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# INDECOMPOSABLE REPRESENTATIONS OF THE EUCLIDEAN ALGEBRA $\mathfrak{e}(3)$ FROM IRREDUCIBLE REPRESENTATIONS OF $\mathfrak{sl}(4, \mathbb{C})$

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**Abstract**

The Euclidean group $E(3)$ is the noncompact, semidirect product group $E(3) \cong \mathbb{R}^3 \rtimes \mathrm{SO}(3)$. It is the Lie group of orientation-preserving isometries of three-dimensional Euclidean space. The Euclidean algebra $\mathfrak{e}(3)$ is the complexification of the Lie algebra of $E(3)$. We embed the Euclidean algebra $\mathfrak{e}(3)$ into the simple Lie algebra $\mathfrak{sl}(4, \mathbb{C})$ and show that the irreducible representations $V(m, 0, 0)$ and $V(0, 0, m)$ of $\mathfrak{sl}(4, \mathbb{C})$ are $\mathfrak{e}(3)$-indecomposable, thus creating a new class of indecomposable $\mathfrak{e}(3)$-modules. We then show that $V(0, m, 0)$ may decompose.

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## 1. Introduction

The Euclidean group $E(3)$ is the noncompact, semidirect product group $E(3) \cong \mathbb{R}^3 \rtimes \mathrm{SO}(3)$. It is the Lie group of orientation-preserving isometries of three-dimensional Euclidean space. The Euclidean algebra $\mathfrak{e}(3)$ is the complexification of the Lie algebra of $E(3)$. Its finite-dimensional irreducible representations are not very interesting, but classifying its indecomposable representations remains a significant challenge. We remind the reader that a representation is *irreducible* if it has no proper subrepresentations. It is *indecomposable* if it is not isomorphic to a direct sum of two nonzero subrepresentations.

Although a full classification of $\mathfrak{e}(3)$-indecomposable representations remains elusive, constructing large classes of indecomposable representations that may be classified is a viable option. Towards this end, in the current paper we embed the Euclidean algebra $\mathfrak{e}(3)$ into the simple Lie algebra $\mathfrak{sl}(4, \mathbb{C})$ and examine certain irreducible representations of $\mathfrak{sl}(4, \mathbb{C})$ to determine whether or not they remain indecomposable upon restriction to $\mathfrak{e}(3)$ under this embedding.

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This direction of research has been pursued, for instance, by Douglas and Premat [3], who showed that irreducible $\mathfrak{sl}(3, \mathbb{C})$-modules remain $\mathfrak{e}(2)$-indecomposable, and later by Casati *et al.* [2], who established that irreducible $\mathfrak{sl}(3, \mathbb{C})$- and $\mathfrak{so}(5, \mathbb{C})$-modules remain indecomposable modules of the Diamond Lie algebra under appropriate embeddings. The Diamond Lie algebra is a central extension of the Poincaré Lie algebra in two dimensions.

In the current paper, we show that the irreducible representations $V(m, 0, 0)$, and $V(0, 0, m)$ of $\mathfrak{sl}(4, \mathbb{C})$ remain $\mathfrak{e}(3)$-indecomposable for all nonnegative integers $m$, thus creating a new class of $\mathfrak{e}(3)$-indecomposable modules. We then present examples in low dimension, based upon which we will conjecture that $V(0, m, 0)$ is not indecomposable for any nonnegative integer $m$.

The paper is organized as follows. In Section 2 we describe the basis and commutation relations of $\mathfrak{e}(3)$. Section 3 records information about the simple Lie algebra $\mathfrak{sl}(4, \mathbb{C})$, and its irreducible representations that will be employed in the following section. In Section 4 we embed $\mathfrak{e}(3)$ into $\mathfrak{sl}(4, \mathbb{C})$, and show that the $\mathfrak{sl}(4, \mathbb{C})$ irreducible representations $V(m, 0, 0)$ and $V(0, 0, m)$ remain $\mathfrak{e}(3)$-indecomposable under this embedding. The final section includes the presentation of examples illustrating the decomposition of $V(0, m, 0)$.

