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GALOIS MODULE STRUCTURE OF HOLOMORPHIC DIFFERENTIALS

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For a finite cyclic p–extension L/K of a rational function field $K = k(x)$ over an algebraically closed field k of characteristic $p > 0$ such that every ramified prime divisor is fully ramified, we find a basis of the $k[G]$–module $\Omega_L(0)$ of holomorphic differentials of L. We use this basis, which is similar to the Boseck-Garcia basis in the elementary abelian case, to find the $k[G]$–module structure of $\Omega_L(0)$ in terms of indecomposable modules.

1. INTRODUCTION

Let K/k be a field of algebraic functions of one variable, k algebraically closed. For a finite Galois extension L/k of K/k, the set of holomorphic differentials $\Omega_L(0)$ of L, is a $k[G]$–module, where $G = \mathrm{Gal}(L/K)$. We are interested in the structure of $\Omega_L(0)$ in terms of indecomposable $k[G]$–modules.

In the classical case, that is, when k is the field of complex numbers, this structure was determined by Chevalley and Weil [**2**]. In characteristic p this is an open problem. For wildly ramified extensions the structure has been obtained for general finite cyclic p–extensions [**8**]. For non–cyclic extensions, the answer is not known even for the general simplest case $G \cong C_p \times C_p$, where C_p denotes the cyclic group of p elements.

When G is an elementary abelian p–group, $K = k(x)$ is a rational function field, every prime divisor in K ramified in L is fully ramified and it has only one ramification number, the structure of $\Omega_L(0)$ was obtained in [**5**]. In that paper the main tool was a modification of the Boseck-Garcia bases of $\Omega_L(0)$ [**1, 3**] to make it convenient for the analysis of the Galois action. The result was formally the same as in the complementary cyclic case of [**8**]. The elementary abelian and the cyclic cases are complementary in the sense that in the cyclic case the genus grows very quickly [**6**] and in the elementary abelian case the genus is bounded by the Castelnuovo-Severi inequality.

Since the formal algebraic structure of $\Omega_L(0)$ in both cases, the cyclic and the elementary abelian cases, is the same, it should be possible to find a general result for

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arbitrary abelian p–extensions L/K. The first step must be to find a unified proof for the cyclic and the elementary abelian cases. This is the motivation for this note.

Here we obtain, in a special case, the Galois module structure of $\Omega_L(0)$ for the p–cyclic case. We consider the case when $K = k(x)$ and every ramified prime divisor of K in L is fully ramified. First we find a new k–basis of $\Omega_L(0)$ similar to the Boseck-Garcia basis. This basis is different from the one found by Madden in [4]. This is the main result of Section 2 (Theorem 1). Once we have this basis we analyze the Galois action in a way similar to how it was done in [5]. The structure of $\Omega_L(0)$ is the main result of Section 3 (Theorem 5).

We observe that even though this proof is similar to the one in the elementary abelian case, the indecomposable modules that appear in the decomposition in each case are completely different.

2. Bases of holomorphic differentials

Let k be an algebraically closed field of characteristic $p > 0$. Let $K = k(x)$ be a rational function field of one variable over k. We consider a cyclic extension L/K of degree p^n, $n \geqslant 1$, such that every ramified prime divisor of K is fully ramified in L. Let $\mathrm{Gal}(L/K) = \langle\sigma\rangle \cong W_n(\mathbb{F}_p)$, the ring of Witt vectors of length n over the finite field of p elements $\mathbb{F}_p$. The extension L/K is given by a Witt equation

$$y^p \dot{-} y = \beta \tag{1}$$

where $y = (y_1, \dots, y_n) \in W_n(L)$, $\beta = (\beta_1, \dots, \beta_n) \in W_n(K)$ and $L = K(y) = K(y_1, \dots, y_n)$ (see [9]).

Without loss of generality, we may assume that $\wp_\infty$, the infinite prime divisor of K, is unramified in L. For a Witt vector $X = (X_1, \dots, X_n \mid X^{(1)}, \dots, X^{(n)})$, $X^{(i)}$, $1 \leqslant i \leqslant n$, will denote the "ghost" components of X. We have

$$\left\{\begin{array}{rcl} X^{(1)} & = & X_1 \\ X^{(2)} & = & X_1^p + pX_2 \\ \vdots & & \\ X^{(t)} & = & X_1^{p^{t-1}} + pX_2^{p^{t-2}} + \dots + p^{t-1}X_t = \displaystyle\sum_{j=1}^{t} p^{j-1}X_j^{p^{t-j}} \\ \vdots & & \\ X^{(n)} & = & X_1^{p^{n-1}} + pX_2^{p^{n-2}} + \dots + p^{n-1}X_n \end{array}\right. \tag{2}$$

Let $1 = (1, 0, \dots, 0 \mid 1, 1, \dots, 1)$. The Galois action is given by $\sigma y = y \dot{+} 1 = (\sigma y_1, \dots, \sigma y_n \mid y^{(1)} + 1, \dots, y^{(n)} + 1)$.

Therefore we have $y^{(i)} + 1 = (\sigma y)^{(i)}$. It follows that

$$\begin{aligned} & y_1^{p^{i-1}} + py_2^{p^{i-2}} + \dots + p^{j-1}y_j^{p^{i-j}} + \dots + p^{i-1}y_i + 1 = \\ & \qquad = (\sigma y_1)^{p^{i-1}} + p(\sigma y_2)^{p^{i-2}} + \dots + p^{j-1}(\sigma y_j)^{p^{i-j}} + \dots + p^{i-1}(\sigma y_i). \end{aligned} \tag{3}$$

From (3), we obtain

$$\sigma y_1 = y_1 + 1, \text{ and for } i \geqslant 2,$$

(4) $$\sigma y_i = y_i + \frac{1}{p^{i-1}}\left(y_1^{p^{i-1}} + 1 - (\sigma y_1)^{p^{i-1}}\right) + \frac{1}{p^{i-1}}\sum_{j=2}^{i-1} p^{j-1}\left(y_j^{p^{i-j}} - (\sigma y_j)^{p^{i-j}}\right).$$

Proposition 1. *For any $1 \leqslant s \leqslant n$ we have*

(5) $$\sigma y_s = y_s + f_s(y_1, \dots, y_{s-1})$$

for some $f_s(T_1, \dots, T_{s-1}) \in \mathbb{Z}[T_1, \dots, T_{s-1}]$.

