will soon be in every college library. The printing is pleasing and the book is moderately priced.

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Calculus of variations and partial differential equations of the first order. Part II : Calculus of variations, by C. Carathéodory. (Translated from the German by Robert B. Dean). Holden-Day Inc., San Francisco, 1967. xvi + 398 pages. U.S. $10.75.

The original (1935) German version of this work consists of two parts but was published as a single volume. The long-awaited translation follows the trend of a more recent - essentially unchanged - edition; the first translated volume, entitled "Partial Differential Equations of the First Order", appeared in 1966.

The underlying philosophy of the second volume can best be described by briefly summarizing Carathéodory's views on the overall trends in the development of the calculus of variations at the time of writing. Three distinct approaches are recognized : the first is the variational calculus, begun by Lagrange, which is now a part of the tensor calculus; the second is the theory of Tonelli, which reveals the more delicate relationships between the minimum problem and set theory; and thirdly, the trend originated by Euler, which depends on the close association between the calculus of variations on the one hand, and the theory of differential equations on the other, and which is accordingly oriented towards differential geometry and physical applications. It is the latter trend with which the book under review is chiefly concerned, one of its main objectives being the inclusion of the theory of Weierstrass within this context.

After an elementary, but extremely rigorous survey of the theory of extreme values of functions of several variables, in particular of quadratic forms subject to constraints, the fundamentals of the simplest problem in the calculus of variations are treated from a local point of view. Already at this stage the book departs significantly from the well-trodden path of its many predecessors : the treatment depends on Carathéodory's brilliant use of the concept of equivalent integrals, which not only leads directly to the so-called fundamental equations of the calculus of variations, but which also suggests the immediate introduction of appropriate canonical variables. (It is, perhaps, worth remarking that later writers have referred to this approach as "der Königsweg von Carathéodory".) A subsequent chapter deals with the corresponding theory for parameter-invariant problems (theory of Weierstrass); again particular stress is laid on an essentially new canonical formalism and a certain class of Hamiltonian functions. One of the most remarkable features of the book is the thoroughness with which many diverse and intrinisically important examples are treated : this is done in particular in the subsequent chapter on positive definite variational problems, in which also differential-geometric concepts such as the indicatrix, figuratrix, transversality and families of geodesically equidistant hypersurfaces are described. The next chapter is devoted to quadratic variational problems and consequently to the theory of the second variation and the accessory problem. Again, the corresponding canonical formalism plays a special role in the construction of the relevant fields of extremals. Naturally, this involves a detailed discussion of focal points, focal surfaces, conjugate points and the envelope theorems.

Up to this stage the book is primarily concerned with curves of which it is merely expected that sufficiently small subarcs display the required extremum properties. The general boundary value problem, according to which a curve possessing these attributes is required to pass through two given fixed points, is therefore now investigated in detail, both locally and globally. By means of natural examples, the fundamental difficulties associated with this problem are revealed; nevertheless remarkably strong general local and global theorems are subsequently obtained, most of which are published here for the first time. This is followed by a short chapter on the (global) theory of closed extremals and periodic variational problems, which represents an extension of the work of Poincaré and Hadamard. The final chapter is concerned with the problem of Lagrange which results from the imposition of constraints. Firstly, it is shown that the usual formulation of such problems is entirely unsatisfactory owing to the fact that constraints often render the variation of admissible curves of comparison impossible; thus a new formulation is devised, again in terms of the method of equivalent integrals, which is treated once more by means of an appropriate canonical formalism. Secondly, a description of the author's intricate and conceptually difficult theory of what he calls the 'class' of a problem of Lagrange is given : this is concerned (roughly) with the determination of the dimensionality of distinguished fields of extremals of such problems.

Although 33 years have elapsed since the first appearance of this book, there is no doubt that there is no single modern work which can claim to have superseded this masterpiece of Carathéodory. A great deal of new work has resulted from the book, but even so the latter has not yet been fully exploited. It is to be expected that the new, competent translation under review will serve to stimulate further activities in this direction.

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Linear and quasi-linear elliptic equations, by O. Ladyzhenskaya and N. Ural'tseva. Academic Press, New York, 1968. xviii + 495 pages. U.S. $24.

The 19th problem of Hilbert asked whether every solution of an elliptic equation with smooth (e.g. analytic) coefficients was itself smooth (analytic); the 20th problem was to prove that variational problems (and their associated Euler equations) always have solutions, provided that "solution" is understood in a sufficiently general sense. (In particular, "solution" should presumably have the same meaning for both problems.) These two related problems have determined much of the subsequent research in partial differential equations. For the case of linear equations of arbitrary order, with continuous coefficients, questions concerning existence and regularity of generalized solutions were completely resolved by 1960 (at the latest); for a good survey see Partial Differential Equations, by L. Bers, F. John and M. Schechter, Interscience, New York (1964). For equations with discontinuous coefficients, and especially for non-linear equations, we have only partial solutions. Second order equations in two space variables can be treated, for example, by the techniques of quasi-conformal mappings; cf. Generalized Analytic Functions, by I. N. Vekua, Pergamon Press, London (1962).

The book under review is concerned with linear and quasi-linear equations of second order in n variables, having discontinuous coefficients. As is well known, the shape of the theory must depend strongly on whether the equation is in "divergence" form

$$Lu = \Sigma D_i(a_{ij}(x, u, u_x)D_i u) + \dots = f$$

or "non-divergence" form