EMBEDDING PROBLEMS IN SEMIRING THEORY

by MIREILLE P. GRILLET

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The main object of this paper is to give various constructions of the universal semiring of a partial semiring and apply them to find conditions for the embedding of partial semirings into semirings.

First we define in a natural way a multivalued multiplication on an additive semigroup A having a generating subset which is also a multiplicative semigroup and describe all semiring congruences of A (Theorem 1.2); this result directly applies to the construction of free semirings (see also [4]), of the universal skew-ring of a semiring, and of the universal semiring of a multiplicative semigroup. Our main result (Theorem 3.1) gives detailed descriptions of the universal semiring R of a partial semiring A (= set with two associative operations) which is embeddable into a multiplicative semigroup. Further distributivity conditions allow especially simple descriptions of R and embedding conditions. In case S is a multiplicative semigroup containing a semiring A as an ideal, it follows that S is embeddable into a semiring whose addition extends the addition of A if and only if the two following conditions hold:

i) $s(a+b)t = sat + sbt$ for all $s, t \in S^1$ and $a, b \in A$;

ii) for all $w_i, w'_j \in A$,

$$\sum_{i=1}^{i=n} \left(\sum_{j=1}^{j=p} w_i w'_j \right) = \sum_{j=1}^{j=p} \left(\sum_{i=1}^{i=n} w_i w'_j \right).$$

These conditions hold for instance if S is an ideal extension of A determined by a partial homomorphism. Let A be a prime ideal of S and define a left (right) S-translation of A to be a finite sum of mappings of the form: $x \rightsquigarrow sx (x \rightsquigarrow xs)$, where $s \in S$. Then S is embeddable into a semiring whose addition extends the addition of A if and only if every S-translation of A is an additive homomorphism of A. We are indebted to Professor L. Fuchs for suggesting the above result which generalizes the main result of [2], and to Professor P. A. Grillet for suggesting the applications to ideal extensions of semirings which complete the paper.

1. Extension of a multiplication to an additive semigroup

Let A be an additive semigroup and S be a generating subset of A which is also a multiplicative semigroup. It is possible to define a multivalued multiplication on A by the formula:

$$(1) \qquad (x_1 + \cdots + x_n)(y_1 + \cdots + y_p) = \sum_{i=1}^{i=n} \left(\sum_{j=1}^{j=p} x_i y_j \right),$$

for all $x_i,\ y_j \in S$. We call this multivalued multiplication the *extension of the multiplication of* S *to* A. Clearly if it is well-defined, it is also associative and distributive on the right with respect to the addition; however it is not in general distributive on the left with respect to the addition. It is easy to check that we have the following:

PROPOSITION 1.1. *Let* A *be an additive semigroup and* S *be a generating subset of* A *which is also a multiplicative semigroup. Then* A *is a semiring under its addition and the extension of the multiplication of* S *if and only if*

$$(2) \qquad \sum_{i=1}^{i=n} \left(\sum_{j=1}^{j=p} x_i y_j \right) = \sum_{j=1}^{j=p} \left(\sum_{i=1}^{i=n} x_i y_j \right)$$

for all $x_i,\ y_j \in S$ *and*

$$(3) \qquad \sum_{i=1}^{i=n} s x_i = \sum_{j=1}^{j=p} s x'_j, \qquad \sum_{i=1}^{i=n} x_i s = \sum_{j=1}^{j=p} x'_j s$$

for all $s, x_i, x'_j \in S$ *such that* $\sum_{i=1}^{i=n} x_i = \sum_{j=1}^{j=p} x'_j$.

Let now A be any additive semigroup on which a multivalued multiplication has been defined. We call *semiring congruence* of A an additive congruence of A such that the induced multiplication is well-defined and that the quotient set is a semiring under the induced operations. The following result describes such congruences in our situation.

THEOREM 1.2. *Let* A *be an additive semigroup and* S *be a generating subset of* A *which is also a multiplicative semigroup. Write* $A = F/\varepsilon$, *where* F *is the free semigroup on the set* A *and* ε *a relation on* F, *and consider on* A *the multiplication induced by the multiplication of* S. *Then, if* ρ *is a relation on* A *such that* $\rho \cup \varepsilon$ *admits the multiplication by elements of* S, *the semiring congruence generated by* ρ *is the additive congruence generated by* $\rho \cup \delta$, *where* δ *is the set of all pairs of the form*:

$$(4) \qquad \left(\sum_{i=1}^{i=n} \left(\sum_{j=1}^{j=p} x_i y_j \right),\ \sum_{j=1}^{j=p} \left(\sum_{i=1}^{i=n} x_i y_j \right) \right),$$

where $x_i,\ y_j \in S$.

PROOF. Denote by $\bar{\rho}$ the additive congruence generated by $\rho \cup \delta$, by $A' = A/\bar{\rho}$ the quotient set of A by $\bar{\rho}$, and by p_ρ the canonical projection of A onto A'.