Proof: For $s = 1$, let $f_1 = 1$. From (4) we obtain (formally)

$$\sigma y_2 = y_2 + \frac{1}{p}(y_1^p + 1 - (y_1 + 1)^p) = y_2 - \frac{1}{p}\sum_{t=1}^{p-1}\binom{p}{t} y_1^t$$

$$= y_2 - \sum_{t=1}^{p-1}\frac{1}{t}\binom{p-1}{t-1} y_1^t = y_2 + f_2(y_1), \quad f_2(T) \in \mathbb{Z}[T].$$

We assume that (5) holds for $1 \leqslant j \leqslant s < n$, that is,

$$\sigma y_j = y_j + f_j(y_1, \dots, y_{j-1}), \text{ for some } f_j(T_1, \dots, T_{j-1}) \in \mathbb{Z}[T_1, \dots, T_{j-1}].$$

For $s + 1$ we obtain from (4):

$$\sigma y_{s+1} = y_{s+1} + \frac{1}{p^s}\left(y_1^{p^s} + 1 - (\sigma y_1)^{p^s}\right) + \frac{1}{p^s}\sum_{j=2}^{s} p^{j-1}\left(y_j^{p^{s+1-j}} - (\sigma y_j)^{p^{s+1-j}}\right)$$

$$= y_{s+1} + \frac{1}{p^s}\left(y_1^{p^s} + 1 - (y_1 + 1)^{p^s}\right)$$

$$+ \frac{1}{p^s}\sum_{j=2}^{s} p^{j-1}\left(y_j^{p^{s+1-j}} - (y_j + f_j(y_1, \dots, y_{j-1}))^{p^{s+1-j}}\right)$$

$$= y_{s+1} - \sum_{t_1=1}^{p^s-1}\frac{1}{p^s}\binom{p^s}{t_1} y_1^{t_1}$$

$$+ \sum_{j=2}^{s}\frac{1}{p^{s+1-j}}\Bigg[-\sum_{t_j=1}^{p^{s+1-j}-1}\binom{p^{s+1-j}}{t_j} y_j^{t_j} f_j(y_1, \dots, y_{j-1})^{p^{s+1-j}-t_j}$$

$$- \quad f_j(y_1, \dots, y_{j-1})^{p^{s+1-j}}\Bigg]$$

$$= y_{s+1} - \sum_{t_1=1}^{p^s-1}\frac{1}{t_1}\binom{p^s-1}{t_1-1} y_1^{t_1}$$

$$- \sum_{j=2}^{s}\sum_{t_j=1}^{p^{s+1-j}-1}\left(\frac{1}{t_j}\binom{p^{s+1-j}-1}{t_j-1}\right) y_j^{t_j} f_j(y_1, \dots, y_{j-1})^{p^{s+1-j}-t_j}$$

$$- \left. \frac{1}{p^{s+1-j}} f_j(y_1, \dots, y_{j-1})^{p^{s+1-j}} \right) . \tag{6}$$

Therefore $\sigma y_{s+1} = y_{s+1} + f_{s+1}(y_1, \dots, y_s)$, with $f_{s+1}(T_1, \dots, T_s) \in \mathbb{Q}[T_1, \dots, T_s]$.

Now, the only possible denominators in (6) are powers of p. From [**9**] we have that this power is 0. Thus $f_{s+1}(T_1, \dots, T_s) \in \mathbb{Z}[T_1, \dots, T_s]$. ∎

We also have a different generation of L/K given by Madden [**4**]. This generation is given as follows: $L = K(\widetilde{y}_1, \dots, \widetilde{y}_n)$ such that

$$\widetilde{y}_i^p - \widetilde{y}_i = \gamma_i \in K_{i-1} = K(\widetilde{y}_1, \dots, \widetilde{y}_{i-1}) = K(y_1, \dots, y_{i-1}) \tag{7}$$

and γ_i is in normal form for every prime divisor of K_{i-1}, that is, for every prime divisor $\wp$ of K_{i-1}, $v_\wp(\gamma_i) = \chi_i$ with $\chi_i \geqslant 0$ or $\chi_i < 0$ and relatively prime to p.

Furthermore $\widetilde{y}_i$ can be chosen such that $\sigma^{p^{i-1}} \widetilde{y}_i = \widetilde{y}_i + 1$. We also have $\sigma^{p^{i-1}} y_i = y_i + 1$. Therefore $\sigma^{p^{i-1}}(\widetilde{y}_i - y_i) = \widetilde{y}_i - y_i$. Hence $\widetilde{y}_i - y_i \in K_{i-1}$ and $\widetilde{y}_i = y_i + c$, $c \in K_{i-1} = K[y_1, \dots, y_{i-1}] = K[\widetilde{y}_1, \dots, \widetilde{y}_{i-1}]$. That is, $c = g_{i-1}(y_1, \dots, y_{i-1})$, $g_{i-1}(T_1, \dots, T_{i-1}) \in k(x)[T_1, \dots, T_{i-1}]$. Thus,

Proposition 2. *We have that* $\sigma \widetilde{y}_i = \widetilde{y}_i + p_i(\widetilde{y}_1, \dots, \widetilde{y}_{i-1})$ *with* $p_i(T_1, \dots, T_{i-1}) \in k(x)[T_1, \dots, T_{i-1}]$.

Let $\wp$ be a prime divisor of K fully ramified in L/K. Let $\lambda_1, \cdots, \lambda_n$ be the lower ramification numbers of $\wp$ (see [**7**]). Then, the ramification groups satisfy

$$\begin{aligned} G = G_0 = \cdots = G_{\lambda_1} \supsetneqq G_{\lambda_1+1} = \cdots = G_{\lambda_2} \supsetneqq \cdots \supsetneqq G_{\lambda_{n-1}+1} \\ = \cdots = G_{\lambda_n} \supsetneqq G_{\lambda_n+1} = \{e\}, \end{aligned}$$

with $G_{\lambda_i} \cong \mathbb{Z}/p^{n+1-i}\mathbb{Z}$.

The differential exponent of the prime divisor $\mathfrak{P}$ in L above $\wp$ is given by

$$\begin{aligned} \alpha = \sum_{i=0}^{\infty} (|G_i| - 1) &= (\lambda_1 + 1)(p^n - 1) + (\lambda_2 - \lambda_1)(p^{n-1} - 1) \\ &\qquad + \cdots + (\lambda_n - \lambda_{n-1})(p - 1) \\ &= (\lambda_1 + 1)(p^n - 1) + \sum_{j=1}^{n-1} \lambda_{j+1}(p^{n-j} - 1) - \sum_{j=1}^{n-1} \lambda_j (p^{n-j} - 1) \\ &= (p-1) \sum_{j=1}^{n} p^{n-j} \lambda_j + (p^n - 1). \end{aligned} \tag{8}$$

Schmid [**6**] defined the following invariants: for fixed $\wp$, we can choose a Witt equation $z^p \dot{-} z = \delta$, $(\delta_j)_{K_{j-1}} = A_j / \wp^{\gamma_j}$ with $\gamma_j \leqslant 0$ or $\gamma_j > 0$ and $(\gamma_j, p) = 1$ for all $j = 1, \dots, n$.

Let $s_j = \max\{\gamma_j, 0\}$, and $M_j = \max\{p^{j-\nu}s_j \mid 1 \leqslant \nu \leqslant j\}$. Then

$$\alpha = (p-1)\sum_{j=1}^{n}(M_j+1)p^{j-1}. \tag{9}$$

From (8) and (9) we obtain

$$\lambda_j = p^{j-1}M_j - (p-1)\sum_{\nu=1}^{j-1}p^{\nu-1}M_\nu. \tag{10}$$

We have

Proposition 3.

$$\lambda_j \geqslant p(p-1)\sum_{\nu=1}^{j-1}p^{j-1-\nu}\lambda_\nu.$$

Proof: See [4]. ∎

From (1) and (7) it follows that we can choose $\beta_1 = \gamma_1 \in K$ such that

$$\beta_1 = \frac{h(x)}{(x-a_1)^{m_1}\cdots(x-a_r)^{m_r}},\ (m_i,p)=1,\ \deg h(x) \leqslant m_1+\cdots+m_r.$$

Recall that we are considering that the infinite prime is unramified.

