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Nilpotent groups acting fixed point freely on solvable groups

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All groups considered are finite. We enlarge the class of groups for which the conjecture below is known to hold, to include all nilpotent groups A such that every proper subgroup of A is $Z_p \sim Z_p$ free for all primes p.

CONJECTURE. *If a nilpotent group A acts fixed point freely on a solvable group G where $(|A|, |G|) = 1$, then the Fitting height of G is bounded above by the number of primes, counting multiplicities, dividing $|A|$.*

1. Introduction

Using his Theorems A and B of [1], Berger proves in [2, Section III] that the conjecture is true whenever A is $Z_p \sim Z_p$ free for all primes p. Here Z_n denotes the cyclic group of order n and $X \sim Y$ the (standard, restricted) wreath product. Camina and Lockett [3] use some partial results of Gross [4, 5] to show that the conjecture holds for D_8, the smallest group not covered by Berger's result. We here modify Berger's work, obtaining Theorems A* and B* from which follows:

MAIN THEOREM. *The conjecture is true for every nilpotent group A, such that every proper subgroup of A is $Z_p \sim Z_p$ free for all primes p.*

In particular, it is true for $Z_p \sim Z_p$ (which is D_8 when $p = 2$).

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2. Preliminaries

We assume the following situation throughout this paper, except when otherwise indicated.

HYPOTHESIS. *(i)* AG *is a solvable group with normal subgroup* $G \neq 1$ *and nilpotent complement* A *, where* $(|A|, |G|) = 1$.

(ii) k *is a field of characteristic* $c \nmid |A|$.

(iii) V *is a sum of isomorphic copies of a faithful, irreducible* $k[AG]$ *module.*

(iv) $A = A^0 > A^1 > \ldots > A^n = 1$ *is some central series for* A *, and* m *is such that* $C_V(A^m) = (0)$ *but* $C_V(A^{m+1}) \neq (0)$.

We use the following definition and theorems of Berger.

DEFINITION [1, page 321]. Let H be a solvable group with normal subgroup L . We call R an L *support subgroup of* H if

(a) R is a normal r-subgroup of H for some prime r ,

(b) R contains a unique maximal H-invariant subgroup $R^* < R$, and

(c) $L/C_L(R/R^*)$ is a non-trivial H-chief factor, and $C_L(X/Y) = L$ for all H-chief factors X/Y of R^* .

THEOREM A [1, page 337]. *Assume* A *is* $Z_p \sim Z_p$ *free for all primes* p . *If* $R \leq G$ *is an abelian normal subgroup of* AG *, then there is a subgroup* $D \leq A^m$ *such that*

(a) $C_V(A^{m+1}D) = (0)$ *, and*

(b) $C_R(D) \geq C_R(A^{m+1})$.

THEOREM B [1, page 331]. *Assume* A *is* $Z_p \sim Z_p$ *free for all primes* p . *If* R *is an* L *support subgroup of* AG *, where* $L \leq G$ *, then there is a subgroup* $D \leq A^m$ *such that*

(a) $C_V(A^{m+1}D) = (0)$,

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(b) $C_{G/F(G)}(D) \geq C_{G/F(G)}(A^{m+1})$, *and*

(c) $C_{R/R^*}(D) \geq C_{R/R^*}(A^{m+1})$.

We wish to replace the assumption that A is $Z_p \sim Z_p$ free with the weaker requirement that every proper subgroup of A is $Z_p \sim Z_p$ free. We will need to make the additional assumption that $C_G(A) = 1$. This will enable us to apply the lemma.

LEMMA. *Assume AG is a solvable group with normal subgroup $G \neq 1$ and complement A , where $(|A|, |G|) = 1$ and $C_G(A) = 1$. Suppose V is an irreducible $\mathrm{k}[AG]$ module, where k is an algebraically closed field, and that G acts non-trivially on V . Then there is a proper subgroup $A_0 < A$ and a $\mathrm{k}[A_0G]$ module U , such that $V \simeq U|^{AG}$.*

REMARK. Here we do not assume the Hypothesis.