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Since $\rho \cup \varepsilon$ admits the multiplication by elements of S by assumption and δ trivially does so, it follows easily that $\bar{\rho}$ admits the multiplication by elements of S. In particular the restriction to $S' = p_\rho(S)$ of the induced multiplication is well-defined and S' is a multiplicative semigroup under the induced multiplication. Furthermore it is clear that S' generates $(A', +)$ and that the induced multiplication on A' is exactly the extension of the multiplication of S' to A'.

Next we observe that the fact that $\bar{\rho}$ contains δ implies that (2) holds in A', and the fact that $\bar{\rho}$ admits the multiplication by elements of S is nothing but condition (3) in A', (with now s, t, x_i, $y_j \in S'$). Thus by Proposition 1.1, A' is a semiring under the induced operations, whence $\bar{\rho}$ is a semiring congruence. Also any semiring congruence containing ρ certainly contains δ, and therefore contains $\bar{\rho}$. It follows that $\bar{\rho}$ is the semiring congruence of A generated by ρ.

COROLLARY 1.3. *Let A be an additive semigroup and S be a generating subset of A which is also a multiplicative semigroup. Then the greatest homomorphic image of A which is a semiring under the induced addition and the multiplication induced by the extension of the multiplication of S to A is the quotient of A by the additive congruence generated by $\delta \cup \varepsilon'$, where*

$$\varepsilon' = \{(sat, sbt);\ s, t \in S,\ (a, b) \in \varepsilon\}.$$

In case S is the free (multiplicative) semigroup on a set X, say S_X, and A is the free (additive) semigroup W_X on S_X, then the greatest homomorphic image of W_X which is a semiring under the induced operations is simply the quotient R_X of W_X by the additive congruence generated by δ. It is easy to check that R_X is the free semiring over X.

We recall that, if A is an additive semigroup with a generating subset S which is also a multiplicative semigroup, then the additive congruence generated by δ is the transitive closure of the set δ' of all pairs having the form (x, y) or (y, x), where either $x = y$ or

$$(5) \qquad x = u + \sum_{i=1}^{i=n} \left(\sum_{j=1}^{j=p} x_i y_j \right) + v, \quad y = u + \sum_{j=1}^{j=p} \left(\sum_{i=1}^{i=n} x_i y_j \right) + v,$$

where x_i, $y_j \in S$ and u, $v \in A^0$. For convenience, if two formal sums of elements of S are given by the expressions (5) (where now u, v represent formal sums of elements of S^0), we shall say that each of them results from the other by *δ-commutation*. The following characterization of free semirings (Theorem 3 of [4]) is immediate on the above construction.

THEOREM 1.4. *Let R be a semiring and X be a non-empty subset of R. Then R is a free semiring over X if and only if every element of R can be written as a finite sum of products of elements of X uniquely up to finitely many δ-commutations.*

Another application of Theorem 1.2 is the following construction of the universal skew-ring (= semiring whose additive semigroup is a group) of a semiring A with zero.

Recall that the universal group R of $(A, +)$ can be described as follows: let F be the additive semigroup obtained by adjoining to $(A, +)$ an "indeterminate" ${}^-p$ for each non-zero element p of A. Then R is the quotient of F by the congruence $\bar{\rho}$ generated by the set ρ of all pairs of the form:

$$(p + {}^-p, 0),\ ({}^-p + p, 0),\ ({}^-p + 0, {}^-p),\ (0 + {}^-p, {}^-p), \tag{6}$$

where $p \in A - \{0\}$. Let ${}^-A = \{{}^-p;\ p \in A - \{0\}\}$ and $S = A \cup {}^-A$; we can extend the multiplication of A to an associative multiplication on S by setting

$$p({}^-q) = ({}^-p)q = {}^-(pq),\ ({}^-p)({}^-q) = pq \text{ and } 0({}^-p) = ({}^-p)0 = 0$$

for all $p, q \in A - \{0\}$. Then we consider on F the extension of the multiplication of S.

By distributivity on A we have:

$$ps + pq + rs + rq = (p + r)(s + q) = ps + rs + pq + rq,$$

whence by cancellativity modulo $\bar{\rho}$, $(pq + rs, rs + pq) \in \bar{\rho}$ for all $p, q, r, s \in A$. It is easy to see that in fact this is also true for all $p, q, r, s \in S$, so that clearly $\delta \subseteq \bar{\rho}$. Since obviously $\rho \cup \varepsilon$ admits the multiplication by elements of S, by Theorem 1.2 the multiplication induced by the extension of the multiplication of S to F is a well-defined operation which gives to R a structure of skew-ring. It easily follows that we have the following:

PROPOSITION 1.5. *Let A be a semiring with zero and R be the universal group of $(A, +)$. Then R is a skew-ring under its addition and the multiplication defined by*:
(a) $\alpha(a)\alpha(b) = \alpha(ab)$,

$$\alpha(a)(-\alpha(b)) = (-\alpha(a))\alpha(b) = -\alpha(a)\alpha(b),\ (-\alpha(a))(-\alpha(b)) = \alpha(a)\alpha(b);$$

(b)
$$\left(\sum_{i=1}^{i=n} x_i\right)\left(\sum_{j=1}^{j=p} y_j\right) = \sum_{i=p}^{i=n}\left(\sum_{j=1}^{j=p} x_i y_j\right),$$

for all $a, b \in A$ and $x_i, y_j \in \alpha(A) \cup {}^-\alpha(A)$, where α is the canonical mapping of A into R and ${}^-\alpha(A) = \{-\alpha(a);\ a \in A\}$. Furthermore, for every homomorphism f of A into a skew-ring R', there exists a unique homomorphism f' of R into R' such that $f = f' \circ \alpha$.

COROLLARY 1.6. *Let A be a semiring with zero. Then A is embeddable into a skew-ring if and only if its additive semigroup is embeddable into a group.*

2. The universal semiring of a multiplicative semigroup

Given a multiplicative semigroup S, we call *universal semiring of* S a pair (R^S, j^S) consisting of a semiring R^S and a homomorphism j^S of S into $(R^S, \cdot)$ having the following property: for every homomorphism f of S into the multiplicative semigroup of a semiring R', there exists a unique homomorphism f' of R into R' such that $f = f' \circ j^S$.

Throughout this section, S is an arbitrary multiplicative semigroup which we can represent as the semigroup generated by some set X subject to some relations μ, i.e. $S = S_X/\mu$ where S_X is the free multiplicative semigroup over X.

Consider the free additive semigroup W_S on the set S. Clearly it is the quotient of the free additive semigroup W_X over the set S_X by the congruence $\bar{\mu}_W$ generated by μ, the canonical projection p_μ^W being given by:

$$p_\mu^W(w_1 + \cdots + w_n) = p_\mu(w_1) + \cdots + p_\mu(w_n)$$

for all $w_i \in S_X$, where p_μ is the canonical projection of S_X onto S. Furthermore the extensions of the multiplications of S and S_X to W_S and W_X respectively are well-defined operations. It is clear on formula (1) that p_μ^W is a multiplicative homomorphism, whence $\bar{\mu}_W$ is a multiplicative congruence on W_X. Note that the additive congruence generated by $\mu \cup \delta$ on W_X is simply the transitive closure of $\bar{\mu}_W \cup \bar{\delta}$.

In the other hand, we may also consider on the free semiring R_X over X the additive congruence $\bar{\mu}_R$ generated by μ; by Theorem 1.2 it is also a multiplicative congruence and the canonical projection p_μ^R of R_X onto $R_X/\bar{\mu}_R$ is a semiring homomorphism.

The isomorphism theorems for semigroups applied to the additive structure yield that the quotient of W_X by the additive congruence generated by $\mu \cup \delta$ is isomorphic to $R_X/\bar{\mu}_R$, and also to $W_S/\bar{\delta}$; since all congruences involved are also multiplicative congruences, these isomorphisms are in fact semiring isomorphisms. We identify these isomorphic semirings and denote by R^S the resulting semiring and by j^S the restriction of S of the canonical projection p_δ^S of W_S onto R^S. Clearly j^S is one-to-one whic allows us to consider S as a subset of R^S. To show that R^S has the universal property described above is then routine.

THEOREM 2.1. *The pair* (R^S, j^S) *described above is the universal semiring of the semigroup* S.

As an immediate consequence of the above construction and Theorem 1.4, we obtain:

THEOREM 2.2. *Let* R *be a semiring and* S *be a multiplicative subsemigroup of* R. *Then* R *is the universal semiring of* S *if and only if every element of* R *is written as a sum of elements of* S *uniquely up to a finite number of* δ*-communications.*

For semirings with a commutative addition, the results are considerably simpler. With the obvious definition of the universal semiring with a commutative addition of a semigroup, we have:

THEOREM 2.3. *Let S be a multiplicative semigroup and R_c^S be the free commutative additive semigroup over the set S provided with the extension of the multiplication of S. Then R_c^S together with the canonical inclusion j_c^R of S into R_c^S is the universal semiring with a commutative addition of S.*

3. The universal semiring of a partial semiring

We call *partial semiring* a set A with two partial operations of partial semigroups called addition and multiplication. The *universal semiring R of a partial semiring A* is a semiring R together with a partial homomorphism j of A into R having the following property: for every partial homomorphism f of A into a semiring R', there exists a unique homomorphism f' of R into R' such that $f = f' \circ j$.