## 2. The Euclidean algebra $\mathfrak{e}(3)$

The *Euclidean algebra* $\mathfrak{e}(3)$ is the complexification of the Lie algebra of the Euclidean Lie group $E(3)$. For a more detailed discussion of $E(3)$, and the calculation of its Lie algebra we refer the reader to [5]. The Euclidean algebra $\mathfrak{e}(3)$ has basis $E, H, F, P_0, P_\pm$, and nonzero commutation relations,

$$\begin{aligned} &[H, E] = 2E, && [H, F] = -2F, && [E, F] = H, \\ &[H, P_\pm] = \pm 2P_\pm, && [E, P_0] = -P_+, && [F, P_0] = -P_-, \\ &[F, P_+] = -2P_0, && [E, P_-] = -2P_0. && \end{aligned} \tag{2.1}$$

One can easily see that $\langle E, H, F\rangle \cong \mathfrak{sl}(2, \mathbb{C})$, and that $\langle P_0, P_\pm\rangle$ is an abelian ideal of $\mathfrak{e}(3)$.

## 3. The simple Lie algebra $\mathfrak{sl}(4, \mathbb{C})$ and its irreducible representations

The special linear algebra $\mathfrak{sl}(4, \mathbb{C})$ is the 15-dimensional Lie algebra of traceless $4 \times 4$ matrices with complex entries. It is the simple Lie algebra of type $A_3$. Let $\{x_i, y_i, h_j, 1 \le i \le 6, 1 \le j \le 3\}$ be the Chevalley basis of $\mathfrak{sl}(4, \mathbb{C})$ defined by

$$\begin{aligned} &ah_1 + bh_2 + ch_3 + dx_1 + ex_2 + fx_3 + gx_4 + hx_5 + ix_6 \\ &\quad + d'y_1 + e'y_2 + f'y_3 + g'y_4 + h'y_5 + i'y_6 \\ &= \begin{pmatrix} a & d & -g & i \\ d' & b-a & e & -h \\ -g' & e' & c-b & f \\ i' & -h' & f' & -c \end{pmatrix}. \end{aligned} \tag{3.1}$$

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For $i = 1, 2$, or 3, define $\Lambda_i \in \mathfrak{h}*$ by $\Lambda_i(h_j) = \delta_{ij}$. For each $\lambda = m_1\Lambda_1 + m_2\Lambda_2 + m_3\Lambda_3 \in \mathfrak{h}*$ with nonnegative integers $m_1, m_2, m_3$ there exists a finite-dimensional irreducible $\mathfrak{sl}(4, \mathbb{C})$-module $V(m_1, m_2, m_3)$ which can be realized as the quotient of the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}(4, \mathbb{C}))$ by the left ideal $J_\lambda$, generated by $x_i$, $h_i - \lambda(h_i)$, $y_i^{1+\lambda(h_i)}$, $1 \le i \le 3$ (here the action of $\mathcal{U}(\mathfrak{sl}(4, \mathbb{C}))$ on itself and on $V(m_1, m_2, m_3)$ is given by left multiplication). We will denote the element $1 + J_\lambda$ of $V(m_1, m_2, m_3)$ by $v_\lambda$, or simply $v$ if there is no ambiguity.

We describe here a basis of irreducible $\mathfrak{sl}(4, \mathbb{C})$ representations due to Littelman [7] (as reported in [1] in a more general setting).

THEOREM 3.1 [7]. *For nonnegative integers $m_1, m_2, m_3$, let $V(m_1, m_2, m_3)$ be the finite-dimensional irreducible representation of $\mathfrak{sl}(4, \mathbb{C})$ with highest weight $\lambda = m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3$. Then the following is a basis of $V(m_1, m_2, m_3)$:*

$$\mathcal{B}_\lambda = \{y_1^{(a_1^1)} y_2^{(a_2^2)} y_1^{(a_1^2)} y_3^{(a_3^3)} y_2^{(a_2^3)} y_1^{(a_1^3)} v\}, \quad \text{where } y_i^{(a)} = \frac{y_i^a}{a!}, \tag{3.2}$$

*subject to the following constraints:*

$$\begin{aligned} 0 \le a_1^3 &\le m_1, \\ a_1^3 \le a_2^3 &\le m_2 + a_1^3, \\ a_2^3 \le a_3^3 &\le m_3 + a_2^3, \\ 0 \le a_1^2 &\le m_1 - 2a_1^3 + a_2^3, \\ a_1^2 \le a_2^2 &\le m_2 + a_1^3 + a_1^2 - 2a_2^3 + a_3^3, \\ 0 \le a_1^1 &\le m_1 - 2(a_1^3 + a_1^2) + a_2^3 + a_2^2. \end{aligned} \tag{3.3}$$