We may assume without loss of generality that $a_i \neq 0$ for all $1 \leqslant i \leqslant r$. Let $(x-a_i)_{k(x)} = \wp_i/\wp_\infty$. Then $\wp_1, \ldots, \wp_r$ are precisely the ramified prime divisors of K and they are fully and wildly ramified. We denote by $\lambda(1,i), \ldots, \lambda(n,i)$ the lower ramification numbers of $\wp_i$ and let $M(1,i), \ldots, M(n,i)$ be the corresponding Schmid's invariants.

We have $\lambda(1,i) = m_i$, $1 \leqslant i \leqslant r$. Let $g_\mu(x) = \prod_{i=1}^{r}(x-a_i)^{m_i^{(\mu)}}$, where $\mu = \mu_1 + \mu_2 p + \cdots + \mu_{n-1}p^{n-2} + \mu_n p^{n-1}$, $0 \leqslant \mu_j \leqslant p-1$, $m_i^{(\mu)} \in \mathbb{Z}$, and let $t^{(\mu)} = \sum_{i=1}^{r} m_i^{(\mu)}$.

Set

$$\omega_{\mu,\nu} = x^\nu g_\mu^{-1}(x)\tilde{y}_1^{\mu_1}\cdots\tilde{y}_n^{\mu_n}\,dx. \tag{11}$$

It is easy to see that $\omega_{\mu,\nu}$ is a holomorphic differential for $0 \leqslant \nu \leqslant t^{(\mu)} - 2$ and

$$0 \leqslant m_i^{(\mu)} \leqslant \sum_{j=1}^{n}\frac{p^{n-j}\lambda(j,i)(p-1-\mu_j)+p^{n-j}(p-1)}{p^n}$$
$$= \frac{\sum_{j=1}^{n}p^{n-j}\lambda(j,i)(p-1-\mu_j)+(p^n-1)}{p^n},\quad 1 \leqslant i \leqslant r.$$

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Furthermore, if $\mu = p^n - 1$, that is, if $\mu_1 = \ldots = \mu_n = p - 1$, then

$$\left[\frac{\sum_{j=1}^{n}\left(p^{n-j}\lambda(j,i)(p-1-\mu_j)\right) + (p^n - 1)}{p^n}\right] = \left[\frac{p^n - 1}{p^n}\right] = 0.$$

Let

$$\mathcal{A} = \left\{ \omega_{\mu,\nu} = x^\nu g_\mu^{-1}(x)\widetilde{y}_1^{\mu_1}\cdots\widetilde{y}_n^{\mu_n}dx \mid \mu = \mu_1 + \mu_2 p + \cdots + \mu_{n-1}p^{n-2} + \mu_n p^{n-1}, \right.$$

(12) $$0 \leqslant \mu \leqslant p^n - 2, 0 \leqslant \nu \leqslant t^{(\mu)} - 2, t^{(\mu)} = \sum_{i=1}^{r} m_i^{(\mu)},\ m_i^{(\mu)}$$

$$\left. = \left[\frac{\sum_{j=1}^{n}\left(p^{n-j}\lambda(j,i)(p-1-\mu_j)\right) + (p^n - 1)}{p^n}\right], g_\mu(x) = \prod_{i=1}^{r}(x - a_i)^{m_i^{(\mu)}} \right\}.$$

Then $\mathcal{A}$ is a system of k–linearly independent holomorphic differentials.

The main result of this section is

THEOREM 1. *The set $\mathcal{A}$ given in* (12) *is a basis of the holomorphic differentials of the field L.*

PROOF: It suffices to show that the cardinality of $\mathcal{A}$ equals the genus g_L of L.

Now, for each $0 \leqslant \mu \leqslant p^n - 2$, we have $t^{(\mu)} - 1$ choices of $0 \leqslant \nu \leqslant t^{(\mu)} - 2$. Therefore $|\mathcal{A}| = \sum_{\mu=0}^{p^n-2}(t^{(\mu)} - 1)$.

Thus

$$|\mathcal{A}| = \sum_{\mu=0}^{p^n-2}\left(\sum_{i=1}^{r}\left(\left[\frac{\left(\sum_{j=1}^{n}p^{n-j}\lambda(j,i)(p-1-\mu_j)\right) + (p^n - 1)}{p^n}\right]\right) - 1\right)$$

(13) $$= \sum_{i=1}^{r}\sum_{\mu=0}^{p^n-2}\left[\frac{\left(\sum_{j=1}^{n}p^{n-j}\lambda(j,i)(p-1-\mu_j)\right) + (p^n - 1)}{p^n}\right] - (p^n - 1).$$

Let $\xi_j = p - 1 - \mu_j$. Then, when μ_j runs over $0 \leqslant \mu_j \leqslant p - 1$, ξ_j runs over the same set.

We have $\xi = \xi_1 + \xi_2 p + \cdots + \xi_{n-1}p^{n-2} + \xi_n p^{n-1} = (p-1-\mu_1) + (p-1-\mu_2)p + \cdots + (p-1-\mu_n)p^{n-1} = p^n - 1 - \mu$ and for $0 \leqslant \mu \leqslant p^n - 2$ we have $1 \leqslant \xi \leqslant p^n - 1$.

Therefore

$$|\mathcal{A}| = \sum_{i=1}^{r} \sum_{\xi=1}^{p^n-1} \left[\frac{\left(\sum_{j=1}^{n} p^{n-j}\lambda(j,i)\xi_j\right) + (p^n - 1)}{p^n} \right] - (p^n - 1). \tag{14}$$

We set

$$S_i^{(\xi)} = \frac{\left(\sum_{j=1}^{n} p^{n-j}\lambda(j,i)\xi_j\right) + (p^n - 1)}{p^n}, \quad 1 \leqslant i \leqslant r. \tag{15}$$

We write $\sum_{j=1}^{n} p^{n-j}\lambda(j,i)\xi_j + (p^n - 1) = Q_i^{(\xi)} p^n + R_i^{(\xi)}$ with

$$Q_i^{(\xi)} = \left[\frac{\left(\sum_{j=1}^{n} p^{n-j}\lambda(j,i)\xi_j\right) + (p^n - 1)}{p^n} \right],$$ the integral part of $S_i^{(\xi)}$, and

$$\frac{R_i^{(\xi)}}{p^n} = \left\{ \frac{\left(\sum_{j=1}^{n} p^{n-j}\lambda(j,i)\xi_j\right) + (p^n - 1)}{p^n} \right\} \in [0, 1)$$ the fractional part of $S_i^{(\xi)}$.

Let $T_1^{(i)} = \sum_{\xi=1}^{p^n-1} \left(\left(\sum_{j=1}^{n} p^{n-j}\lambda(j,i)\xi_j \right) + (p^n - 1) \right) \Big/ p^n$ and $T_2^{(i)} = \sum_{\xi=1}^{p^n-1} (R_i^{(\xi)}/p^n)$.