In this section, we give some constructions of the universal semiring of a partial semiring A under the assumption that $(A, \cdot)$ is embeddable into a semigroup. This last condition is not essential, but brings important simplifications and is justified by the fact that it is obviously necessary for the embedding of A into a semiring, so that our results still apply to the general problem of embedding a partial semiring into a semiring.

Let A be a partial semiring such that $(A, \cdot)$ is embeddable into a semigroup and S be the universal semigroup of $(A, \cdot)$. With the partial addition defined only in A and coinciding with the addition of A, S becomes a partial semiring which clearly has the same universal semiring as A. We can describe the universal semigroup W of $(S, +)$ as the quotient of the free semigroup W_S over the set S by the congruence $\bar{\alpha}$ generated by the set α of all pairs of the form:

$$(i(a) + i(b),\ i(a + b)), \tag{6}$$

where $a, b \in S$ are such that $a + b$ is defined in S (or equivalently in A), and i is the canonical inclusion of S into W_S. We denote by p_α the canonical projection of W_S onto W and by k its restriction to S.

Since S is a generating subset of W_S which is a multiplicative semigroup, we can consider on W_S the extension of the multiplication of S, which is a well defined operation. In general α does not admit the multiplication by elements of S, not even its restriction to $k(S)$. Let α' be the set of all pairs of the form:

$$(si(a)t + si(b)t,\ si(a + b)t), \tag{7}$$

where $a, b \in S$ are such that $a + b$ is defined in S and $s, t \in S^1$. Then α' is clearly the smallest relation on W_S containing α which admits the multiplication by elements of S and by Theorem 1.2, the quotient of W_S by the additive congruence generated by $\alpha' \cup \delta$ is a semiring. By the isomorphism theorems for semigroups, this semiring is isomorphic to the quotient of the universal semiring $R^S = W_S/\bar{\delta}$ of $(S, \cdot)$ by the additive congruence α'_R generated by the set α'_R of all pairs of the form:

$$(8) \qquad (sj_S(a)t + sj_S(b)t,\ sj_S(a+b)t),$$

where $a, b \in S$ are such that $a + b$ is defined in S, and $s, t \in S^1$. Let p_α^R be the canonical projection of R_S onto $R = R_S/\bar{\alpha}'_R$ and j be its restriction to S. It is routine to check that the pair (R, j) has the universal property of a universal semiring of A. Thus we have:

THEOREM 3.1. *Let A be a partial semiring such that $(A, \cdot)$ is embeddable into a semigroup, and S be the universal semigroup of $(A, \cdot)$ provided with the partial addition of A. Then the following properties are equivalent:*

i) *R is the universal semiring of A;*

ii) *R is the quotient of the free additive semigroup on the set S provided with the extension of the multiplication of S by the additive congruence generated by $\alpha' \cup \delta$, where α' is the set of all pairs of the form* (7);

iii) *R is the quotient of the universal semiring of $(S, \cdot)$ by the additive congruence generated by the set of all pairs of the form* (8).

In trying to describe the universal semiring of A as the quotient of the universal semigroup W of $(S, +)$, we meet a difficulty which is that the induced multiplication on $k(S)$ is in general a multivalued operation. However the induced multiplication on W is still perfectly described by the formulae:

$$(9) \qquad k(s)k(t) = k(st) \quad \text{for all } s, t \in S,$$

$$(10) \qquad \left(\sum_{i=1}^{i=n} w_i\right)\left(\sum_{j=1}^{j=p} w'_j\right) = \sum_{i=1}^{i=n}\left(\sum_{j=1}^{j=p} w_i w'_j\right),$$

for all $w_i, w'_j \in k(S)$. Now if ε' is the set of all pairs of the form:

$$(11) \qquad (s'k(a)t' + s'k(b)t',\ s'k(a+b)t'),$$

where $a, b \in S$ are such that $a + b$ is defined in S and $s', t' \in k(S)^1$, then another application of the isomorphism theorems for semigroups shows the following:

THEOREM 3.2. *Let A be a partial semiring such that $(A, \cdot)$ is embeddable into a semigroup, and S be the universal semigroup of $(A, \cdot)$ provided with the partial addition of A. Then the universal semiring of A is the quotient of the universal additive semigroup of $(S, +)$ provided with the multivalued multiplication defined by* (9) *and* (10) *by the additive congruence generated by $\varepsilon' \cup \delta$, where ε' is the set of all pairs of the form* (11).

If now we assume that S is a multiplicative semigroup with a partial addition satisfying the condition:

(D′) For all $x, y, z \in S$ such that $y + z$ is defined, $xy + xz$ and $yx + zx$ are also defined and $x(y + z) = xy + xz$, $(y + z)x = yx + zx$ hold.

The construction of the universal semiring of S is greatly simplified. Indeed (D′) implies that α admits the multiplication by elements of S, whence $\alpha = \alpha'$; it follows that the multiplication of $k(S)$ is well-defined so that $k(S)$ is a multiplicative semigroup.

