_0))} = \emptyset$, this leads one to a contradiction. The proof for the positive leaf is similar. $\square$

**Theorem 3.9.** *Let $M$ be a $\mathbb{B}$-manifold of real dimension $2n$. We have the following.*

(1) *If for every $x \in M$ there exist global functions $F_1, \dots, F_n \in \mathcal{B}^\infty(M)$ such that $(\mathrm{d}_x F_1, \dots, \mathrm{d}_x F_n)$ is a basis of $T_x^{*1,0}(M)$, then the foliations $T^-(M)$ and $T^+(M)$ are regular. Hence, the leaves of $T^-(M)$ and $T^+(M)$ are closed submanifolds of $M$.*

(2) *If $M$ is an embedded $\mathbb{B}$-submanifold of an affine space $(\mathbb{B}^N, I)$, then the quotient manifolds $M/T^-(M)$, $M/T^+(M)$ are Hausdorff spaces.*

**Proof.** (1) By virtue of the hypothesis, for every $x \in M$ there exist an open neighbourhood $U$ of $x$, and global functions $F_i \in \mathcal{B}^\infty(M)$, $1 \leqslant i \leqslant n$, such that $(F_i|_U)$ is a $\mathbb{B}$-coordinate system. From § 2.4, we can write $F_i = (1+\mathrm{j})f_i + (1-\mathrm{j})g_i$, $1 \leqslant i \leqslant n$, where $f_i$ and $g_i$ are first integrals of $T^-(M)$ and $T^+(M)$, respectively. Since

$$\mathrm{d}F_1 \wedge \cdots \wedge \mathrm{d}F_n = 2^{n-1}(1+\mathrm{j})\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n + 2^{n-1}(1-\mathrm{j})\mathrm{d}g_1 \wedge \cdots \wedge \mathrm{d}g_n,$$

substituting $f_i - f_i(x)$ for $f_i$ and $g_i - g_i(x)$ for $g_i$, and shrinking $U$ if necessary, we can assume that $(f_1, \dots, f_n, g_1, \dots, g_n)$ is a cubic coordinate system centred at $x \in M$, which is flat [**13**, I.§ 2] with respect to $T^+(M)$. Similarly, $(g_1, \dots, g_n, f_1, \dots, f_n)$ is a cubic coordinate system centred at $x \in M$, which is flat with respect to $T^-(M)$. We have to prove that if a leaf $\mathcal{F}^+$ of $T^+(M)$ intersects $U$, then $\mathcal{F}^+ \cap U$ is contained in a unique $n$-dimensional slice of $(U; f_1, \dots, f_n, g_1, \dots, g_n)$. As $\mathcal{F}^+$ is connected and $g_1, \dots, g_n$ are first integrals of $T^+(M)$, we conclude that $g_i|_{\mathcal{F}^+}$ is constant, i.e. there exist $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that $g_i|_{\mathcal{F}^+} = \lambda_i$, $1 \leqslant i \leqslant n$, and the result follows. The last assertion follows from [**13**, I.§ 4.Theorem VII].

(2) Let $\phi : M \to \mathbb{B}^N$ be a $\mathbb{B}$-embedding. By virtue of Theorem 3.5 and part (1) of the present theorem, we know that the foliations induced by $T^-(M)$, $T^+(M)$ are regular. Let $p^\mp : \mathbb{B}^N \to \mathbb{B}^N/T^\mp(\mathbb{B}^N) = \mathbb{R}^N$ be the canonical projections. We prove that $p^\mp(\phi(M))$ are embedded submanifolds of $\mathbb{R}^N$. In fact, as $\phi$ is an embedding, for every $z_0 \in M$ there exist open neighbourhoods $U$, $V = V^- \times V^+$ of $z_0$, $\phi(z_0)$ in $M$, $\mathbb{B}^N$, respectively, and a $\mathbb{B}$-coordinate system $F_1, \dots, F_N \in \mathcal{B}^\infty(V)$ such that $\phi(U) = \{w \in V : F_{n+1}(w) = \cdots = F_N(w) = 0\}$. From § 2.4 we have $F_i(u, v) = (1+\mathrm{j})f_i(u) + (1-\mathrm{j})g_i(v)$, so

$$p^-(\phi(U)) = \{u \in V^- : f_{n+1} = \cdots = f_N(u) = 0\}. \tag{3.1}$$

Hence with the topology induced from that of $\mathbb{R}^N$, $p^-(\phi(M))$ can be covered by coordinate open subsets $(V^-; f_1, \dots, f_N)$ so that (3.1) holds true, thus proving that $p^-(\phi(M))$ is an embedded submanifold in $\mathbb{R}^N$ [**3**, 16.8.1.1], and similarly for $p^+(\phi(M))$. Moreover, the maps $p^\mp \circ \phi : M \to (p^\mp \circ \phi)(M)$ are surjective submersions as $\phi_*$ transforms $T^\mp(M)$ into $T^\mp(\mathbb{B}^N)$, hence $\mathrm{Ker}(p^- \circ \phi)_* = T^+(M)$, and thus $\dim \mathrm{Im}(p^- \circ \phi)_* = n = \dim(p^- \circ \phi)(M)$. As $\phi(\mathcal{F}_z^\mp) \subseteq \mathcal{F}_{\phi(z)}^\mp$, we conclude that $p^\mp(\phi(M))$ can be identified with the quotient manifolds $M/T^\mp(M)$. $\square$