Then $\sum_{\xi=1}^{p^n-1} Q_i^{(\xi)} = T_1^{(i)} - T_2^{(i)}$.

It follows that

$$\begin{aligned} T_1^{(i)} &= \frac{1}{p^n} \sum_{j=1}^{n} p^{n-j}\lambda(j,i) \sum_{\xi=1}^{p^n-1} \xi_j + \frac{p^{2n} - 2p^n + 1}{p^n} \\ &= \frac{1}{p^n} d_i \left(p^{n-1} \sum_{t=1}^{p-1} t \right) + \left(p^n - 2 + \frac{1}{p^n} \right) \end{aligned}$$

where $d_i = \sum_{j=1}^{n} p^{n-j}\lambda(j,i)$ and $\sum_{\xi=1}^{p^n-1} \xi_j = p^{n-1} \sum_{t=1}^{p-1} t$ since each value $\xi_j \in \{1, \ldots, p-1\}$ is taken for arbitrary $\xi_1, \ldots, \xi_{j-1}, \xi_{j+1}, \ldots, \xi_n \in \{1, \ldots, p-1\}$ and $\xi_j = 0$ does not contribute to the sum.

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Therefore

$$T_1^{(i)} = \frac{1}{p^n} d_i p^{n-1} \frac{p(p-1)}{2} + \left(p^n - 2 + \frac{1}{p^n}\right) = \frac{d_i(p-1)}{2} + \left(p^n - 2 + \frac{1}{p^n}\right). \tag{16}$$

To evaluate $\sum_{\xi=1}^{p^n-1} (R_i^{(\xi)}/p^n) = \sum_{\xi=1}^{p^n-1} \left\{ \left(\left(\sum_{j=1}^{n} p^{n-j} \lambda(j,i) \xi_j \right) + (p^n - 1) \right) \Big/ p^n \right\}$, we note that for $1 \leqslant j \leqslant n$, $0 \leqslant \xi_j \leqslant p-1$ we have to consider the set

$$\left\{ \sum_{j=1}^{n} p^{n-j} \lambda(j,i) \xi_j \bmod p^n \right\}_{\xi=1}^{p^n-1}.$$

Let $f(\xi) = f(\xi_1, \dots, \xi_n) = \sum_{j=1}^{n} p^{n-j} \lambda(j,i) \xi_j \bmod p^n$.

If $f(\xi) = f(\xi')$, then $f(\xi - \xi') = 0$, that is

$$\sum_{j=1}^{n} p^{n-j} \lambda(j,i) \xi_j \equiv \sum_{j=1}^{n} p^{n-j} \lambda(j,i) \xi'_j \bmod p^n$$

so that $\sum_{j=1}^{n} p^{n-j} \lambda(j,i)(\xi_j - \xi'_j) \equiv 0 \bmod p^n$.

We will see that $\xi_j = \xi'_j$ for all j. It suffices to prove that if $|b_j| < p$, $b_j \in \mathbb{Z}$ and $\sum_{j=1}^{n} p^{n-j} \lambda(j,i) b_j \equiv 0 \bmod p^n$, then $b_j = 0$ for all j. We have that if $p^{n-1}\lambda(1,i)b_1 + p^{n-2}\lambda(2,i)b_2 + \cdots + p\lambda(n-1,i)b_{n-1} + \lambda(n,i)b_n \equiv 0 \bmod p^n$, then $p \mid \lambda(n,i)b_n$. Since $(\lambda(j,i), p) = 1$, $p \mid b_n$. Therefore $b_n = 0$.

Continuing in this way, we obtain $b_1 = \cdots = b_n = 0$. Therefore $f\colon \{1, \dots, p^n - 1\} \to \{1, \dots, p^n - 1\}$ is bijective. It follows that

$$T_2^{(i)} = \sum_{\xi=1}^{p^n-1} \frac{R_i^{(\xi)}}{p^n} = \sum_{\xi=1}^{p^n-1} \left\{ \frac{\left(\sum_{j=1}^{n} p^{n-j} \lambda(j,i) \xi_j \right) + (p^n - 1)}{p^n} \right\}$$

$$= \sum_{\xi=1}^{p^n-1} \left\{ \frac{\xi + p^n - 1}{p^n} \right\} = \frac{p^n}{2} - \frac{3}{2} + \frac{1}{p^n}. \tag{17}$$

From (16) and (17) we obtain

$$\sum_{\xi=1}^{p^n-1} Q_i^{(\xi)} = T_1^{(i)} - T_2^{(i)} = \frac{1}{2}\left(d_i(p-1) + p^n - 1\right). \tag{18}$$

From (14) and (18) it follows that

$$(19)\qquad |\mathcal{A}| = \sum_{i=1}^{r} \frac{1}{2}\big(d_i(p-1) + (p^n - 1)\big) - (p^n - 1) = \frac{(r-2)(p^n-1)}{2} + \frac{p-1}{2}\sum_{i=1}^{r} d_i.$$

Now from (8) we have that the differential exponent of the prime divisor $\mathfrak{P}_i$ in L above $\wp_i$ is given by

$$(20)\qquad \alpha_i = (p-1)\sum_{j=1}^{n} p^{n-j}\lambda(j,i) + (p^n - 1) = (p-1)d_i + (p^n - 1)$$

that is, the different of the extension L/K is given by

$$(21)\qquad \mathfrak{D}_{L/K} = \mathfrak{P}_1^{\alpha_1} \cdots \mathfrak{P}_r^{\alpha_r}.$$

Therefore, since $g_K = 0$, we obtain from the Riemann-Hurwitz formula

$$\begin{aligned} g_L &= 1 + (g_K - 1)[L : K] + \frac{1}{2}\deg\left(\mathfrak{D}_{L/K}\right) \\ &= 1 - p^n + \frac{1}{2}(p-1)\sum_{i=1}^{r} d_i + \frac{r(p^n-1)}{2} \\ (22)\qquad &= \frac{p-1}{2}\sum_{i=1}^{r} d_i + \frac{(r-2)(p^n-1)}{2}. \end{aligned}$$

The result follows form (19) and (22). ∎

3. Galois action on $\Omega_L(0)$

We use the k-basis of $\Omega_L(0)$ given in (12). In [5] the key fact in order to obtain the Galois module structure was that given $\omega_{\mu,\nu}$ a basis element, then

$$(23)\qquad \sigma\omega_{\mu,\nu} = \omega_{\mu,\nu} + \sum_{\mu' < \mu}\ \sum_{\nu' \leqslant t(\mu') - 2} c_{\mu',\nu'}\omega_{\mu',\nu'}.$$

We want to show that, in the present case, we have the same property.

First, we have

Proposition 4. *For any $0 \leqslant \mu \leqslant p^n - 1$ we have*

$$(\sigma - 1)^{\mu}\widetilde{y}_1^{\mu_1} \cdots \widetilde{y}_n^{\mu_n} = (\sigma - 1)^{\mu} y_1^{\mu_1} \cdots y_n^{\mu_n} = \mu_1! \cdots \mu_n!.$$

Proof: [8, Lemma 1, p.109]. ∎

Note that

$$\{\widetilde{w}_\mu = \widetilde{y}_1^{\mu_1} \cdots \widetilde{y}_n^{\mu_n} \mid 0 \leqslant \mu \leqslant p^n - 1\}$$

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and

$$\{w_\mu = y_1^{\mu_1} \cdots y_n^{\mu_n} \mid 0 \leqslant \mu \leqslant p^n - 1\}$$

are field bases for the extension L/K.