THEOREM 3.3. *Let S be a multiplicative semigroup with a partial addition such that* (D′) *holds. Then the following properties are equivalent*:

i) *R is the universal semiring of S*;

ii) *R is the quotient of the universal semiring of $(S, \cdot)$ by the additive congruence generated by the set α_R of all pairs of the form*: $(j_S(a) + j_S(b), j_S(a + b))$, *where $a, b \in S$ are such that $a + b$ is defined in S*;

iii) *R is the quotient of the free additive semigroup on S provided with the extension of the multiplication of S by the additive congruence generated by $a \cup \delta$*;

iv) *R is the quotient of the universal additive semigroup of $(S, +)$ provided with the extension of the multiplication of $k(S)$ by the additive congruence generated by δ.*

We do not try to write the embedding conditions which could be deduced from Theorem 3.1 and 3.2 because they would be too complicated. Under condition (D′) we have:

THEOERM 3.4. *Let S be a multiplicative semigroup with a partial addition such that* (D′) *holds. Then S is embeddable into a semiring if and only if, whenever two sums of elements of S differ only by a finite number of δ-commutations and the dispositions of parentheses and are defined in S, then these two sums are equal.*

PROOF. Immediate from Theorem 3.3.iv).

This result applies for instance to partial semirings of endomorphisms of (additive) semigroups. Another interesting consequence is as follows:

Let S be a multiplicative semigroup with a partial addition which satisfies the following condition (which is obviously necessary for the embedding of S into a semiring):

(C) Whenever $sat = ua'v$, $sbt = ub'v$ for some $s, t, u, v \in S^1$ and $a, a', b, b' \in S$ such that $a + b$, $a' + b'$ are defined in S, then $s(a + b)t = u(a' + b')v$.

Then it is possible to extend the addition of S to an addition $*$ of S (which is clearly well-defined) by setting:

(12) $x * y = s(a+b)t$ whenever $x = sat$, $y = sbt$ for some $s, t \in S^1$ and $a, b \in S$ such that $a + b$ is defined in S.

Then S becomes a partial semiring S^* which obviously satisfies D($'$). Furthermore any partial homomorphism f of S into a semiring R' is also a partial homomorphism of S^* into R' so that S and S^* have same universal semiring. Thus by Theorem 3.4 we have:

PROPOSITION 3.5. *Let S be a multiplicative semigroup with a partial addition. Then S is embeddable into a semiring if and only if* (C) *holds, and whenever two sums of elements of S differ only by a finite number of δ-commutations and the dispositions of parentheses and are defined in S^*, these two sums are equal.*

Finally we state the simpler results in the case when we consider only semirings with a commutative addition. In these statements, it is understood that j^S and k used in formulae (8), $\cdots$, (11) should be replaced by the canonical mappings j_c^S and k_c of S into R_c^S and W_c respectively.

THEOREM 3.6. *Let A be a partial semiring such that $(A, \cdot)$ is embeddable into a semigroup, and let S be the universal semigroup of $(A, \cdot)$ provided with the partial addition defined in A. Then the following properties are equivalent:*

i) *R_c is the universal semiring with a commutative addition of A;*

ii) *R_c is the quotient of the universal semiring with a commutative addition of $(S, \cdot)$ by the additive congruence generated by the set of all pairs of the form* (8);

iii) *R_c is the quotient of the universal commutative additive semigroup W_c of $(S, +)$ provided with the multiplication defined by* (9) *and* (10) *by the additive congruence generated by the set of all pairs of the form* (11).

THEOREM 3.7. *Let S be a multiplicative semigroup with a partial addition such that* (D$'$) *holds. Then the universal semiring with a commutative addition of S is the universal commutative additive semigroup W_c of $(S, +)$ provided with the multiplication defined by* (9) *and* (10).

COROLLARY 3.8. *Let S be a multiplicative semigroup with a partial addition such that* (D$'$) *holds. Then S is embeddable into a semiring with a commutative addition if and only if $(S, +)$ is embeddable into a commutative semigroup.*

4. Applications to ideal extensions of semirings

1. We now concentrate on the case when S is a multiplicative semigroup containing a semiring A as an ideal. An easy application of Theorem 3.4 gives the following:

THEOREM 4.1. *Let S be a multiplicative semigroup which contains a semiring A as an ideal. Then S is embeddable into a semiring whose addition extends the addition of A if and only if the two following conditions hold:*

i) $s(a+b)t = sat + sbt$ *for all* $s,t \in S^1$ *and* $a,b \in A$;

ii) $$\sum_{i=1}^{i=n}\left(\sum_{j=1}^{j=p} w_i w'_j\right) = \sum_{j=1}^{j=p}\left(\sum_{i=1}^{i=n} w_i w'_j\right) \text{ for } all\ w_i, w'_j \in A$$

such that $w_i w'_j \in A$.