Proof. Now $V \simeq W|^{AG}$, where W is a primitive $\mathrm{k}[H]$ module for some subgroup $H \leq AG$. $H \cap G$ is a normal Hall subgroup of H and so has a complement in H . Then, replacing H and W by appropriate conjugates, we may assume $H = A_0G_0$, where $A_0 = A \cap H$ and $G_0 = G \cap H$.

Assume $A_0 = A$. Since $C_V(G) \neq V$ and V is irreducible, we have $C_V(G) = (0)$ and $C_{G_0}(W) < G_0$. Choose $\bar{N} = N/C_{G_0}(W) \trianglelefteq G_0/C_{G_0}(W)$ minimal A invariant. Then $\bar{N}$ is abelian and so, since W is primitive, $\bar{N}$ acts on W via scalar multiplication. In particular, the action of $\bar{N}$ on W commutes with the action of A on W , so that

$$[N, A] \leq C_{AG_0}(W) \cap N \leq C_{G_0}(W) .$$

That is, $C_{\bar{N}}(A) = \bar{N}$. But A acts fixed point freely on N and hence on $\bar{N}$; that is $C_{\bar{N}}(A) = 1$, so $\bar{N} = 1$, a contradiction. Then we must have $A_0 < A$. Setting $U = W|^{A_0G}$ then gives the desired conclusion.

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3. The theorems

THEOREM A*. *Assume every proper subgroup of* A *is* $Z_p \sim Z_p$ *free for all primes* p *and that* $C_G(A) = 1$. *If* $R \le G$ *is an abelian normal subgroup of* AG , *then there is a subgroup* $D \le A^m$ *such that*

(a) $C_V(A^{m+1}D) = (0)$, *and*

(b) $C_R(D) \ge C_R(A^{m+1})$.

THEOREM B*. *Assume every proper subgroup of* A *is* $Z_p \sim Z_p$ *free for all primes* p *and that* $C_G(A) = 1$. *If* R *is an* L *support subgroup of* AG , *where* $L \le G$, *then there is a subgroup* $D \le A^m$ *such that*

(a) $C_V(A^{m+1}D) = (0)$,

(b) $C_{G/F(G)}(D) \ge C_{G/F(G)}(A^{m+1})$, *and*

(c) $C_{R/R^*}(D) \ge C_{R/R^*}(A^{m+1})$.

The proofs of these theorems are essentially duplications of the first parts of Berger's proofs of his Theorems A and B (see [*1*], pages 331-339). The proofs of the two theorems are extremely similar. For that reason the theorems are proved together.

Proof. Step 1. Clearly we may assume that V is irreducible.

Step 2. We may assume k is algebraically closed.

Let $\hat{k}$ be an algebraic closure of k and let $\hat{V} = \hat{k} \otimes_k V$. Then

$$\hat{V} \simeq \hat{V}_1 \oplus \hat{V}_2 \oplus \ldots \oplus \hat{V}_t ,$$

the $\hat{V}_i$'s the homogeneous $\hat{k}[AG]$ components, which are all algebraically conjugate. AG must be faithful on each of the V_i's and, by [*1*, 1.10], $C_{\hat{V}}(A^j) = \hat{k} \otimes C_V(A^j)$. Then for some (and hence all) i ,

$$C_{\hat{V}_i}(A^m) = (0) \quad \text{and} \quad C_{\hat{V}_i}(A^{m+1}) \ne (0) .$$

Thus, the hypotheses of the theorem are satisfied for $\hat{V}_i$, AG, $\hat{\mathrm{k}}$, and m (all i) and the conclusion for each i implies the conclusion for V, AG, k , and m . Then it suffices to prove the theorems for algebraically closed fields.

Step 3. We may now apply the lemma. That is, $V \simeq U|^{AG}$, where U is a $\mathrm{k}[A_0G]$ module and A_0 is a proper subgroup of A . We may take A_0 maximal in A and so normal of prime index p . Let $\{x_1 = 1, x_2, \ldots, x_p\}$ be a transversal for A_0 in A . Then

$$V \simeq x_1 \otimes U \oplus x_2 \otimes U \oplus \ldots \oplus x_p \otimes U .$$

Let $N = \ker(A_0G \to \operatorname{aut} U)$. For $T \le A_0G$, let $\overline{T}$ denote TN/N . Write $A_0^j = A_0 \cap A^j$. By [1, 1.11], $C_U\left(A_0^m\right) = (0)$ and $C_U\left(A_0^{m+1}\right) \neq (0)$.