Let $\omega_{\mu,\nu} \in \mathcal{A}$. From Proposition 2 we have that

$$\begin{aligned} \sigma\omega_{\mu,\nu} &= x^\nu g_\mu^{-1}(x)\sigma\widetilde{y}_1^{\mu_1}\cdots\sigma\widetilde{y}_n^{\mu_n}dx \\ &= x^\nu g_\mu^{-1}(x)(\widetilde{y}_1+1)^{\mu_1}\left(\widetilde{y}_2+p_2(\widetilde{y}_1)\right)^{\mu_2}\cdots\left(\widetilde{y}_n+p_n(\widetilde{y}_1,\dots,\widetilde{y}_{n-1})\right)^{\mu_n}dx \end{aligned} \tag{24}$$

with $p_i(T_1,\dots,T_{i-1}) \in k(x)[T_1,\dots,T_{i-1}]$.

Therefore

$$\sigma\omega_{\mu,\nu} = \omega'_{\mu,\nu} + \omega''_{\mu,\nu} \tag{25}$$

where

(26)

$$\begin{aligned} \omega'_{\mu,\nu} &= x^\nu g_\mu^{-1}(x)(\widetilde{y}_1+1)^{\mu_1}\left(\widetilde{y}_2+p_2(\widetilde{y}_1)\right)^{\mu_2}\cdots\left(\widetilde{y}_{n-1}+p_{n-1}(\widetilde{y}_1,\dots,\widetilde{y}_{n-2})\right)^{\mu_{n-1}}\widetilde{y}_n^{\mu_n}dx, \\ \omega''_{\mu,\nu} &= \sum_{t=0}^{\mu_n-1} x^\nu g_\mu^{-1}(x)(\widetilde{y}_1+1)^{\mu_1}\left(\widetilde{y}_2+p_2(\widetilde{y}_1)\right)^{\mu_2} \\ &\qquad\cdots\left(\widetilde{y}_{n-1}+p_{n-1}(\widetilde{y}_1,\dots,\widetilde{y}_{n-2})\right)^{\mu_{n-1}}\binom{\mu_n}{t}p_n(\widetilde{y}_1,\dots,\widetilde{y}_{n-1})^{\mu_n-t}\widetilde{y}_n^t dx. \end{aligned}$$

Since $\beta \in W_n(K)$, it follows from Proposition 2 and (24) that

$$\omega''_{\mu,\nu} = \sum_{\mu',\nu} x^\nu g_\mu^{-1}(x)c_{\mu',\nu}\widetilde{y}_1^{\mu'_1}\cdots\widetilde{y}_n^{\mu'_n}dx \tag{27}$$

with $c_{\mu',\nu} \in k(x) = K$.

Let $\mu'_j > p$ in (27) for some μ'_j. Then, from (7) we have that

$$\widetilde{y}_j^p = \widetilde{y}_j + \gamma_j \tag{28}$$

and $\gamma_j \in K_{j-1}$ with $v_\wp(\gamma_j) \geqslant 0$ for all unramified prime divisors $\wp$ in K_{j-1}. From (25), (26), (27), and (28) it follows that

$$\sigma\omega_{\mu,\nu} = \left(\sum_{\xi=0}^{p^n-1} c_\xi(x)\widetilde{y}_1^{\xi_1}\cdots\widetilde{y}_n^{\xi_n}\right)dx \tag{29}$$

with $c_\xi(x) \in k(x)$. We shall prove that each term $c_\xi(x)\widetilde{y}_1^{\xi_1}\cdots\widetilde{y}_n^{\xi_n}dx$ is holomorphic.

Lemma 1. *Let $\wp$ be a prime divisor in K with $\mathfrak{P}^{(1)},\dots,\mathfrak{P}^{(s)}$ the prime divisors in L dividing $\wp$. Let v_i be the valuation corresponding to $\mathfrak{P}^{(i)}$, $1 \leqslant i \leqslant s$. Assume that $\widetilde{y}_j^p - \widetilde{y}_j = b_j$ with b_j in normal form for each prime divisor of K_{j-1} divided by some $\mathfrak{P}^{(i)}$. If $z = \sum_{\xi=0}^{p^n-1} c_\xi(x)\widetilde{y}_1^{\xi_1}\cdots\widetilde{y}_n^{\xi_n}$, then*

$$\min_{1\leqslant i\leqslant s} v_i(z) = \min_{\substack{1\leqslant \ell\leqslant s\\ 0\leqslant\xi\leqslant p^n-1}} v_\ell\left(c_\xi(x)\widetilde{y}_1^{\xi_1}\cdots\widetilde{y}_n^{\xi_n}\right).$$

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PROOF: [8, Lemma 2, p.109]. ∎

LEMMA 2. *We have that each individual term $c_\xi(x)\widetilde{y}_1^{\xi_1}\cdots\widetilde{y}_n^{\xi_n}dx$ in (29) is holomorphic.*

PROOF: From (7), we have that $\widetilde{y}_i^p - \widetilde{y}_i = \gamma_i$ where γ_i is in normal form for every prime divisor of K_{i-1}. Therefore, from Lemma 1 we obtain that if $\mathfrak{P}_i$ is a ramified prime divisor and $\wp_i$ is the corresponding prime divisor below it, then

$$v_{\mathfrak{P}_i}(\sigma\omega_{\mu,\nu}) = \min_{\mathfrak{P}|\wp_i} v_{\mathfrak{P}}(\sigma\omega_{\mu,\nu}) = v_{\mathfrak{P}_i}\left(\sum_{\xi=0}^{p^n-1} c_\xi(x)\widetilde{y}_1^{\xi_1}\cdots\widetilde{y}_n^{\xi_n}\right) + v_{\mathfrak{P}_i}(dx) + v_{\mathfrak{P}_i}\left(\mathfrak{D}_{L/K}\right)$$
$$= \min_{0\leqslant\xi\leqslant p^n-1} v_{\mathfrak{P}_i}\left(c_\xi(x)\widetilde{y}_1^{\xi_1}\cdots\widetilde{y}_n^{\xi_n}\right) + v_{\mathfrak{P}_i}\left(\mathfrak{D}_{L/K}\right) = \min_{0\leqslant\xi\leqslant p^n-1} v_{\mathfrak{P}_i}\left(c_\xi(x)\widetilde{y}_1^{\xi_1}\cdots\widetilde{y}_n^{\xi_n}dx\right).$$

Since $v_{\mathfrak{P}_i}(\sigma\omega_{\mu,\nu}) \geqslant 0$, it follows that $\min_{0\leqslant\xi\leqslant p^n-1} v_{\mathfrak{P}_i}\left(c_\xi(x)\widetilde{y}_1^{\xi_1}\cdots\widetilde{y}_n^{\xi_n}dx\right) \geqslant 0$.