The $\mathfrak{sl}(4, \mathbb{C})$ irreducible representations $V(m, 0, 0)$, and $V(0, 0, m)$ are the focus of the current paper. Note that $V(m, 0, 0) \cong V(0, 0, m)^*$. In the special case $V(m, 0, 0)$, the basis relations of Equation (3.3) for $\mathcal{B}_{(m,0,0)}$ reduce to

$$\begin{aligned} 0 \le a_1^3 = a_2^3 = a_3^3 &\le m, \\ 0 \le a_1^2 = a_2^2 &\le m - a_1^3, \\ 0 \le a_1^1 &\le m - a_1^3 - a_1^2. \end{aligned} \tag{3.4}$$

The following lemma will be used below.

LEMMA 3.2. *Suppose that $0 \le a + b + c \le m$. Then the element $y_1^a y_4^b y_6^c v \in V(m, 0, 0)$ is a nonzero scalar multiple of the element $y_1^{(a)} y_2^{(b)} y_1^{(b)} y_3^{(c)} y_2^{(c)} y_1^{(c)} v \in \mathcal{B}_{(m,0,0)}$.*

PROOF. From Equation (3.4), we can see that the element $y_1^{(a)} y_2^{(b)} y_1^{(b)} y_3^{(c)} y_2^{(c)} y_1^{(c)} v$ with $0 \le a + b + c \le m$ is a member of $\mathcal{B}_{(m,0,0)}$.

We first show that $y_2^b y_1^b y_6^c v$ is a nonzero scalar multiple of $y_4^b y_6^c v$. Since $[y_1, y_6] = [y_2, y_6] = [y_4, y_6] = 0$, it suffices to show that $y_2^b y_1^b v$ is a nonzero scalar multiple of $y_4^b v$. Let $b = 1$; using the fact that $y_2 v = 0$, we obtain

$$y_2 y_1 v = -y_4 v. \tag{3.5}$$

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Assume that $y_2^i y_1^i v = -\alpha_i y_4^b v$ for all $i$ such that $1 \le i \le b-1 < m$, with $\alpha_i$ a nonzero scalar. Then, using $[y_2, y_4] = 0$ and $[y_1, y_2] = y_4$,

$$\begin{aligned} y_2^b y_1^b v &= y_2^{b-1} y_1 y_2 y_1^{b-1} v - \alpha_{b-1} y_4^b v \\ &= y_2^{b-2} y_1 y_2^2 y_1^{b-1} v - 2\alpha_{b-1} y_4^b v \\ &\ \vdots \\ &= y_2 y_1 y_2^{b-1} y_1^{b-1} v - (b-1)\alpha_{b-1} y_4^b v \\ &= -\alpha_{b-1} y_4^b v - (b-1)\alpha_{b-1} y_4^b v \\ &= -b\alpha_{b-1} y_4^b v. \end{aligned} \tag{3.6}$$

We now show that $y_3^c y_2^c y_1^c v$ is a nonzero scalar multiple of $y_6^c v$, proceeding by induction on $c$. If $c = 1$, using the fact that $y_2 v = y_3 v = 0$,

$$y_3 y_2 y_1 v = -y_3 y_4 v = -y_6 v. \tag{3.7}$$

Assume that $y_3^i y_2^i y_1^i v = \beta_i y_6^i v$ for all $i$ such that $1 \le i < c < m$, where $\beta_i$ is an nonzero scalar. We show that it holds for $i = c$. Note that from the above work we have $y_2^c y_1^c v = \alpha y_6^c v$ for a nonzero scalar $\alpha$, and that $[y_3, y_6] = 0$, so

$$\begin{aligned} \frac{1}{\alpha} y_3^c y_2^c y_1^c v &= y_3^c y_4^c v \\ &= y_3^{c-1} y_4 y_3 y_4^{c-1} v - \beta_{c-1} y_6^c v \\ &= y_3^{c-2} y_4 y_3^2 y_4^{c-1} v - 2\beta_{c-1} y_6^c v \\ &\ \vdots \\ &= y_3 y_4 y_3^{c-1} y_4^{c-1} v - (c-1)\beta_{c-1} y_6^c v \\ &= -c\beta_{c-1} y_6^c v. \end{aligned} \tag{3.8}$$

We have shown that $y_3^c y_2^c y_1^c v$ is a nonzero scalar multiple of $y_6^c v$, and that $y_2^b y_1^b y_6^c v$ is a nonzero scalar multiple of $y_4^b y_6^c v$, from which the result follows. □