We are now in a position to apply Berger's Theorems A and B.

Step 4 (a). Applying Theorem A to U , $\overline{A_0G}$, $\overline{R}$, and m we obtain a subgroup $\overline{D} \le \overline{A}_0^m$ such that

(a) $C_U\left(A_0^{m+1}D\right) = (0)$, and

(b) $C_{\overline{R}}(D) = C_{\overline{R}}\left(A_0^{m+1}\right)$.

Replacing D by $DN \cap A_0^m$, we may take $D \le A^m$.

Step 4 (b). By [1, 3.8], there exist $H_0 \le R$ and $L_0 \le L$, such that $\overline{H}_0$ is an $\overline{L}_0$ support subgroup of $\overline{A_0G}$. ($H_0 \le R$ is minimal A_0G invariant such that $H_0 \nleq R^*(N \cap R)$, $H_0^* = H_0 \cap R^* = H_0 \cap R^*(N \cap R)$, and $L_0/C_L(H_0/H_0^*)$ is an A_0G chief factor.) Applying Theorem B to U , $\overline{A_0G}$, $\overline{H}_0$, and m , we obtain a subgroup $D \le A^m$ such that

(a) $C_U\left(A_0^{m+1}D\right) = (0)$,

(b) $C_{\overline{G}/F(\overline{G})}(D) \ge C_{\overline{G}/F(\overline{G})}\left(A_0^{m+1}\right)$, and

(c) $C_{\overline{H}_0/\overline{H}_0^*}(D) \geq C_{\overline{H}_0/\overline{H}_0^*}\left(A_0^{m+1}\right)$.

Step 5. $C_V\left(A^{m+1}D\right) = (0)$.

Since A_0^m/A_0^{m+1} is central in A , $A_0^{m+1}D \trianglelefteq A$. Setting $U_i = x_i \otimes U$, we have $C_{U_i}\left(A_0^{m+1}D\right) = x_i \otimes C_U\left(A_0^{m+1}D\right) = (0)$, so that $C_V\left(A^{m+1}D\right) \leq C_V\left(A_0^{m+1}D\right) = (0)$.

Step 6 (a). $C_R(D) \geq C_R\left(A^{m+1}\right)$.

Set $M = N \cap R$ and $M_i = x_iMx_i^{-1} = \ker\left(R \to \operatorname{aut} U_i\right)$. From (b) of Step 4 (a), we conclude that $C_{R/M}\left(A^{m+1}\right) = C_{R/M}\left(A^{m+1}D\right)$. Recalling that A^{m+1} and $A^{m+1}D$ are normal in A , this implies

$$C_{R/M_i}\left(A^{m+1}\right) = C_{R/M_i}\left(A^{m+1}D\right) \leq C_{R/M_i}(D) ,$$

all i . Then $C_R\left(A^{m+1}\right) \leq C_R(D)M_i$, all i , and so

$$C_R\left(A^{m+1}\right) \leq C_R(D)\left(\bigcap_i M_i\right) = C_R(D) .$$

This concludes the proof of Theorem A*. We need now consider only Theorem B*.

Step 6 (b). $C_{G/F(G)}(D) \geq C_{G/F(G)}\left(A^{m+1}\right)$.

Set $N_i = x_iNx_i^{-1} = \ker\left(G \to \operatorname{aut} U_i\right)$ and $M_i/N_i = F\left(G/N_i\right)$. Since $\bigcap_i N_i = 1$, we have $\bigcap_i M_i = F(G)$. From (b) of Step 4 (b), we conclude that $C_{G/M_1}\left(A^{m+1}\right) = C_{G/M_1}\left(A^{m+1}D\right)$. As in Step 6 (a), this yields $C_G\left(A^{m+1}\right) \leq C_G(D)F(G)$, so that $C_{G/F(G)}\left(A^{m+1}\right) \leq C_{G/F(G)}(D)$.