Hence $v_{\mathfrak{P}_i}\left(c_\xi(x)\widetilde{y}_1^{\xi_1}\cdots\widetilde{y}_n^{\xi_n}dx\right) \geqslant 0$, $0 \leqslant \xi \leqslant p^n - 1$.

Similarly, for any prime divisor $\mathfrak{P}$ of L, we obtain

(30) $$v_{\mathfrak{P}}\left(c_\xi(x)\widetilde{y}_1^{\xi_1}\cdots\widetilde{y}_n^{\xi_n}dx\right) \geqslant 0, \quad 0 \leqslant \xi \leqslant p^n - 1.$$

Therefore $c_\xi(x)\widetilde{y}_1^{\xi_1}\cdots\widetilde{y}_n^{\xi_n}dx$ is a holomorphic differential for all $0 \leqslant \xi \leqslant p^n - 1$. ∎

The next result is the key fact to obtain the $k[G]$-module structure of $\Omega_L(0)$. This corresponds to (23).

THEOREM 2. *For any $\omega_{\mu,\nu} \in \mathcal{A}$ we have*

(31) $$\sigma\omega_{\mu,\nu} = \omega_{\mu,\nu} + \sum_{\mu'<\mu}\sum_{\nu'\leqslant t(\mu')-2} c_{\mu',\nu'}\omega_{\mu',\nu'}$$

where $c_{\mu',\nu'} \in k$.

PROOF: For $\mu' < \mu$ we have

$$g_\mu^{-1}(x) = g_{\mu'}^{-1}(x)\frac{g_{\mu'}(x)}{g_\mu(x)} = g_{\mu'}^{-1}(x)\prod_{i=1}^{r}(x-a_i)^{m_i^{(\mu')}-m_i^{(\mu)}}.$$

Using Proposition 3, it can be proved that $m_i^{(\mu')} \geqslant m_i^{(\mu)}$. Hence

(32) $$g_\mu^{-1}(x) = \left(\sum_{\ell=0}^{t(\mu')-t(\mu)} B_\ell x^\ell\right)g_{\mu'}^{-1}(x), \quad B_\ell \in k.$$

We have that the terms of $\omega''_{\mu,\nu}$ in (26) are of the form $\widetilde{y}_1^{b_1}\cdots\widetilde{y}_{n-1}^{b_{n-1}}\widetilde{y}_n^j$ with $0 \leqslant b_i \leqslant p-1$, $j \leqslant \mu_n - 1 < \mu_n$. Therefore, $\mu' = b_1 + b_2p + \cdots + b_{n-1}p^{n-2} + jp^{n-1} \leqslant (p-1)(1+p+\cdots+p^{n-2}) + (\mu_n-1)p^{n-1} = \mu_np^{n-1} - 1 < \mu_np^{n-1} \leqslant \mu$.

Now we consider the terms of $\omega'_{\mu,\nu}$ in (26). We have

$$x^\nu g_\mu^{-1}(x)(\widetilde{y}_1+1)^{\mu_1}\big(\widetilde{y}_2+p_2(\widetilde{y}_1)\big)^{\mu_2}\cdots\big(\widetilde{y}_{n-1}+p_{n-1}(\widetilde{y}_1,\dots,\widetilde{y}_{n-2})\big)^{\mu_{n-1}}\widetilde{y}_n^{\mu_n}\,dx$$
$$= x^\nu g_\mu^{-1}(x)\widetilde{y}_1^{\mu_1}\cdots\widetilde{y}_n^{\mu_n}\,dx+\sum_{\mu'<\mu}\sum_{\nu'}c_{\mu',\nu'}\omega_{\mu',\nu'}=\omega_{\mu,\nu}+\sum_{\mu'<\mu}\sum_{\nu'}c_{\mu',\nu'}\omega_{\mu',\nu'}.$$

Collecting terms, we obtain that

$$\sigma\omega_{\mu,\nu}=\omega_{\mu,\nu}+\sum_{\mu'<\mu}\sum_{\nu'}c_{\mu',\nu'}\omega_{\mu',\nu'}. \tag{33}$$

It remains to show that $\nu'\leqslant t^{(\mu')}-2$.

Note that,

$$\sum_{\nu'}c_{\mu',\nu'}\omega_{\mu',\nu'}=\sum_{\nu'}c_{\mu',\nu'}x^{\nu'}\omega_{\mu',0}=h_{\mu'}^{(\nu)}(x)\omega_{\mu',0},$$

where

$$h_{\mu'}^{(\nu)}(x)=\sum_{\nu'}c_{\mu',\nu'}x^{\nu'}. \tag{34}$$

That is, (33) is of the form

$$\sigma\omega_{\mu,\nu}=\omega_{\mu,\nu}+\sum_{\mu'=0}^{\mu-1}h_{\mu'}^{(\nu)}(x)\omega_{\mu',0}. \tag{35}$$

Thus (31) will follow from

$$\deg h_{\mu'}^{(\nu)}(x)\leqslant t^{(\mu')}-2. \tag{36}$$

From (35) we obtain

$$(\sigma-1)\omega_{\mu,\nu}=h_{\mu-1}^{(\nu)}(x)\omega_{\mu-1,0}+\sum_{\mu'=0}^{\mu-2}h_{\mu'}^{(\nu)}(x)\omega_{\mu',0}.$$

From Proposition 4, it follows that

$$(\sigma-1)^\mu\omega_{\mu,\nu}=x^\nu g_\mu^{-1}(x)\mu_1!\cdots\mu_n!dx$$

(37) and

$$(\sigma-1)^\mu\omega_{\mu',\nu'}=0 \text{ for } \mu'<\mu.$$

Therefore

$$\begin{aligned}\mu_1!\mu_2!\cdots\mu_n!x^\nu g_\mu^{-1}(x)dx&=(\sigma-1)^\mu\omega_{\mu,\nu}\\&=(\sigma-1)^{\mu-1}h_{\mu-1}^{(\nu)}(x)\omega_{\mu-1,0}+(\sigma-1)^{\mu-1}\left(\sum_{\mu'=0}^{\mu-2}h_{\mu'}^{(\nu)}(x)\omega_{\mu',0}\right)\\&=h_{\mu-1}^{(\nu)}(x)g_{\mu-1}^{-1}(x)(\mu-1)_{(1)}!\cdots(\mu-1)_{(n)}!dx\end{aligned}$$

where $\mu - 1 = (\mu-1)_{(1)} + (\mu-1)_{(2)}p + \cdots + (\mu-1)_{(n)}p^{n-1}$, $0 \leqslant (\mu-1)_{(i)} \leqslant p-1$.

Therefore $h^{(\nu)}_{\mu-1}(x) = \Big((\mu_1!\mu_2!\cdots\mu_n!)/((\mu-1)_{(1)}!\cdots(\mu-1)_{(n)}!)\Big)x^{\nu}(g_{\mu-1}(x)/g_{\mu}(x))$ is a polynomial of degree $\nu + t^{(\mu-1)} - t^{(\mu)} \leqslant t^{(\mu-1)} - 2$.