## 4. Representations of $\mathfrak{e}(3)$ from irreducible representations of $\mathfrak{sl}(4, \mathbb{C})$

We may embed $\mathfrak{e}(3)$ into $\mathfrak{sl}(4, \mathbb{C})$ as follows:

$$\begin{aligned} \phi : \mathfrak{e}(3) &\hookrightarrow \mathfrak{sl}(4, \mathbb{C}) \\ E &\mapsto x_2 + x_6 \\ H &\mapsto h_1 + 2h_2 + h_3 \\ F &\mapsto y_2 + y_6 \\ P_+ &\mapsto -2x_4 \\ P_0 &\mapsto x_1 - y_3 \\ P_- &\mapsto 2y_5. \end{aligned} \tag{4.1}$$

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In this section we will show that $V(m, 0, 0)$, and $V(0, 0, m)$ are $\mathfrak{e}(3)$-indecomposable under the embedding $\phi$. Since $V(0, 0, m)^* \cong V(m, 0, 0)$, the following proposition reduces this to showing that $V(m, 0, 0)$ is $\mathfrak{e}(3)$-indecomposable.

PROPOSITION 4.1. *Suppose that $V$ is a finite-dimensional representation of $\mathfrak{e}(3)$. Then $V$ is indecomposable if and only if its dual (that is, contragredient) $V^*$ is indecomposable.*

PROOF. Suppose that the representation $V$ decomposes: $V = V_1 \oplus V_2$. Then it is easy to see that the natural decomposition $V^* = V_1^* \oplus V_2^*$ of vector spaces is in fact a decomposition of representations. The converse follows from the fact that $V^{**} \cong V$. □

The following lemmas will be used to establish the indecomposability of $V(m, 0, 0)$ and $V(0, 0, m)$ in Theorem 4.6 below.

LEMMA 4.2. *Let $v$ be the maximal vector of $V(m, 0, 0)$, and $0 \le i + j + k \le m$; then*

$$H \cdot y_1^i y_4^j y_6^k v = (m - 2(j + k)) y_1^i y_4^j y_6^k v, \tag{4.2}$$

$$E \cdot y_1^i y_4^j y_6^k v = \eta_1(i, j, k) y_1^i y_4^j y_6^{k-1} v + \eta_2(i, j, k) y_1^{i+1} y_4^{j-1} y_6^k v, \tag{4.3}$$

$$P_+ \cdot y_1^i y_4^j y_6^k v = \eta_+(i, j, k) y_1^i y_4^{j-1} y_6^k v, \tag{4.4}$$

$$P_0^i \cdot y_1^i v = \Pi_{t=1}^i t(m - t + 1) v, \tag{4.5}$$

*where*

$$\eta_1(i, j, k) = k(m - i - j - k + 1), \quad \eta_2(i, j, k) = -j, \tag{4.6}$$

$$\eta_+(i, j, k) = -2j(m - i - j - k + 1). \tag{4.7}$$

PROOF. We prove only Equations (4.2) and (4.3). The other equations are proved in a similar fashion. Since $[H, y_1] = 0$, $[H, y_4] = -2y_4$, and $[H, y_6] = -2y_6$, Equation (4.2) follows from a simple count of weights:

$$H \cdot y_1^i y_4^j y_6^k v = (m - 2(j + k)) y_1^i y_4^j y_6^k v. \tag{4.8}$$

Since

$$[E, y_1] = x_5, \quad [E, y_4] = x_3 - y_1, \quad [E, y_6] = h_1 + h_2 + h_3,$$
$$[x_5, y_1] = [x_5, y_4] = 0, \quad [x_5, y_6] = -y_1 \quad \text{and} \quad [x_3, y_6] = y_4,$$

we have

$$\begin{aligned} E \cdot y_1^i y_4^j y_6^k v &= y_1^i E y_4^j y_6^k v + i y_1^{i-1} y_4^j x_5 y_6^k v \\ &= y_1^i y_4^j E y_6^k v + j y_1^i y_4^{j-1} x_3 y_6^k v - j y_1^{i+1} y_4^{j-1} y_6^k v - ik y_1^i y_4^j y_6^{k-1} v \\ &= \sum_{l=0}^{k-1} y_1^i y_4^j y_6^l h_1 y_6^{k-1-l} v + \sum_{l=0}^{k-1} y_1^i y_4^j y_6^l h_3 y_6^{k-1-l} v \\ &\quad - jk y_1^i y_4^j y_6^{k-1} v - j y_1^{i+1} y_4^{j-1} y_6^k v - ik y_1^i y_4^j y_6^{k-1} v \end{aligned}$$

