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# PSEUDO-INTEGRALITY

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ABSTRACT. Let $R$ be an integral domain. An element $u$ of the quotient field of $R$ is said to be pseudo-integral over $R$ if $uI_v \subseteq I_v$ for some nonzero finitely generated ideal $I$ of $R$. The set of all pseudo-integral elements forms an integrally closed (but not necessarily pseudo-integrally closed) overring $\tilde{R}$ of $R$. It is shown that $(\widetilde{R[X]}) = \tilde{R}[X]$, where $X$ is a family of indeterminates; pseudo-integrality is analyzed in rings of the form $D+M$; and an example is given to show that pseudo-integrality does not behave well with respect to localization.

**Introduction.** Throughout this note, $R$ will be an integral domain with quotient field $K$. We wish to introduce and study a new type of integrality which is intermediate between almost integrality and ordinary integrality. Our definition requires the so-called $v$-operation. Denote by $\mathcal{F}(R)$ the set of nonzero fractional ideals of $R$. For $I \in \mathcal{F}(R)$, set $I^{-1} = \{x \in K : xI \subseteq R\}$. The $v$-operation on $R$ is the map from $\mathcal{F}(R)$ into inself given by $I \longrightarrow I_v \equiv (I^{-1})^{-1}$. The nonzero ideal $I$ is said to be *divisorial*, or a *$v$-ideal*, if $I = I_v$. The $v$-operation is an example of a star-operation; the reader is referred to [6, Sections 32 and 34] for a discussion of the properties of the $v$-operation, which we shall use freely.

It is well-known that an element $u \in K$ is almost integral over $R \Leftrightarrow$ there is a nonzero ideal $I$ of $R$ for which $uI \subseteq I$. Simlarly, $u$ is integral over $R \Leftrightarrow uI \subseteq I$ for some nonzero finitely generated ideal $I$ of $R$. We now define an element $u$ of $K$ to be *pseudo-integral* over $R$ if $uI_v \subseteq I_v$ (equivalently, $uI^{-1} \subseteq I^{-1}$) for some nonzero finitely generated ideal $I$ of $R$. It is clear that $u$ integral $\Rightarrow$ $u$ pseudo-integral $\Rightarrow$ $u$ almost integral.

We denote by $\tilde{R}$ the set of elements of $K$ which are pseudo-integral over $R$. In his thesis [12], B. G. Kang also studies the ring $\tilde{R}$, but his results are for the most part specialized, while our goal is a systematic study of pseudo-integrality.

In the first section, we show that $\tilde{R}$ is an integrally closed overring of $R$. Part of our motivation for studying pseudo-integrality arises from the fact that the integral closure of a domain often has desirable properties. For example, the integral closure of a Noetherian domain is a Krull domain. A question that has been open for several years is the following: is the integral closure of a one-dimensional coherent domain necessarily Prüfer? (cf. [9]). Along these lines, we show, as a consequence of the fact that $\tilde{R}$ is integrally closed, that there is a large class of rings $T$ for which $\tilde{T}$ is a Prüfer $v$-multiplication domain. (Kang

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[12, Theorem 5.10] has proved an equivalent result by different methods.) In addition, we prove that pseudo-integrality behaves well under passage to the polynomial ring. We close Section 1 with a discussion of pseudo-integrality in $D + M$ examples.

It is well-known that almost integrality fails to be transitive, that the complete integral closure of a domain need not be completely integrally closed, and that a localization of a completely integrally closed domain need not be completely integrally closed. In Section 2 we show that pseudo-integrality has corresponding faults.

Finally, in Section 3, we show that in fact pseudo-integrality exhibits behavior that is even more miscreant than that of almost integrality. We provide an example showing that an element which is pseudo-integral over $R$ need not be pseudo-integral over an overring $T$ of $R$, even when $T$ is pseudo-integral over $R$. We also observe that the intersection of pseudo-integrally closed domains need not be pseudo-integrally closed.