That is,

$$\deg h^{(\nu)}_{\mu-1}(x) = \nu + t^{(\mu-1)} - t^{(\mu)}. \tag{38}$$

Now from Lemma 2 we have that

$$h^{(\nu)}_{\mu'}(x)\omega_{\mu',0} \in \Omega_L(0). \tag{39}$$

In particular, if we let $\nu = t^{(\mu)} - 2$, we have that

$$\sigma\omega_{\mu,t^{(\mu)}-2} = \omega_{\mu,t^{(\mu)}-2} + \sum_{\mu'=0}^{\mu-1} h^{(t^{(\mu)}-2)}_{\mu'}(x)\omega_{\mu',0}.$$

It follows that $\deg h^{(t^{(\mu)}-2)}_{\mu'} \leqslant t^{(\mu')} - 2$.

For $0 \leqslant \nu \leqslant t^{(\mu)} - 2$, $x^{-\nu+t^{(\mu)}-2}\omega_{\mu,\nu} = \omega_{\mu,t^{(\mu)}-2}$. Hence, on the one hand

$$\begin{aligned}\sigma(x^{-\nu+t^{(\mu)}-2}\omega_{\mu,\nu}) &= x^{-\nu+t^{(\mu)}-2}\sigma\omega_{\mu,\nu}\\ &= \omega_{\mu,t^{(\mu)}-2} + \sum_{\mu'=0}^{\mu-1} x^{-\nu+t^{(\mu)}-2}h^{(\nu)}_{\mu'}(x)\omega_{\mu',0}.\end{aligned} \tag{40}$$

On the other hand

$$\sigma(x^{-\nu+t^{(\mu)}-2}\omega_{\mu,\nu}) = \sigma(\omega_{\mu,t^{(\mu)}-2}) = \omega_{\mu,t^{(\mu)}-2} + \sum_{\mu'=0}^{\mu-1} h^{(t^{(\mu)}-2)}_{\mu'}(x)\omega_{\mu',0}. \tag{41}$$

From (40) and (41) it follows that

$$h^{(\nu)}_{\mu'}(x) = x^{\nu-t^{(\mu)}+2}h^{(t^{(\mu)}-2)}_{\mu'}(x), \nu = 0, 1, \ldots, t^{(\mu)} - 2.$$

Thus

$$\deg h^{(\nu)}_{\mu'} = \nu + 2 - t^{(\mu)} + \deg h^{(t^{(\mu)}-2)}_{\mu'}. \tag{42}$$

Therefore $\deg h^{(\nu)}_{\mu'} = \nu+2-t^{(\mu)}+\deg h^{t^{(\mu)}-2}_{\mu'} \leqslant \nu+2-t^{(\mu)}+t^{(\mu')}-2 = \nu+t^{(\mu')}-t^{(\mu)} \leqslant t^{(\mu')} - 2$.

The result follows. ∎

Theorem 3. *Let $\mathcal{A}_{\mu,\nu} := k[G]\omega_{\mu,\nu} \cong k[G]/\mathrm{ann}(\omega_{\mu,\nu})$. Then, as $k[G]$-modules, we have that $\mathcal{A}_{\mu,\nu} \cong k[G]/(\sigma-1)^{\mu+1}$ and $\mathcal{A}_{\mu,\nu}$ is an indecomposable $k[G]$-module.*

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PROOF: We have from Proposition 4 that $(\sigma-1)^{\mu}\omega_{\mu,\nu} = \mu_1!\cdots\mu_n! x^{\nu} g_{\mu}^{-1}(x)dx \neq 0$ and $(\sigma-1)^{\mu+1}\omega_{\mu,\nu} = 0$. Hence $(\sigma-1)^{\mu+1} \in \operatorname{ann}(\omega_{\mu,\nu})$.

Let $\phi\colon k[G]/(\sigma-1)^{\mu+1} \to \mathcal{A}_{\mu,\nu}$ be the natural epimorphism. It follows that $\dim_k \mathcal{A}_{\mu,\nu} \leqslant \dim_k k[G]/(\sigma-1)^{\mu+1} = \mu+1$.

It is well known that the indecomposable $k[G]$-modules are $k[G]/(\sigma-1)^s$, $s = 0,1,\dots p^n-1$. From the Krull-Schmidt-Azumaya Theorem, we obtain the $k[G]$ decomposition

$$\mathcal{A}_{\mu,\nu} \cong \bigoplus_{i=0}^{p^n-1} L(i)^{\alpha_i} \tag{43}$$

with $L(i) = k[G]/(\sigma-1)^i$.

Since $(\sigma-1)^{\mu}\mathcal{A}_{\mu,\nu} \neq 0$, from (43) we obtain that $\alpha_{\mu'} \neq 0$ for some $\mu' \geqslant \mu+1$. Therefore $\dim_k \mathcal{A}_{\mu,\nu} \geqslant \alpha_{\mu'} \dim_k L(\mu') \geqslant \mu+1$. The result follows. □

REMARK 1. As a direct consequence of Theorem 3 we can find the $k[G]$-module structure of $\Omega_L(0)$ as in [**8**]. For if $\omega \in \Omega_L(0)$,

$$\omega = \sum_{\mu,\nu} D_{\mu,\nu}\omega_{\mu,\nu}, \quad D_{\mu,\nu} \in k,$$

then

$$(\sigma-1)^s\omega \neq 0 \text{ and } (\sigma-1)^{s+1}\omega = 0 \iff \omega = \sum_{\nu} D_{s,\nu}\omega_{s,\nu} + \sum_{\mu<s}\sum_{\nu} D_{\mu,\nu}\omega_{\mu,\nu}$$

with some $D_{s,\nu} \neq 0$. It follows that $\dim_k(\Omega_{s+1}/\Omega_s) = t^{(s)} - 1$ where

$$\Omega_i = \{\omega \in \Omega_L(0) \mid (\sigma-1)^i\omega = 0\}.$$

From here the structure of $\Omega_L(0)$ follows using the filtration $\{\Omega_i\}_{i=1}^{p^n}$ of $\Omega_L(0)$ as it was done in [**8**]. Our goal is to give a different proof, one that is similar to the one for the elementary abelian case as it was done in [**5**].

Now let $N_{\mu,\nu}$ be the k-vector space given by

$$N_{\mu,\nu} = \begin{cases} \langle \omega_{\mu',\nu'} \mid 0 \leqslant \mu' \leqslant \mu,\ 0 \leqslant \nu' \leqslant \nu + t^{(\mu')} - t^{(\mu)} - 1\rangle & \text{if } \nu \geqslant 1, \\ \langle \omega_{\mu',\nu'} \mid 0 \leqslant \mu' \leqslant \mu-1,\ 0 \leqslant \nu' \leqslant \nu + t^{(\mu')} - t^{(\mu)} - 1\rangle & \text{if } \nu = 0. \end{cases} \tag{44}$$

From Theorem 2 we have that $N_{\mu,\nu}$ is a $k[G]$-submodule of

$$M_{\mu,\nu} := \langle \omega_{\mu',\nu'} \mid 0 \leqslant \mu' \leqslant \mu,\ 0 \leqslant \nu' \leqslant \nu + t^{(\mu')} - t^{(\mu)}\rangle. \tag{45}$$

Furthermore, $\dim_k M_{\mu,\nu}/N_{\mu,\nu} = \mu+1 = \dim_k \mathcal{A}_{\mu,\nu}$ and $\mathcal{A}_{\mu,\nu}$ is a $k[G]$-submodule of $M_{\mu,\nu}$.