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$$
\begin{aligned}
&= \left(km - k^2 + \frac{k(k+1)}{2}\right) y_1^i y_4^j y_6^{k-1} v \\
&\quad + \left(-k^2 + \frac{k(k+1)}{2}\right) y_1^i y_4^j y_6^{k-1} v \\
&\quad - jk y_1^i y_4^j y_6^{k-1} v - j y_1^{i+1} y_4^{j-1} y_6^k v - ik y_1^i y_4^j y_6^{k-1} v \qquad (4.9)\\
&= (km - k^2 + k - jk - ik) y_1^i y_4^j y_6^{k-1} v - j y_1^{i+1} y_4^{j-1} y_6^k v \\
&= k(m - k + 1 - j - i) y_1^i y_4^j y_6^{k-1} v - j y_1^{i+1} y_4^{j-1} y_6^k v \\
&= \eta_1(i, j, k) y_1^i y_4^j y_6^{k-1} v + \eta_2(i, j, k) y_1^{i+1} y_4^{j-1} y_6^k v.
\end{aligned}
$$

This concludes the proof. □

LEMMA 4.3.

$$\dim(V(m, 0, 0)) = \sum_{i=0}^{\lfloor m/2 \rfloor} (m - 2i + 1)^2, \qquad (4.10)$$

*where $\lfloor m/2 \rfloor$ is the largest integer less than or equal to $m/2$.*

PROOF. Recall Weyl's character formula for the dimension of $V(\lambda)$ [6]:

$$\dim(V(\lambda)) = \frac{\Pi_{\alpha>0}\langle \lambda + \delta, \alpha \rangle}{\Pi_{\alpha>0}\langle \delta, \alpha \rangle}. \qquad (4.11)$$

The positive roots of $A_3$ are $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_1 + \alpha_2$, $\alpha_2 + \alpha_3$, and $\alpha_1 + \alpha_2 + \alpha_3$. Accordingly, for $\lambda = m\lambda_1$, the denominator is $1 \cdot 1 \cdot 1 \cdot 2 \cdot 2 \cdot 3 = 12$, while the numerator is $(m + 1) \cdot 1 \cdot 1 \cdot (m + 2) \cdot 2 \cdot (m + 3)$. Thus,

$$\dim(V(m, 0, 0)) = \frac{(m+1)(m+2)(m+3)}{6}. \qquad (4.12)$$

It is then not difficult to show that

$$\frac{(m+1)(m+2)(m+3)}{6} = \sum_{i=0}^{\lfloor m/2 \rfloor} (m - 2i + 1)^2. \qquad (4.13)$$

For odd $m$, this follows easily from the familiar formula

$$\sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}.$$

Subtracting this result from the whole sum then recovers the result for even $m$. □

LEMMA 4.4. *The $H$-maximal vectors that occur in $V(m, 0, 0)$ have weights $m - 2M$, where $0 \le M \le \lfloor m/2 \rfloor$. A basis for the $H$-highest weight vectors of $H$-weight $m - 2M$*

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*is given by*

$$w(M,i)=\sum_{j=0}^{M}\alpha_j(M,i)y_1^{i+j}y_4^{M-j}y_6^{j}v, \tag{4.14}$$

*for* $0\le i\le m-2M$*, where the nonzero scalars* $\alpha_j(M,i)$ *are defined recursively by*

$$\begin{gathered}\alpha_M(M,i)=1,\\ \alpha_j(M,i)=\frac{-\alpha_{j+1}(M,i)\eta_1(i+j+1,\,M-j-1,\,j+1)}{\eta_2(i+j,\,M-j,\,j)},\quad 0\le j<M.\end{gathered}\tag{4.15}$$

PROOF. We first show the $w(M,i)$ are linearly independent. First note that, by Lemma 3.2, each summand in $w(M,i)$ is a nonzero basis vector (up to a nonzero scalar multiple) in $\mathcal{B}_{(m,0,0)}$. By Lemma 4.2, the weight of $w(M,i)$ is $m-2M$. So it suffices to check that $w(M,i)$ are linearly independent for fixed $M$, where $0\le M\le\lfloor m/2\rfloor$, and all $i$ such that $0\le i\le m-2M$. This, however, follows easily by noting that the leading term $y_1^i y_4^M v$ of $w(M,i)$ occurs as a summand in $w(M,i')$ if and only if $i=i'$.