**Definition 3.10.** A $\mathbb{B}$-manifold is said to be a *Stein* $\mathbb{B}$*-manifold* if $M$ satisfies the following three conditions (cf. [**8**, 5.1.3]):

($\alpha$) $M$ is $\mathbb{B}$-convex, i.e. $\hat{K} = \{x \in M : |F(x)| \leqslant \sup_K |F|,\ \forall F \in \mathcal{B}^\infty(M)\}$ is compact for every compact subset $K \subseteq M$;

($\beta$) if $x_1 \neq x_2$ are two points in $M$, then there exists $F \in \mathcal{B}^\infty(M)$ such that $F(x_1) \neq F(x_2)$; and

($\gamma$) for every $x \in M$, there exist $n$ functions $F_1, \dots, F_n \in \mathcal{B}^\infty(M)$ which form a coordinate system at $x$.

**Remark 3.11.** From Theorem 3.5, it follows that every $\mathbb{B}$-manifold $M$ admitting an immersion into an affine space $\mathbb{B}^N$, satisfies the condition ($\gamma$).

**Remark 3.12.** $(\mathbb{B}^N, I)$ is a Stein $\mathbb{B}$-manifold: ($\beta$) and ($\gamma$) are trivial and ($\alpha$) follows from the formula $\hat{K} = p^-(K) \times p^+(K)$, where $p^\mp : \mathbb{B}^N = \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ are the canonical projections, as follows from Example 2.11.

**Remark 3.13.** As in the complex case (see, for example, [**7**, VII.A.Examples (6)]) and taking the first remark above into account, it follows that any closed $\mathbb{B}$-submanifold of $\mathbb{B}^N$ is a Stein $\mathbb{B}$-submanifold.

## 4. The embedding theorem

**Theorem 4.1.** *A connected* $\mathbb{B}$*-manifold* $M$ *is embeddable as a closed* $\mathbb{B}$*-submanifold of an affine space* $(\mathbb{B}^N, I)$ *if and only if* $M$ *is a Stein* $\mathbb{B}$*-manifold and the quotient manifolds* $M^- = M/T^-(M)$, $M^+ = M/T^+(M)$ *are Hausdorff spaces.*

**Proof.** Assume that $M$ is a closed $\mathbb{B}$-submanifold of $\mathbb{B}^N$. From Remark 3.13 above it follows that $M$ is a Stein $\mathbb{B}$-manifold and from Theorem 3.9 (2) we also conclude that $M^\mp$ are Hausdorff spaces. Conversely, assume that both conditions of the statement hold true for $M$. Then, by virtue of Theorem 3.1 we know that the map $\varphi : M^- \times M^+$, $\varphi(x) = (\mathcal{F}_x^-, \mathcal{F}_x^+)$ is an open $\mathbb{B}$-embedding. We first prove that $\varphi$ is a $\mathbb{B}$-diffeomorphism. For every subset $S \subset M$, we set $S^- = p^-(S)$, $S^+ = p^+(S)$, where $p^\mp : M \to M^\mp$ stand for the canonical mappings. As $\varphi$ is an open embedding, we identify $M$ to $\varphi(M)$, thus assuming that $\varphi$ is the inclusion. If there exists a relatively compact open subset $U \subset M$, such that $\bar{U}^- \times \bar{U}^+ \not\subset M$ (and hence $(\bar{U}^- \times \bar{U}^+) \cap M \neq \emptyset$, as $\bar{U} \subset \bar{U}^- \times \bar{U}^+$ and $(\bar{U}^- \times \bar{U}^+) \cap (\mathbb{B}^N - M) \neq \emptyset$), then $M$ is not $\mathbb{B}$-convex. In fact, picking a point $x \in \partial M \cap (\bar{U}^- \times \bar{U}^+)$ we obtain a sequence $x_i \in (\bar{U}^- \times \bar{U}^+) \cap M$ converging to $x$. Therefore, we have a sequence in $\hat{\bar{U}}$ that has no convergent subsequence in $\hat{\bar{U}}$, and, accordingly, $\hat{\bar{U}}$ is not compact. It thus follows that for every relatively compact open subset $U \subset M$, we have $\bar{U}^- \times \bar{U}^+ \subseteq M$. Let $(U_i)_{i \in \mathbb{N}}$ be a sequence of relatively compact subsets such that $\bar{U}_i \subset U_{i+1}$, $M = \bigcup_{i \in \mathbb{N}} \bar{U}_i$. Given $x^- \in M^-$, $y^+ \in M^+$, let $x, y \in M$ be two points such that $p^-(x) = x^-$, $p^+(y) = y^+$, so that $x = (x^-, x^+)$, $y = (y^-, y^+)$. For a large enough $i$, we have $x, y \in \bar{U}_i$, and since $\bar{U}_i^- \times \bar{U}_i^+ \subset M$, we conclude that $(x^-, y^+) \in M$, and hence $M = M^- \times M^+$. According to Whitney's Embedding Theorem, $M^-$ and $M^+$