## 1. **The Good.**

DEFINITION. Let $R$ be a domain with quotient field $K$. An element $x \in K$ is *pseudo-integral over* $R$ if $xI_v \subseteq I_v$ (equivalently, $x \in I_v : I_v$) for some nonzero finitely generated ideal $I$ of $R$.

As we show in Proposition 1.1 below, the set of pseudo-integral elements is an overring of $R$, which we denote by $\tilde{R}$ and call the *pseudo-integral closure* of $R$. We shall also use $R'$ to denote the integral closure of $R$ and $R^*$ to denote the complete integral closure of $R$. It is clear that $R' \subseteq \tilde{R} \subseteq R^*$.

PROPOSITION 1.1. *$\tilde{R}$ is the directed union of the overrings $I_v : I_v$, the union being taken over all nonzero finitely generated ideals $I$ of $R$. In particular, $\tilde{R}$ is a ring.*

PROOF. It is clear from the definition that $\tilde{R} = \cup(I_v : I_v)$. That the union is directed follows from $(I_v : I_v) \cup (J_v : J_v) \subseteq (IJ)_v : (IJ)_v$. Finally, since each $(I_v : I_v)$ is a ring, $\tilde{R}$ is also a ring. ∎

Recall that a domain $R$ is said to be *essential* if it is the intersection of valuation overrings each of which is a localization of $R$. It is well-known that an essential domain is a $v$-domain; a domain $R$ is a *$v$-domain* if $(II^{-1})_v = R$ (equivalently, $(II^{-1})^{-1} = R$) for each nonzero finitely generated ideal $I$ of $R$. It is natural to call a domain $R$ *pseudo-integrally closed* if $R = \tilde{R}$. From the equation $(II^{-1})^{-1} = I^{-1} : I^{-1} = I_v : I_v$ one easily obtains that a domain $R$ is pseudo-integrally closed $\Leftrightarrow$ $R$ is a $v$-domain. Thus essential domains provide examples of pseudo-integrally closed domains. In [13, 14] Nagata gives an example of a completely integrally closed (hence pseudo-integrally closed) one-dimensional quasi-local domain which is not a valuation domain. Thus pseudo-integrally closed domains need not be essential.

We next point out situations in which the pseudo-integral closure of a domain coincides with either the integral closure or the complete integral closure. Of course, if the domain $R$ is Noetherian, then $R' = \tilde{R} = R^*$. Now recall that if $R$ is a domain, then $R'$ is the directed union of $\{(I : I) : I$ is a nonzero finitely generated ideal of $R\}$. Thus if $R$

has the property that $I^{-1}$ is finitely generated for each nonzero finitely generated ideal $I$ of $R$, that is, if $R$ is *quasi-coherent* [2], then $R' = \tilde{R}$. Of course, this implies that $R' = \tilde{R}$ for each coherent domain $R$. On the other hand, it is easy to see that $\tilde{R} = R^*$ if $R$ has the property that for each nonzero ideal $I$ of $R$ there is a finitely generated ideal $J$ of $R$ for which $I_v = J_v$. It is known [16, Théorème 1] that Mori domains, domains satisfying the ascending chain condition on divisorial ideals, possess this property. Hence we can restate the well-known characterization of Krull domains as completely integrally closed Mori domains as follows: a domain $R$ is a Krull domain $\Leftrightarrow$ $R$ is a pseudo-integrally closed Mori domain (cf. [20]).

It is known that if $R$ is a domain, then $R^*$ is integrally closed [18, p. 76] but need not be completely integrally closed [8, Example 1]. We now provide the first step in the proof that pseudo-integral closure behaves similarly.