We now check that $E\cdot w(M,i)=0$. Using Lemma 4.2,

$$\begin{aligned}
&E\cdot w(M,i)\\
&\quad=\sum_{j=0}^{M}\alpha_j(M,i)E\cdot y_1^{i+j}y_4^{M-j}y_6^{j}v\\
&\quad=\sum_{j=1}^{M-1}\alpha_j(M,i)(\eta_1(i+j,\,M-j,\,j)y_1^{i+j}y_4^{M-j}y_6^{j-1}v\\
&\qquad+\eta_2(i+j,\,M-j,\,j)y_1^{i+j+1}y_4^{M-j-1}y_6^{j}v)\\
&\qquad+\alpha_0(M,i)\eta_2(i,\,M,\,0)y_1^{i+1}y_4^{M-1}v\\
&\qquad+\alpha_M(M,i)\eta_1(i+M,\,0,\,M)y_1^{i+M}y_6^{M-1}v\\
&\quad=(\alpha_0(M,i)\eta_2(i,\,M,\,0)+\alpha_1(M,i)\eta_1(i+1,\,M-1,\,1))y_1^{i+1}y_4^{M-1}v\\
&\qquad+\sum_{j=1}^{M-1}(\alpha_j(M,i)\eta_2(i+j,\,M-j,\,j)\\
&\qquad+\alpha_{j+1}(M,i)\eta_1(i+j+1,\,M-j-1,\,j+1))y_1^{i+j+1}y_4^{M-j-1}y_6^{j}v\\
&\qquad+(\alpha_{M-1}(M,i)\eta_2(i+M-1,\,1,\,M-1)\\
&\qquad+\alpha_M(M,i)\eta_1(i+M,\,0,\,M))y_1^{i+M}y_6^{M-1}v\\
&\quad=\Bigg(\frac{-\alpha_1(M,i)\eta_1(i+1,\,M-1,\,1)}{\eta_2(i,\,M,\,0)}\eta_2(i,\,M,\,0)\\
&\qquad+\alpha_1(M,i)\eta_1(i+1,\,M-1,\,1)\Bigg)y_1^{i+1}y_4^{M-1}v
\end{aligned}$$

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$$
\begin{aligned}
&+ \sum_{j=1}^{M-1} \bigg( \frac{-\alpha_{j+1}(M, i)\eta_1(i+j+1, M-j-1, j+1)}{\eta_2(i+j, M-j, j)} \\
&\times \eta_2(i+j, M-j, j) \\
&+ \alpha_{j+1}(M, i)\eta_1(i+j+1, M-j-1, j+1) \bigg) y_1^{i+j+1} y_4^{M-j-1} y_6^{j} v \\
&+ \bigg( -\frac{\eta_1(i+M, 0, M)}{\eta_2(i+M-1, 1, M-1)} \eta_2(i+M-1, 1, M-1) \\
&+ \eta_1(i+M, 0, M) \bigg) y_1^{i+M} y_6^{M-1} v \\
= {}& 0.
\end{aligned}
$$

We thus have

$$\langle w(M, i)\rangle \cong_{\mathfrak{sl}(2,\mathbb{C})} V(m-2M), \tag{4.16}$$

for each $i$ and $M$ such that $0 \le M \le \lfloor m/2 \rfloor$ and $0 \le i \le m - 2M$. By linear independence of the $w(M, i)$, we have a direct sum $\mathfrak{sl}(2, \mathbb{C})$-subrepresentation of $V(m, 0, 0)$:

$$\bigoplus_{i=0}^{\lfloor m/2 \rfloor} (m-2i+1)\langle w(M, i)\rangle \cong_{\mathfrak{sl}(2,\mathbb{C})} \bigoplus_{i=0}^{\lfloor m/2 \rfloor} (m-2i+1) V(m-2i). \tag{4.17}$$

By dimension considerations, Lemma 4.3 and the fact that $\dim(V(m-2i)) = m-2i+1$,

$$V(m, 0, 0) \cong_{\mathfrak{sl}(2,\mathbb{C})} \bigoplus_{i=0}^{\lfloor m/2 \rfloor} (m-2i+1) V(m-2i). \tag{4.18}$$