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can be embedded as closed submanifolds of $\mathbb{R}^{2n+1}$. We then have a closed $\mathbb{B}$-embedding $M \to \mathbb{B}^{2n+1}$ defined as the composite map $M \xrightarrow{\varphi} M^- \times M^+ \hookrightarrow \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} = \mathbb{B}^{2n+1}$, thus finishing the proof. $\square$

**Remark 4.2.** If the real dimension of $M$ is $2n$, then we can take $N = 2n + 1$.

**Remark 4.3.** As an example, it follows from the above theorem that the unit open disc $\Delta(0,1)$ in $\mathbb{B}$ and the unit open square $(-1,1)^2$ in $\mathbb{B}$ are not $\mathbb{B}$-diffeomorphic as $\Delta(0,1)$ cannot be embedded into any $\mathbb{B}^N$ for condition $(\alpha)$ in Definition 3.10 does not hold true for $\Delta(0,1)$, whereas $(-1,1)^2$ is evidently $\mathbb{B}$-diffeomorphic to $\mathbb{B}$.

**Remark 4.4.** We also remark that, as the above proof shows, every closed $\mathbb{B}$-submanifold of $(\mathbb{B}^N, I)$ is a product manifold $X \times Y$, $\dim X = \dim Y$, with the $\mathbb{B}$-structure given in Example 2.11.

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## References

1. W. Bertram, Complexifications of symmetric spaces and Jordan theory, *Trans. Am. Math. Soc.* **353** (2001), 2531–2556.
2. V. Cruceanu, P. Fortuny and P. M. Gadea, A survey on paracomplex geometry, *Rocky Mt. J. Math.* **26** (1996), 83–115.
3. J. Dieudonné, *Éléments d'analyse*, vol. III (Gauthier-Villars, Paris, 1970).
4. S. Erdem, Paracomplex projective models and harmonic maps into them, *Beitr. Algebra Geom.* **40** (1999), 385–398.
5. P. M. Gadea and J. Muñoz Masqué, $A$-differentiability and $A$-analyticity, *Proc. Am. Math. Soc.* **124** (1996), 1437–1443.
6. P. M. Gadea, J. Grifone and J. Muñoz Masqué, Manifolds modelled over free modules over the double numbers, *Acta Math. Hungar.* **101** (2003), in press.
7. R. C. Gunning and H. Rossi, *Analytic functions of several complex variables* (Prentice Hall, Englewood Cliffs, NJ, 1965).
8. L. Hörmander, *An introduction to complex analysis in several variables* (North-Holland, Amsterdam, 1973).
9. S. Kaneyuki, On a remarkable class of homogeneous symplectic manifolds, *Proc. Jpn Acad.* A **67** (1991), 128–131.
10. S. Kaneyuki and M. Kozai, Paracomplex structures and affine symmetric spaces, *Tokyo J. Math.* **8** (1985), 81–98.
11. P. W. Ketchum, Analytic functions of hypercomplex variables, *Trans. Am. Math. Soc.* **30** (1928), 641–667.
12. G. 'Olaffson, Symmetric spaces of Hermitian type, *Diff. Geom. Applic.* **1** (1991), 195–233.
13. R. S. Palais, *A global formulation of the Lie theory of transformation groups*, Memoirs of the American Mathematical Society, vol. 22 (1957).
14. V. V. Shurygin, Manifolds over algebras and their application to the geometry of jet bundles, *Usp. Mat. Nauk* **48** (1993), 75–106 (also: *Russ. Math. Surv.* **48** (1993), 75–104).
15. W. C. Waterhouse, Analyzing some generalized analytic functions, *Expo. Math.* **10** (1992), 183–192.

16. W. C. Waterhouse, Differentiable functions on algebras and the equation $\mathrm{grad}(w) = M\,\mathrm{grad}(v)$, *Proc. R. Soc. Edinb.* A **122** (1992), 353–361.
17. T. Wazewsky, Sur l'équation $Pp + Qq = 0$, *Mathematika* **8** (1934), 103–116.
18. T. Wazewsky, Sur l'équation $Pp + Qq = 0$, *Mathematika* **9** (1935), 179–182.
19. R. Z. Yeh, Analysis and applications of holomorphic functions in higher dimensions, *Trans. Am. Math. Soc.* **345** (1994), 151–177.
20. R. Z. Yeh, Errata to 'Analysis and applications of holomorphic functions in higher dimensions', *Trans. Am. Math. Soc.* **347** (1995), 1081.