THEOREM 1.2. *Let $T$ be an overring of the domain $R$, and let $x$ be an element of $K$. Suppose that $T$ is pseudo-integral over $R$ and that $x$ is integral over $T$. Then $x$ is pseudo-integral over $R$. In particular, $\tilde{R}$ is integrally closed.*

PROOF. From the equation of integrality satisifed by $x$, there are elements $u_1, \ldots, u_p \in T$ with $x$ integral over $S = R[u_1, \ldots, u_p]$. Since each $u_i$ is pseudo-integral over $R$, there is a nonzero finitely generated ideal $J_i$ of $R$ such that $u_i(J_i)_v \subseteq (J_i)_v$. Let $J = \prod J_i$; then $u_i J_v \subseteq J_v$ for each $i$. Since $(J_v : J_v)$ is a ring, it follows that $S \subseteq (J_v : J_v)$. Now, since $x$ is integral over $S$, there is a nonzero finitely generated ideal $I = Sz_1 + \cdots + Sz_q$ of $S$ with $xI \subseteq I$. Let $A = Jz_1 + \cdots + Jz_q$. Then $A$ is a finitely generated ideal of $R$, and we shall complete the proof by showing that $x \in (A_v : A_v)$. First note that $JSz_j \subseteq J_v z_i$ for each $i$. Thus $JI \subseteq J_v z_1 + \cdots + J_v z_q \subseteq (Jz_1 + \cdots + Jz_q)_v = A_v$. Since $xz_i \in I$ for each $i$, we therefore have $xA = x(Jz_1 + \cdots + Jz_q) \subseteq JI \subseteq A_v$. It follows that $x \in (A_v : A_v)$, as desired. ∎

Recall that a *Prüfer v-multiplication domain*(PVMD) is a domain in which the fractional $v$-ideals of finite type form a group under $v$-multiplication. Equivalently, a PVMD is a $v$-domain $R$ in which $I^{-1}$ is a $v$-ideal of finite type for each nonzero finitely generated ideal $I$ of $R$. In [11] the authors introduced the notion of a UMT-domain. To define this property, let $R$ be a domain and consider the polynomial ring $R[X]$. A nonzero prime ideal $P$ of $R[X]$ is said to be an *upper to zero* if $P \cap R = (0)$; equivalently, $P$ is an upper to zero if $P$ is the contraction of a nonzero prime from $K[X]$. Then $R$ is a *UMT-domain* if each upper to zero contains an element $f$ with $c(f)_v = R$, where $c(f)$ is the ideal generated by the coefficients of $f$. It was observed [11, Proposition 3.2] that a domain $R$ is a PVMD $\Leftrightarrow$ $R$ is an integrally closed UMT-domain, and it was shown [11, Proposition 3.3] that the integral closure of a quasi-coherent UMT-domain is a PVMD. The question of whether the hypothesis of quasi-coherence is necessary was left open. As an application of Theorem 1.2, we prove that the pseudo-integral closure of a UMT-domain is a PVMD. We need one further idea. An overring $T$ of a domain $R$ is said to be *t-linked over $R$* if $(T : (T : IT)) = T$ for each nonzero finitely generated ideal $I$ of $R$ for which $I_v = R$. In [5] it was shown that, for every domain $R$, $\tilde{R}$ is t-linked over $R$.

PROPOSITION 1.3. *If $R$ is a UMT-domain, then $\tilde{R}$ is a PVMD.*

PROOF. Since (by Theorem 1.2) $\tilde{R}$ is integrally closed, it suffices to prove that $\tilde{R}$ is a UMT-domain. Accordingly, let $P$ be an upper to zero in $\tilde{R}[X]$. Then $P \cap R[X]$ is an upper to zero in $R[X]$, whence $P \cap R[X]$ contains an element $f$ with $c_R(f)_v = R$. Let $I = c_R(f)$. Then $I\tilde{R} = c_{\tilde{R}}(f)$, and the conclusion follows from the fact that $\tilde{R}$ is t-linked over $R$. ∎

Next, we wish to show that pseudo-integrality behaves well under passage to the polynomial ring, that is, that $(\widetilde{R[X]}) = \tilde{R}[X]$. In fact, we shall do this in the more general context of semigroup rings. The reader is referred to [7] for background and terminology. Thus let $S$ be a commutative, additive, cancellative, torsion-free monoid, and let $G$ be the quotient group of $S$. For each fractional ideal $J$ of $S$, let $J^{-1} = \{g \in G : g + J \subseteq S\}$ and $J_v = (J^{-1})^{-1}$. We then define $\tilde{S}$ to be the set of elements $g$ of $G$ such that $g + J_v \subseteq J_v$ for some finitely generated ideal $J$ of $S$.