Theorem 4. *For any (μ,ν) such that $\omega_{\mu,\nu} \in \mathcal{A}$ we have $M_{\mu,\nu} \cong N_{\mu,\nu} \oplus \mathcal{A}_{\mu,\nu}$ as $k[G]$-modules.*

Proof: Since $\dim_k M_{\mu,\nu} = \dim_k N_{\mu,\nu} + \dim_k \mathcal{A}_{\mu,\nu}$, it suffices to show that $\mathcal{A}_{\mu,\nu} \cap N_{\mu,\nu} = \{0\}$.

From (37) we obtain that

$$(\sigma-1)^{\mu}\omega_{\mu,\nu} = \mu_1! \cdots \mu_n! x^{\nu} g_{\mu}^{-1}(x)dx = \mu_1! \cdots \mu_n! x^{\nu} g_0^{-1}(x)\frac{g_0(x)}{g_{\mu}(x)}dx$$
$$= x^{\nu} g_0^{-1}(x) x^{t^{(0)}-t^{(\mu)}} \mu_1! \cdots \mu_n! dx + x^{\nu} g_0^{-1}(x)h(x)dx$$

with $\deg h(x) \leqslant t^{(0)} - t^{(\mu)} - 1$.

Therefore $x^{\nu} g_0^{-1}(x)h(x)dx \in N_{\mu,\nu}$ and $x^{\nu+t^{(0)}-t^{(\mu)}} g_0^{-1}(x)\mu_1! \cdots \mu_n! dx \notin N_{\mu,\nu}$. Hence

$$(\sigma-1)^{\mu}\omega_{\mu,\nu} \notin N_{\mu,\nu}. \tag{46}$$

Let $\omega \in \mathcal{A}_{\mu,\nu} \cap N_{\mu,\nu}$, $\omega = f(\sigma)\omega_{\mu,\nu}$. If $(\sigma-1)^{\mu+1} \mid f(\sigma)$, from (37) we obtain that $\omega = h(\sigma)(\sigma-1)^{\mu+1}\omega_{\mu,\nu} = 0$.

Now we assume that $f(x) = (x-1)^t \ell(x) \in k[x]$ with $t \leqslant \mu$ and $\big(\ell(x), x-1\big) = 1$. There exist $a(x)$, $b(x) \in k[x]$ such that

$$1 = \ell(x)a(x) + b(x)(x-1)^{p^n}.$$

Then

$$\ell(\sigma)a(\sigma) + b(\sigma)(\sigma-1)^{p^n} = \ell(\sigma)a(\sigma) = 1.$$

That is, $\ell(\sigma)$ is a unit in $k[G]$. Thus $f(\sigma)\omega_{\mu,\nu} = (\sigma-1)^t\ell(\sigma)\omega_{\mu,\nu} \in N_{\mu,\nu}$. It follows that $(\sigma-1)^t\omega_{\mu,\nu} \in N_{\mu,\nu}$. Therefore $(\sigma-1)^{\mu}\omega_{\mu,\nu} = (\sigma-1)^{\mu-t}(\sigma-1)^t\omega_{\mu,\nu} \in N_{\mu,\nu}$ contrary to (46).

Thus, we have that $(\sigma-1)^{\mu+1} \mid f(\sigma)$ and $\mathcal{A}_{\mu,\nu} \cap N_{\mu,\nu} = \{0\}$. ∎

Now we are ready to prove the main result of this section.

Theorem 5. *Let $K = k(x)$ be a rational function field of one variable over an algebraically closed field k of characteristic $p > 0$. Let L/K be a cyclic extension of degree p^n, $n \geqslant 1$ with Galois group G. If every ramified prime divisor of K is fully ramified in L, then the $k[G]$–module structure of the holomorphic differentials $\Omega_L(0)$ of L in terms of indecomposable modules is*

$$\Omega_L(0) \cong \bigoplus_{i=1}^{p^n-1} L(i)^{d_i}$$

where $L(i) = k[G]/(\sigma-1)^i$, $i = 1, 2, \ldots, p^n-1$,

$$d_{p^n-1} = t^{(p^n-2)} - 1 \text{ and } d_j = t^{(j-1)} - t^{(j)},\ j = 1, 2, \ldots, p^n-2.$$

PROOF: Let $0 \leqslant \mu_0 \leqslant p^n - 2$ be maximal such that $t^{(\mu_0)} - 2 \geqslant 0$ and let $\nu_0 = t^{(\mu_0)} - 2$. From Theorem 4 we have that

$$M_{\mu_0, t^{(\mu_0)}-2} = \Omega_L(0) \cong \mathcal{A}_{\mu_0, t^{(\mu_0)}-2} \oplus N_{\mu_0, t^{(\mu_0)}-2} \tag{47}$$

with $\mathcal{A}_{\mu_0, t^{(\mu_0)}-2} \cong L(\mu_0 + 1)$ (Theorem 3).

We apply the above argument to the module $N_0 = N_{\mu_0, t^{(\mu_0)}-2}$ in place of $\Omega_L(0)$. More precisely, let $0 \leqslant \mu_1 \leqslant p^n - 2$ be maximal such that $t^{(\mu_1)} - 3 \geqslant 0$ and let $\nu_1 = t^{(\mu_1)} - 3$. Then, from (44) and (45) we have

$$N_{\mu_0, t^{(\mu_0)}-2} = M_{\mu_1, t^{(\mu_1)}-3}.$$

From Theorem 4 we have that

$$N_{\mu_0, t^{(\mu_0)}-2} = M_{\mu_1, t^{(\mu_1)}-3} \cong \mathcal{A}_{\mu_1, t^{(\mu_1)}-3} \oplus N_{\mu_1, t^{(\mu_1)}-3} \tag{48}$$

where $\mathcal{A}_{\mu_1, t^{(\mu_1)}-3} \cong k[G]/(\sigma - 1)^{\mu_1+1} = L(\mu_1 + 1)$ (Theorem 3).

We apply again the same argument for $N_{\mu_1, t^{(\mu_1)}-3}$ in (48). That is, from (44) and (45) we have

$$N_{\mu_1, t^{(\mu_1)}-3} = M_{\mu_2, t^{(\mu_2)}-4}$$

where μ_2 is maximal such that $t^{(\mu_2)} - 4 \geqslant 0$.

Finally, we obtain

$$\Omega_L(0) = \bigoplus_{j=0}^{m} \mathcal{A}_{\mu_j, t^{(\mu_j)}-j-2} \tag{49}$$

with $\mathcal{A}_{\mu_j, t^{(\mu_j)}-j-2} \cong L(\mu_j + 1)$.

The number of times that $L(p^n - 1)$ appears in (49) is $t^{(p^n-2)} - 1$ and for $1 \leqslant j \leqslant p^n - 2$, $L(j)$ appears $t^{(j-1)} - t^{(j)}$ times in (49). This completes the proof. ∎

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