This concludes the proof. □

LEMMA 4.5. *Suppose that* $0 \le M \le \lfloor m/2 \rfloor$, $0 \le i \le m - 2M$. *Then*

$$P_+^M \cdot w(M, i) = \alpha_0(M, i)(\Pi_{k=1}^{M} \eta_+(i, k, 0)) y_1^i v. \tag{4.19}$$

PROOF. Equation (4.4) of Lemma 4.2 implies that $P_+^M \cdot y_1^{i+j} y_4^{M-j} y_6^j v = 0$ for $j > 0$ since in this case the exponent of $y_4$ is less than $M$. Further,

$$P_+^M \cdot y_1^i y_4^M v = (\Pi_{k=1}^{M} \eta_+(i, k, 0)) y_1^i v$$

follows from Lemma 4.2. The result follows. □

THEOREM 4.6. *The* $\mathfrak{sl}(4, \mathbb{C})$*-modules* $V(m, 0, 0)$ *and* $V(0, 0, m)$ *are* $\mathfrak{e}(3)$*-indecomposable.*

PROOF. By Proposition 4.1, since $V(m, 0, 0)^* \cong V(0, 0, m)$, it suffices to show that $V(m, 0, 0)$ is $\mathfrak{e}(3)$-indecomposable. Suppose that $V(m, 0, 0)$ decomposes as

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an $\mathfrak{e}(3)$-module:

$$V(m,0,0) \cong_{\mathfrak{e}(3)} V \oplus V'. \tag{4.20}$$

By way of contradiction, suppose that both $V \neq 0$ and $V' \neq 0$. Then, by Lemma 4.4, each of $V$ and $V'$ contains an $H$-highest weight vector:

$$\begin{aligned}\sum_{i=0}^{m-2M} \beta_i w(M,i) &\in V,\\ \sum_{i'=0}^{m-2M'} \beta'_{i'} w(M',i') &\in V',\end{aligned} \tag{4.21}$$

where $0 \leq M,\ M' \leq \lfloor m/2 \rfloor$, and not all $\beta_i$ nor all $\beta''_i$ are zero.

Let $i_{\max}$ be maximal among $i$ such that $\beta_i \neq 0$. Then, using Lemmas 4.2 and 4.5,

$$\begin{aligned}&P_0^{i_{\max}} \cdot \left( P_+^M \cdot \sum_{i=0}^{m-2M} \beta_i w(M,i) \right)\\ &\quad = P_0^{i_{\max}} \cdot \left( \sum_{i=0}^{m-2M} \beta_i \alpha_0(M,i) (\Pi_{k=1}^{M} \eta_+(i,k,0)) y_1^i v \right)\\ &\quad = \beta_{i_{\max}} \alpha_0(M,i_{\max}) (\Pi_{t=1}^{i_{\max}} t(m-t+1)) (\Pi_{k=1}^{M} \eta_+(i,k,0)) v.\end{aligned} \tag{4.22}$$

Hence, since $\beta_{i_{\max}} \alpha_0(M,i_{\max}) (\Pi_{t=1}^{i_{\max}} t(m-t+1)) (\Pi_{k=1}^{M} \eta_+(i,k,0))$ is a nonzero scalar, we see that $v \in V$. Similarly, we may show $v \in V'$, a contradiction. Thus it must be the case that $V(m,0,0)$ is indecomposable. □

## 5. Conclusions

We have shown that the irreducible $\mathfrak{sl}(4,\mathbb{C})$-modules $V(m,0,0)$ and $V(0,0,m)$ are $\mathfrak{e}(3)$-indecomposable under the embedding described above. However, not all $\mathfrak{sl}(4,\mathbb{C})$-modules are $\mathfrak{e}(3)$-indecomposable, as the following examples illustrate. All the examples were calculated with the assistance of the GAP computer algebra system [4].

The $\mathfrak{sl}(4,\mathbb{C})$ representations $V(0,1,0)$ and $V(0,2,0)$, of dimension 6 and 20 respectively, decompose over $\mathfrak{e}(3)$ as follows:

$$\begin{aligned}V(0,1,0) &\cong_{\mathfrak{e}(3)} \langle y_2 v - y_6 v \rangle \oplus \langle v,\ y_1 y_2 v,\ y_5 v,\ y_2 y_6 v,\ y_2 y_6 v \rangle,\\ V(0,2,0) &\cong_{\mathfrak{e}(3)} \langle y_2 y_5 v - y_5 y_6 v + y_5^2 v \rangle\\ &\qquad \oplus \langle y_2 v - y_6 v,\ y_2 y_4 v - y_4 y_6 v,\ y_2 y_5 v - y_5 y_6 v,\\ &\qquad\quad y_2^2 y_6 v - y_2 y_6^2 v,\ y_2^2 v - y_6^2 v \rangle\\ &\qquad \oplus \langle v,\ y_5 v,\ y_2 v - y_6 v,\ y_4 v,\ y_5^2 v,\ y_2 y_5 v + y_5 y_6 v,\\ &\qquad\quad y_4 y_5 v,\ y_2^2 v + y_2 y_6 v + y_6^2 v,\ y_4^2 v,\ y_4 y_6 v + y_2 y_4 y_6 v,\\ &\qquad\quad y_2 y_5 y_6 v,\ y_2^2 y_6 v + y_2 y_6^2 v,\ y_2 y_4 y_6 v,\ y_2^2 y_6^2 v \rangle.\end{aligned} \tag{5.1}$$

Based on these examples we conjecture that $V(0,m,0)$ decomposes for all $m$. Indecomposability in the general case $V(m_1,m_2,m_3)$ is less clear. However, it is clear

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that a class larger than $V(m, 0, 0)$, and $V(0, 0, m)$ does remain $\mathfrak{e}(3)$-indecomposable; for instance, the modules $V(1, 0, 1)$, and $V(1, 1, 0)$ are $\mathfrak{e}(3)$-indecomposable.

It is also interesting to note that $\mathfrak{e}(3)$ may be embedded into other simple Lie algebras. For instance, we may embed $\mathfrak{e}(3)$ into $\mathfrak{so}(5, \mathbb{C})$, the simple Lie algebra of type $B_2$. We are currently investigating the irreducible $\mathfrak{so}(5, \mathbb{C})$ representations restricted to $\mathfrak{e}(3)$. Embedding $\mathfrak{e}(3)$ into $\mathfrak{sl}(4, \mathbb{C})$ was investigated in the present paper since this is a natural generalization of embedding $\mathfrak{e}(2)$ into $\mathfrak{sl}(3, \mathbb{C})$ examined in [3].

Since $\mathfrak{e}(3)$ may be embedded into $\mathfrak{so}(5, \mathbb{C})$, it naturally embeds into $\mathfrak{so}(7, \mathbb{C})$, the simple Lie algebra of type $B_3$. An embedding is given by

$$\begin{aligned} \phi : \mathfrak{e}(3) &\hookrightarrow \mathfrak{so}(7, \mathbb{C}) \\ E &\mapsto x_5 \\ H &\mapsto 2h_2 + h_3 \\ F &\mapsto y_5 \\ P_+ &\mapsto x_9 \\ P_0 &\mapsto \tfrac{1}{2}x_6 \\ P_- &\mapsto x_1. \end{aligned} \tag{5.2}$$

However, irreducible representations of $\mathfrak{so}(7, \mathbb{C})$, even in small dimension, appear to $\mathfrak{e}(3)$-decompose as the following examples in dimensions 7, 21 and 8, respectively, illustrate:

$$\begin{aligned} V_{\mathfrak{so}(7,\mathbb{C})}(1, 0, 0) &\cong_{\mathfrak{e}(3)} \langle v, y_1 v, y_6 v, y_1 y_9 v\rangle \oplus \langle y_4 v\rangle \oplus \langle y_8 v\rangle, \\ V_{\mathfrak{so}(7,\mathbb{C})}(0, 1, 0) &\cong_{\mathfrak{e}(3)} \langle y_4 y_9 v, y_4 v, y_2, y_2 y_6 v, y_2 y_9 v\rangle \\ &\quad \oplus \langle y_8 y_9 v, y_8 v, y_7 v, y_5 y_8 v, y_7 y_9 v\rangle \\ &\quad \oplus \langle v, y_5 v, y_2 y_7 v, y_6 v, -y_2 y_8 v - y_4 y_7 v + y_9 v, \\ &\qquad y_5 y_9 v, y_4 y_8 v, y_2 y_6 v, y_9^2, y_2 y_8 v, y_4 y_7 v\rangle, \\ V_{\mathfrak{so}(7,\mathbb{C})}(0, 0, 1) &\cong_{\mathfrak{e}(3)} \langle v, y_5 v, y_6 v, y_9 v\rangle \oplus \langle y_3 v, y_7, y_8, y_3 y_9 v\rangle. \end{aligned} \tag{5.3}$$

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