COROLLARY 4.2. *Let S be a multiplicative semigroup which contains a semiring A with a commutative addition as an ideal. Then S is embeddable into a semiring with a commutative addition if and only if* $s(a+b)t = sat + sbt$ *for all* $a,b \in A$ *and* $s,t \in S^1$.

Let A be a semiring. We say that a semiring R is an *ideal extension of A* (*determined by a partial homomorphism*) in case A is a subsemiring of R and $(R,\cdot)$ is an ideal extension of $(A,\cdot)$ (determined by a partial homomorphism).

COROLLARY 4.3. *Let S be a multiplicative semigroup which contains a semiring A as an ideal. Assume that S is an ideal extension of* $(A,\cdot)$ *determined by a partial homomorphism. Then S is embeddable into a semiring whose addition extends the addition of A. Furthermore the universal semiring R of S provided with the addition of A is also an ideal extension of A determined by a partial homomorphism.*

PROOF. We know that an ideal extension V of a semigroup T is determined by a partial homomorphism if and only if there exists a homomorphism of V into T whose restriction to T is the identity.

Let A be a semiring and S be an ideal extension of $(A,\cdot)$ determined by a partial homomorphism. Let f be a homomorphism of S into $(A,\cdot)$ whose restriction to A is the identity. Then, for all $s,t \in S^1$ and $a,b \in A$,

$$\begin{aligned} s(a+b)t &= f(s(a+b)t) = f(s)f(a+b)f(t) = \\ &= f(s)af(t) + f(s)bf(t) = sat + sbt, \end{aligned}$$

where $f(1)$ is a formal identity of $(A,\cdot)$. Thus condition i) of Theorem 4.1. holds. A similar reasoning shows that condition ii) holds too. It follows that S is embeddable into a semiring whose addition extends the addition of A.

Trivially A is a multiplicative ideal of the universal semiring R of S provided with the addition of A. Moreover f, as a partial homomorphism of S into A, extends to a homomorphism g of R into A, whose restriction to A is clearly the identity. Therefore R is an ideal extension of A determined by a partial homomorphism.

In general we can express some of the conditions of Theorem 4.1 by means of mappings of A into A which generalize the inner translations used in [2] to give a condition for the embedding of a semiring into a semiring with identity. In this way we obtain simple necessary conditions of embedding which will appear to be also sufficient in case A is a prime ideal of S.

Let A be a semiring which is embeddable as an ideal into a multiplicative semigroup S. We call *left* (*right*) *S-translation* of A any finite pointwise sum of mappings of A into A having the form: $\lambda_s: a \rightsquigarrow sa$ ($\rho_s: a \rightsquigarrow as$) where $s \in S$. We denoteby Λ_S (P_S) the set of all such mappings.

Observe that the S-translations of A are well-defined mappings since A is an ideal of S. Examples show that they are not in general additive homomorphisms, and that Λ_S and P_S are not in general semirings under pointwise addition and composition of mappings (cf. [2]).

PROPOSITION 4.4. *Let A be a semiring which is embeddable as an ideal into a multiplicative semigroup S. Then, if S is embeddable into a semiring whoae addition extends the addition of A, every S-translation of A is an additive homomorphism and Λ_S and P_S are semirings under pointwise addition and composition of mappings.*

PROOF. Assume that S is embeddable into a semiring whose addition extends the addition of A. If first $\lambda = \sum_{i=1}^{i=n} \lambda_{s_i}$ is a left S-translation of A, where $s_i \in S$, then for all $a, b \in A$, we have:

$$\lambda(a+b) = \sum_{i=1}^{i=n} \lambda_{s_i}(a+b) = \sum_{i=1}^{i=n} s_i(a+b) = \sum_{i=1}^{i=n} (s_i a + s_i b) =$$

$$= \sum_{i=1}^{i=n} s_i a + \sum_{i=1}^{i=n} s_i b = \lambda(a) + \lambda(b),$$

by conditions i) and ii) of Theorem 4.1. Therefore λ is an additive homomorphism of A. Similarly any right S-translation of A is also an additive homomorphism. It is then routine to check that Λ_S and P_S are semirings.