Step 7 (b). $C_{R/R^*}(D) \geq C_{R/R^*}\left(A^{m+1}\right)$.

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Recall from Step 4 (b) that $\hat{R} = R/R^*$ is an irreducible AG module and $\overline{H}_0/\overline{H}_0^* \simeq H_0/H_0^* \simeq H_0R^*/R^*$ is an irreducible A_0G module. Then

$$\hat{R}|_{A_0G} \simeq \hat{R}_1 \dotplus \hat{R}_2 \dotplus \ldots \dotplus \hat{R}_t ,$$

where the R_i's are the homogeneous components, and $t = 1$ or p. Further, we may label the $\hat{R}_i$'s so that H_0R^*/R^* is a summand of $\hat{R}_1$, and, if $t = p$, $\hat{R}_i = x_i\hat{R}_i$. From (c) of Step 4 (b), we conclude that $C_{\hat{R}_1}\left(A^{m+1}D\right) = C_{\hat{R}_1}\left(A^{m+1}\right)$. When $t = p$,

$$C_{\hat{R}_i}\left(A^{m+1}D\right) = C_{\hat{R}_i}\left(x_iA^{m+1}Dx_i^{-1}\right) = C_{\hat{R}_i}\left(x_iA^{m+1}x_i^{-1}\right) = C_{\hat{R}_i}\left(A^{m+1}\right) .$$

Thus, whether $t = 1$ or p, $C_{\hat{R}}(D) \geq C_{\hat{R}}\left(A^{m+1}\right)$. This concludes the proof of Theorem B*.

Berger's proof of the conjecture in [2, Section III] now holds for any nilpotent group A, all of whose proper subgroups are $Z_p \sim Z_p$ free for all primes p. Thus we have the Main Theorem.

4. Examples

We now briefly consider nilpotent groups A which satisfy

(*) A involves $Z_q \sim Z_q$ while every proper subgroup of A is $Z_p \sim Z_p$ free for all primes p.

Any such group must be a q group generated by two elements. Clearly $Z_q \sim Z_q$ satisfies (*) and if A satisfies (*), with $N \triangleleft A$ such that $A/N \simeq Z_q \sim Z_q$, then A/M satisfies (*) for any $M \leq N$ normal in A.

We give an example of a class of groups which satisfy (*). Let $D = Z_{q^n} \sim Z_q = B\langle y\rangle$ where B is the base group and $\langle y\rangle = Z_q$ complements B in D. Set

$$N = \left\{\left(x_1, \ldots, x_q\right) : x_i \in qZ_{q^n}, i = 1, 2, \ldots, q\right\}$$

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and

$$M = \langle (q, q, \ldots, q) \rangle .$$

Take $A = D/M$. Then A satisfies (*). Now $D/N \simeq Z_q \sim Z_q$. Suppose $H/K \simeq Z_q \sim Z_q$ where $M \leq K \triangleleft H \leq D$. Choose $\bar{x}, \bar{y} \in H$ such that

$$\bar{x} \mapsto (1, 0, \ldots, 0) \text{ and } \bar{y} \mapsto z ,$$

where $\langle (1, 0, \ldots, 0), z \rangle = Z_q \sim Z_q$. It can be shown that $\bar{x} \in B$, the base group, and $\bar{y} = wy^i$, where $w \in B$ and $q \nmid i$. Let $\bar{x} = (x_1, x_2, \ldots, x_q)$ and $u = x_1 + x_2 + \ldots + x_q$. Now

$$\bar{x} + \bar{y}^{\bar{y}} + \ldots + \bar{x}^{\bar{y}^{q-1}} = (u, u, \ldots, u) \notin K$$

and so is not in M. Then $(u, q) = 1$. Consequently, it can be shown that $\langle \bar{x}, \bar{x}^{\bar{y}}, \ldots, \bar{x}^{\bar{y}^{q-1}} \rangle = B$. Then $H = \langle \bar{x}, \bar{y} \rangle K = D$ and so A satisfies (*).

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