It is convenient to state one preliminary result.

LEMMA 1.4. *Let $R$ be a domain, and let the domain $T$ be a flat extension of $R$. Then $\tilde{R} \subseteq \tilde{T}$. (This can fail without the flatness assumption—see Example 3.2 below.)*

PROOF. Since finite intersections of ideals are preserved by flat extensions, it is easy to see that $T : IT = I^{-1}T$ for each finitely generated ideal $I$ of $R$. The result now follows easily. ∎

THEOREM 1.5. *Let $R$ be a domain and let $S$ be as above. Then $(\widetilde{R[S]}) = \tilde{R}[\tilde{S}]$.*

PROOF. Let $u \in \tilde{R}$ and $g \in \tilde{S}$. Then $uI_v \subseteq I_v$ and $g + J_v \subseteq J_v$ for some finitely generated ideals $I$ of $R$ and $J$ of $S$. It is easy to see that $L = I[J]$ is a finitely generated ideal of the semigroup ring $R[S] = R[\{X^g : g \in S\}]$ and that $L_v = I_v[J_v]$. Now $uX^gL_v \subseteq (uI_v)[g + J_v] \subseteq I_v[J_v] = L_v$, so that $uX^g \in (\widetilde{R[S]})$. Thus $\tilde{R}[\tilde{S}] \subseteq (\widetilde{R[S]})$.

To establish the opposite inclusion, we first observe that $(\widetilde{R[S]}) \subseteq K[G]$ since $K[G]$ is completely integrally closed [7, Corollary 12.6(2)]. Let $k$ be an element of $K[G]$ which is pseudo-integral over $R[S]$. Then $kI_v \subseteq I_v$ for some nonzero finitely generated ideal $I$ of $R[S]$. For $f = \sum a_gX^g \in K[G]$, let $c(f)$ denote the content ideal of $f$, that is, $c(f)$ is the fractional ideal of $R$ generated by the coefficients $a_g$. Write $c(I)$ for the ideal generated by the coefficients of all the elements of $I$. Since $I$ is finitely generated, we may write $c(I) = c(h)$ for some $h \in I$. By the content formula [15], there is a positive integer $m$ for which $c(h)^{m+1}c(k) = c(h)^mc(hk)$. Since $I \subseteq c(I)R[S]$, we have that $I_v \subseteq (c(I)R[S])_v = c(I)_vR[S]$, whence $c(I_v) \subseteq c(I)_v$. Therefore, since $hk \in I_v$, we have $c(I)^{m+1}c(k) = c(h)^{m+1}c(k) = c(h)^mc(hk) \subseteq c(I)^mc(I_v) \subseteq c(I)^mc(I)_v$. Thus $c(k)(c(I)^{m+1})_v \subseteq (c(I)^{m+1})_v$, whence $c(k) \subseteq \tilde{R}$ and $k \in \tilde{R}[G]$.

To complete the proof, it suffices to show that $k \in K[\tilde{S}]$, since $\tilde{R}[G] \cap K[\tilde{S}] = \tilde{R}[\tilde{S}]$. We introduce a little more notation. For $f = \sum a_gX^g \in K[G]$, let $C(f) = (\{X^g\})K[S] = K[J]$, where $J$ is the finitely generated ideal of $S$ generated by $\{g : a_g \neq 0\}$. Define $C(I)$, where $I$ is an ideal of $K[S]$, in the natural way. Now $R[S] \subseteq K[S]$ is flat, so that