THEOREM 4.5. *Let A be a semiring which is embeddable into a multiplicative semigroup S as a prime ideal. Then S is embeddable into a semiring whose addition extends the addition of A if and only if every S-translation oj A is an additive homomorphism of A.*

PROOF. The condition is necessary by Proposition 4.4. Conversely assume that every S-translation of A is an additive homomorphism of A. Then condition i) of Theorem 4.1 holds, since λ_s and ρ_t are additive homomorphisms for all $s, t \in A$. To check ii), let $w_1, \cdots, w_n, w'_1, \cdots, w_p \in S$ be such that $w_i w'_j \in A$ for all $i.j$. Clearly since A is a prime ideal of S, either $w_i \in A$ for all i or $w'_i \in A$ for all j. If for instance $w_i \in A$ for all i, then setting $\rho = \sum_{j=1}^{j=p} \rho_{w'_j}$, we have:

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$$\sum_{i=1}^{i=n}\left(\sum_{j=1}^{j=p} w_i w'_j\right) = \sum_{i=1}^{i=n} \rho(w_i) = \rho\left(\sum_{i=1}^{i=n} w_i\right) = \sum_{j=1}^{j=p}\left(\sum_{i=1}^{i=n} w_i w'_j\right);$$

similarly if all w'_j's belong to A, using the fact that $\lambda = \sum_{i=1}^{i=n}\lambda_{w_i}$ is an additive homomorphism of A, we can complete the proof that ii) of Theorem 4.1 holds. It follows that S is embeddable into a semiring whose addition extends the addition of A.

2. Recall that, with the terminology of [6], if T is a semigroup, the translational hull $\Omega(T)$ of T is the set of all bitranslations of T, i.e. the set of all pairs $\omega = (\lambda, \rho)$ of mappings (linked translations) of T into T such that

$$(13) \qquad (\lambda x)y = \lambda(xy), \; x(y\rho) = (xy)\rho, \; x(\lambda y) = (x\rho)y,$$

for all $x, y \in T$. With the multiplication $(\lambda, \rho)(\lambda', \rho') = (\lambda\lambda', \rho\rho')$, where $(\lambda\lambda')x = \lambda(\lambda' x)$ and $x(\rho\rho') = (x\rho)\rho'$ for all $x \in T$, $\Omega(T)$ is a semigroup. The image $\Pi(S)$ of the homomorphism π of T into $\Omega(T)$ defined by $\pi(a) = (\lambda_a\, \rho_a)$, where $\lambda_a x = ax$, $x\rho_a = xa$ for all $x \in T$, is an ideal of $\Omega(T)$. If V is an ideal extension of T, we denote by $\tau = \tau(V, T)$ the canonical homomorphism of V into $\Omega(T)$ defined by: for each $v \in V$, $\tau(v) = (\lambda_v, \rho_v)$, where $\lambda_v x = vc$, $x\rho_v = xv$ for all $x \in T$. The type of the extension V of T is the image $\tau(V, T)(V)$ of $\tau(V, T)$. The following result which is Theorem 1.7 of [6] characterizes the types of ideal extensions of T.

LEMMA 4.6. *A subset U of $\Omega(T)$ is the type of some ideal extension of T if and only if*

i) *U is a subsemigroup of $\Omega(T)$ and $\Pi(T) \subseteq U$;*

ii) *For any $(\lambda, \rho), (\lambda', \rho') \in U$, λ and ρ' commute.*

Consider now a semiring A. The pointwise sum of left (right) translations of A (i.e. of $(A, \cdot)$) is still a left (right) translation of A, and trivially if (λ, ρ), (λ', ρ') belong to $\Omega(A)$, so does $(\lambda + \lambda', \rho + \rho')$. Thus we can define an addition on $\Omega(A)$ by setting:

$$(14) \qquad (\lambda, \rho) + (\lambda', \rho') = (\lambda + \lambda', \rho + \rho'),$$

for all $(\lambda, \rho), (\lambda', \rho') \in \Omega(A)$. This addition is clearly associative; however distributivity of the multiplication with respect to the addition does not hold in general. Still $\Pi(A)$ is a subsemiring of $\Omega(A)$ and π a semiring homomorphism. More generally, if R is a semiring ideal extension of A, then $\tau(R, A)$ is a homomorphism and the type of R is a subsemiring of $\Omega(A)$. We first describe the types of semiring ideal extensions of a semiring.

THEOREM 4.7. *A subset U of $\Omega(A)$ is the type of some semiring ideal extension of a semiring A if and only if*

i) *U is a subsemiring of $\Omega(A)$ and $\Pi(A) \subseteq U$;*

ii) *For any* $(\lambda, \rho), (\lambda', \rho') \in U$, λ *and* ρ' *commute;*
iii) *For any* $(\lambda, \rho) \in U$, λ *and* ρ *are additive homomorphisms of* A.

PROOF. Let U be the type of some semiring ideal extension R of A. Then i) is clear by the above remark; ii) follows from Lemma 1.7. Finally if $(\lambda, \rho) = \tau(R, A)(v) \in U$ for some $v \in R$, then $\lambda(a+b) = v(a+b) = va + vb = \lambda a + \lambda b$ for all $a, b \in A$, whence λ is an additive homomorphism of A; similarly ρ is an additive homomorphism of A. Thus iii) holds.

Conversely assume that U is a subset of $\Omega(A)$ satisfying the conditions of the statement. Using part of the proof of Lemma 4.6 (see [6]), we obtain that the groupoid S on the set $A \cup U$ with multiplication $*$ defined by:

$$\begin{aligned} x * y &= xy && \text{if } x, y \in A, \\ \omega * \omega' &= \omega \cdot \omega' && \text{if } \omega, \omega' \in U, \\ x * (\lambda, \rho) &= x\rho && \text{if } x \in A,\ (\lambda, \rho) \in U, \\ (\lambda, \rho) * x &= \lambda x && \text{if } x \in A,\ (\lambda, \rho) \in U, \end{aligned}$$

is a semigroup and an ideal extension of $(A, \cdot)$ of type U.

It is clear on the above formulae defining the multiplication of S that A is a prime ideal of S. Let $\lambda = \sum_{i=1}^{i=n} \lambda_{s_i}$, where $s_i \in S$, be any left S-translation of A. Then if $\rho = \sum_{i=1}^{i=n} \rho_{s_i}$, $(\lambda, \rho) = \sum_{i=1}^{i=n} (\lambda_{s_i}, \rho_{s_i}) \in U$ by i). Thus by iii) λ is an additive homomorphism of A. Similarly any right S-translation of A is an additive homomorphism of A. By Theorem 4.5 it follows that S is embeddable into the universal semiring R of S provided with the addition of A.

Clearly A is an ideal of R. Also $r = s_1 + \cdots + s_n$ is any element of R, where $s_i \in S$, then for all $a \in A$ $ra = s_1 a + \cdots + s_n a$, $ar = as_1 + \cdots + as_n$ implies that $\tau(R, A)(r) = \sum_{i=1}^{i=n} (\lambda_{s_i}, \rho_{s_i}) \in U$ by i) since $(\lambda_{s_i}, \rho_{s_i}) \in U$. Therefore R is a semiring ideal extension of A of type U.

Note that condition i) of Theorem 4.7 can be weakened to:

i') U is a multiplicative semigroup of $\Omega(A)$ which is closed under addition, and $\Pi(A) \subseteq U$.

Indeed i') and iii) imply that the distributivity laws hold in U.

In case U is the subsemiring of $\Omega(A)$ generated by $\Pi(A)$ and a bitranslation (λ, ρ), Theorem 4.7 can be improved as follows:

THEOREM 4.8. *Let* $(\lambda, \rho) \in \Omega(A)$ *be such that* λ *and* ρ *commute and that all finite sums* $\bar{\lambda}$ $(\bar{\rho})$ *of mappings of the form*: λ^n (ρ^n), *where* $n > 0$, *or* λ_a (ρ_a), *where* $a \in A$, *are additive homomorphisms of* A. *Then the set* U *of all* $(\bar{\lambda}, \bar{\rho})$ *is a subsemiring of* $\Omega(A)$ *satisfying* ii) *and* iii) *of Theorem* 4.7. *Moreover there exists a semiring ideal extension* R *of* A *of type* U *having the following property*: *for every semiring ideal extension* R' *of* A *whose type contains* (λ, ρ), *there exists a homomorphism of* R *into* R' *whose restriction to* A *is the identity.*

PROOF. It is routine, if tedious to check that U satisfies the conditions of Theorem 4.7. To show the second part of the statement, let S be the groupoid on $A \cup \{s^n;\ n > 0\}$, where s is an indeterminate, with multiplication $*$ defined by:

$$\begin{aligned} x * y &= xy && \text{if } x, y \in A, \\ s^n * s^p &= s^{n+p} && \text{if } n, p > 0, \\ x * s^n &= x\rho^n && \text{if } x \in A,\ n > 0, \\ s^n * x &= \lambda^n x && \text{if } x \in A,\ n > 0. \end{aligned}$$

It is shown in [7] that this multiplication is associative and that S is an ideal extension of $(A, \cdot)$ of type $\Pi(S) \cup (\lambda^n, \rho^n);\ n > 0\}$; also that, for every ideal extension V of $(A, \cdot)$ whose type contains (λ, ρ), there exists a homomorphism f of S into T whose restriction to A is the identity.

Then a reasoning similar to the end of the proof of Theorem 4.7 shows that the universal semiring R of S provided with the addition of A is a semiring ideal extension of A of type U. The proof of the last statement is then routine.

COROLLARY 4.9. *If A is any semiring, the union of all the types of semiring ideal extensions of A is the set of all $(\lambda, \rho) \in \Omega(A)$ such that λ and ρ commute and that all finite sums of mappings of the form λ^n (ρ^n) with $n > 0$, or λ_a (ρ_a) with $a \in A$, are additive homomorphisms.*

PROOF. This is an immediate consequence of Theorem 4.9.

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Department of Mathematics
Kansas State University
Manhattan, Kanass 66502
U.S.A.

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