$(\widetilde{R[S]}) \subseteq (\widetilde{K[S]})$ by Lemma 1.4. Since $k \in (\widetilde{R[S]})$, we therefore have a nonzero finitely generated ideal $I$ of $K[S]$ with $kI_v \subseteq I_v$. There is an element $f \in I$ with $C(I) = C(f) = K[J]$, as just described. We claim that $C(I_v) \subseteq C(I)_v$. This follows since $I_v \subseteq C(I)_v = K[J]_v = K[J_v] \Rightarrow C(I_v) \subseteq C(K[J_vI]) = K[J_v] = K[J]_v = C(I)_v$. As above, $C(k) \subseteq ((C(I)^{m+1})_v : (C(I)^{m+1})_v)$. Now $C(I) = K[J]$ so that $C(I)^{m+1} = K[(m+1)J]$. Thus by [7, Theorem 16.6], we have $C(k) \subseteq K[((m+1)J)_v : ((m+1)J)_v]$. Hence if $k = \sum b_g X^g$, then each $g \in ((m+1)J)_v : ((m+1)J)_v \subseteq \tilde{S}$. Hence $k \in K[\tilde{S}]$, as was to be shown. ∎

COROLLARY 1.6. *$(\widetilde{R[X]}) = \tilde{R}[X]$, where $X = \{X_\alpha\}$ is any set of indeterminates.*

PROOF. $R[X] = R[S]$, where $S = \oplus S_\alpha$ and each $S_\alpha$ is a copy of the additive monoid of nonnegative integers. Define $S^* = \{g \in G : ng + t \in S \text{ for some } t \in S \text{ and all } n \geq 1\}$ (cf. [7, p. 151]). Clearly, $S \subseteq \tilde{S} \subseteq S^*$. Since $(\oplus S_\alpha)^* = \oplus S_\alpha^* = \oplus S_\alpha$, we have that $\tilde{S} = S$. The result now follows from Theorem 1.5. ∎

Let $V$ be a valuation domain of dimension $\geq 2$. Then $V$ contains a nonunit $t$ with $\cap(t^n) \neq (0)$. Hence by [6, Proposition 13.11], $V[[X]]$ is not (pseudo-) integrally closed, and thus $(\widetilde{V[[X]]}) \underset{\neq}{\supset} V[[X]] = \tilde{V}[[X]]$. Thus Corollary 1.6 has no counterpart for power series rings.

We now turn to a discussion of how $D + M$ constructions behave with respect to pseudo-integrality.

LEMMA 1.7. *Let $V$ be a domain of the form $F + M$, where $F$ is a field and $M$ is the maximal ideal of $V$. Let $D$ be a subring of $F$, and let $R = D + M$. Then $\tilde{D} + M \subseteq \tilde{R}$.*

PROOF. We may assume that $D$ is not a field. Let $x = d+m \in \tilde{D}+M$. Then $dI_v \subseteq I_v$ for some nonzero finitely generated ideal $I$ of $D$. It follows that $I + M$ is a finitely generated ideal of $R$ and that $(I + M)_v = I_v + M$ ([3, Theorem 2.1(k)] and [1, Proposition 2.4]). Therefore, since $(d + m)(I_v + M) \subseteq I_v + M, x \in ((I + M)_v : (I + M)_v) \subseteq \tilde{R}$. ∎

PROPOSITION 1.8. *Let $V$ and $R$ be as above, and assume that $V$ is a valuation domain. Then*

*(i) $\tilde{R} = \tilde{D} + M$ if $F$ is the quotient field of $D$, and*

*(ii) $\tilde{R} = V$ if $F$ properly contains the quotient field of $D$.*

PROOF. (i) Suppose that $F$ is the quotient field of $D$. If $D = F$, then $\tilde{R} = V = \tilde{D}+M$. Suppose that $D \underset{\neq}{\subset} F$. Then each nonzero finitely generated fractional ideal $J$ of $R$ has the form $J = c(I + M)$ for some element $c \neq 0$ of the quotient field of $R$ and some nonzero finitely generated (integral) ideal $I$ of $D$ [3, Theorem 2.1(k)]. Thus $J_v : J_v = (I + M)_v : (I + M)_v = (I_v + M) : (I_v + M) = (I_v : I_v) + M$. It follows that $\tilde{R} \subseteq \tilde{D} + M$. The other inclusion follows from Lemma 1.7.

(ii) Now suppose that $F$ properly contains the quotient field $k$ of $D$. Choose $u \in K \setminus k$ and let $I = D+Du$. Then $J = I+M$ is a finitely generated fractional ideal of $R$, and $J_v = V$ [3, Theorem 4.3 and its proof]. Hence $J_v : J_v = V$, whence $V \subseteq \tilde{R}$. To obtain the other

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inclusion, let $A$ be a finitely generated ideal of $R$. By [3, Theorem 2.1(k)] $A = c(B+M)$ for some finitely generated $D$-submodule $B$ of $F$. If $B$ is not a fractional ideal of $D$, then $A_v : A_v = cV : cV = V$ [3, Theorem 4.3(2)]. If $B$ is a fractional ideal of $D$, then, as in the proof of (i) above, $A_v : A_v \subseteq \tilde{D} + M \subseteq V$. Thus $\tilde{R} \subseteq V$, and the proof is complete. ∎

We close this section by making use of the $D+M$ analysis to exhibit a domain $R$ for which $R \underset{\neq}{\subset} R' \underset{\neq}{\subset} \tilde{R} \underset{\neq}{\subset} R^*$.

EXAMPLE 1.9. *Let $F_1 \underset{\neq}{\subset} F_2 \underset{\neq}{\subset} F_3 \underset{\neq}{\subset} F_4$ be fields with $F_2$ algebraic over $F_1$ and $F_2$ algebraically closed in $F_3$. Let $V$ be a discrete rank one valuation domain (DVR) of the form $F_4 + M$, and let $W$ be a DVR of the form $F_3 + N$ and having quotient field $F_4$. Finally, set $R = (F_1 + N) + M$, $R_2 = (F_2 + N) + M$, and $R_3 = W + M$. Then*

*(i) $V$ is the complete integral closure of each of $R, R_2$, and $R_3$;*

*(ii) $R_3$ is the pseudo-integral closure of $R$ and $R_2$, and $R_3$ is pseudo-integrally closed; and*

*(iii) $R_2$ is the integral closure of $R$.*

In particular, $R \underset{\neq}{\subset} R' \underset{\neq}{\subset} \tilde{R} \underset{\neq}{\subset} R^*$.

PROOF. (i) Since $M$ is an ideal of each ring involved and $VM \subseteq M$, it follows that $V$ is contained in the complete integral closure of each ring involved. On the other hand $V$, being a DVR, is completely integrally closed.

(ii) It follows from Proposition 1.8 that $W$ is the pseudo-integral closure of each of the rings $D_1 = F_1 + N$ and $D_2 = F_2 + N$. Then, since each of $D_1, D_2$, and $W$ has quotient field $F_4$, it follows (again from Proposition 1.8) that $\tilde{R} = (\tilde{R_2}) = (\tilde{R_3}) = R_3$.

(iii) This is well-known [3, Theorem 2.1(b)]. ∎

REMARK 1.10. For a concrete example of the situation above, set $F_1 = \mathbb{Q}$, $F_2 = \overline{\mathbb{Q}}$ (= the algebraic closure of $\mathbb{Q}$), $F_3 = \mathbb{C}$, $F_4 = \mathbb{C}((X))$, $V = F_4[[Y]]$ (with $M = YF_4[[Y]]$), and $W = F_3[[X]]$(with $N = XF_3[[X]]$).

2. **The Bad.** We begin this section by showing that pseudo-integrality, like almost integrality, fails to be transitive, and that the pseudo-integral closure of a domain need not be pseudo-integrally closed. In fact, we can use the classic example of Gilmer-Heinzer [8] for our purpose.

EXAMPLE 2.1. Let $k$ be a field, set $R = k\left[\{X^{2n+1}Y^{n(2n+1)}\}_{n=0}^{\infty}\right]$, and set $T = k\left[\{XY^n\}_{n=0}^{\infty}\right]$. Then

(i) $R' = \tilde{R} = R^* = T$;

(ii) $\tilde{T} = T^* = k[X, Y]$; and

(iii) $Y$ is pseudo-integral over $T$ but not over $R$.

PROOF. (i) That $R^* = T$ is [6, Exercise 3, p. 144]. Moreover, $T$ is an integrally closed Mori domain [4, Example 4.6(b)]. Since $T$ is obviously integral over $R$, (i) follows.

(ii) Since $T$ is a Mori domain, $\tilde{T} = T^*$, and it is easy to see that $T^* = k[X, Y]$.

(iii) That $Y$ is pseudo-integral over $T$ follows from $\tilde{T} = T^*$. Finally, $Y$ is not almost integral, much less pseudo-integral, over $R$ since $R^* = T$. ∎

Recall that if $R$ is the ring of entire functions, then $R$ is a completely integrally closed Bézout domain, but that some localizations of $R$ fail to be completely integrally closed (cf. [6, Exercises 17, 19, and 21, pp. 147–148]). (This example is infinite dimensional, but Sheldon [17, Example II] gave a two-dimensional example.) A similar situation exists with respect to pseudo-integrality, as the following result shows.

PROPOSITION 2.2. *Let $R$ be a domain and let $S$ be a multiplicatively closed subset of $R$. Then $(\tilde{R})_S \subseteq (\widetilde{R_S})$. However, proper containment is possible. In particular, the property of being pseudo-integrally closed is not necessarily preserved upon passage to a localization.*

PROOF. That $\tilde{R} \subseteq (\widetilde{R_S})$ follows from Lemma 1.4. (An alternate proof can be based on [19, Lemma 4]. Since $1/s \in (\widetilde{R_S})$ for each $s \in S$, we have $(\tilde{R})_S \subseteq (\widetilde{R_S})$. To show that proper containment is possible, we examine an example of Heinzer [10]. This is an example of an essential domain $R$ (Heinzer uses $D$) containing a prime ideal $P$ such that $R_p$ is not essential. In fact, $R_p = k + M$, where $M$ is the maximal ideal of a valuation ring $W$ of the form $F + M$ and $k \underset{\neq}{\subset} F$ are fields. By Proposition 1.8, $(\widetilde{R_p}) = W$. However, $R$, being essential, is pseudo-integrally closed. Therefore, if we set $S = R \backslash P$, we have that $(\tilde{R})_S = R_S = R_p$, while $(\widetilde{R_S}) = (\widetilde{R_p}) = W$. ∎

REMARK. If $R$ is a Mori domain, then $(\tilde{R})_S = (\widetilde{R_S})$ for each multiplicatively closed subset $S$ of $R$ ([12, Lemma 5.11]).

## 3. **The Ugly.**

REMARK 3.1. The intersection of integrally closed domains (contained in some common field) is integrally closed. Similarly, the intersection of completely integrally closed domains is completely integrally closed. However, the domain $T$ of Example 2.1, being integrally closed, is the intersection of a family of valuation domains inside its quotient field, and, since valuation domains are pseudo-integrally closed, $T$ is the intersection of pseudo-integrally closed domains. Of course, $T$ itself is not pseudo-integrally closed.

EXAMPLE 3.2. *Let $V = F + M$ be a valuation domain with maximal ideal $M$ and with $F$ a field. Let $D$ be a pseudo-integrally closed subring of $F$ such that $D$ contains a field $k$. Suppose that $F$ is the quotient field of $D$. Set $R = k + M$ and $T = D + M$. Then $T$ is pseudo-integral over $R$, but $\tilde{R} \not\subseteq \tilde{T}$.*

PROOF. By Proposition 1.8, $\tilde{R} = V$ and $\tilde{T} = \tilde{D} + M = T$. Thus $V$, hence also $T$, is pseudo-integral over $R$, but $\tilde{R} \not\subseteq \tilde{T}$. ∎

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