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The monostable cooperative system with nonlocal diffusion and free boundaries

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This paper concerns the monostable cooperative system with nonlocal diffusion and free boundaries, which has recently been discussed by Du and Ni [J. Differential equations 308(2021) 369-420 and arXiv:2010.01244]. We here aim at four aspects: the first is to give more accurate estimates for the longtime behaviours of the solution; the second is to discuss the limits of solution pair of a semi-wave problem; the third is to investigate the asymptotic behaviours of the corresponding Cauchy problem; the last is to study the limiting profiles of the solution as one of the expanding rates of free boundaries converges to ∞. Moreover, some epidemic models are given to illustrate their own rich longtime behaviours, which are quite different from those of the relevant existing works.

Keywords: Monostable cooperative system; Nonlocal diffusion; Free boundaries; Accelerated spreading; Spreading speed

2020 *Mathematics Subject Classification:* 35K57, 35R09, 35R20, 35R35, 92D25

1. Introduction and main results

Recently, Du and Ni [10, 11] considered the following monostable cooperative system with nonlocal diffusion and free boundaries

$$\begin{cases} u_{it}=d_i\mathcal{L}_i[u_i](t,x)+f_i(u), \quad t>0,\ x\in(g(t),h(t)), 1\leqslant i\leqslant m_0,\\ u_{it}=f_i(u), \quad t>0,\ x\in(g(t),h(t)),\ m_0+1\leqslant i\leqslant m,\\ u_i(t,g(t))=u_i(t,h(t))=0, \quad t>0,\ 1\leqslant i\leqslant m,\\ g'(t)=-\displaystyle\sum_{i=1}^{m_0}\mu_i\int_{g(t)}^{h(t)}\int_{-\infty}^{g(t)}J_i(x-y)u_i(t,x)\mathrm{d}y\mathrm{d}x, \quad t>0,\\ h'(t)=\displaystyle\sum_{i=1}^{m_0}\mu_i\int_{g(t)}^{h(t)}\int_{h(t)}^{\infty}J_i(x-y)u_i(t,x)\mathrm{d}y\mathrm{d}x, \quad t>0,\\ h(0)=-g(0)=h_0>0, \quad u_i(0,x)=u_{i0}(x), \quad |x|\leqslant h_0,\ 1\leqslant i\leqslant m, \end{cases} \tag{1.1}$$

where $u=(u_1,\cdots,u_m)$, $1\leqslant m_0\leqslant m$, $d_i>0$, $\mu_i\geqslant 0$, $\sum_{i=1}^{m_0}\mu_i>0$, and

$$\mathcal{L}_i[u_i](t,x):=\int_{g(t)}^{h(t)}J_i(x-y)u_i(t,y)\mathrm{d}y-u_i(t,x). \tag{1.2}$$

For $1\leqslant i\leqslant m_0$, kernel functions J_i satisfy

(J) $J_i\in C(\mathbb{R})\cap L^\infty(\mathbb{R})$, $J_i\geqslant 0$, $J_i(0)>0$, $\int_{\mathbb{R}}J_i(x)\mathrm{d}x=1$, J_i is even,

and the initial function $u_0(x)=(u_{10}(x),\cdots,u_{m0}(x))$ satisfies

$$u_{i0}\in C([-h_0,h_0]), \quad u_{i0}(\pm h_0)=0<u_{i0}(x),\ \forall\, x\in(-h_0,h_0),\ i=1,\cdots,m.$$

This model can be used to describe the spreading of some epidemics and the interactions of various species, for example, see [**38**] and [**12**], where similarly to (1.1) the spatial movements of agents are approximated by the nonlocal diffusion operator (1.2) instead of random diffusion (also known as local diffusion). Such kind of free boundary problem was firstly proposed in [**4**] and [**7**]. Especially, it can be seen from [**4**] that the introduction of nonlocal diffusion brings about some different dynamical behaviours from the local version in [**9**], and also gives arise to some technical difficulties. Since these two works [**4**] and [**7**] appeared, some related research has emerged. For example, one can refer to [**8**] for the first attempt to the spreading speed of the model in [**4**], [**5**, **15**, **22**] for the Lotka–Volterra competition and prey–predator models, [**21**, **28**, **29**] for the systems where one species adopts the nonlocal diffusion strategy while the other takes the local diffusion, [**13**, **14**] for high dimensional and radial symmetric version of the model in [**4**], [**20**] for the model with a fixed boundary and a moving boundary, [**19**] for unbounded initial range, [**23**] for the mutualist model, [**34**, **35**] for SIR epidemic model, and [**27**] for the model with seasonal succession.

Before introducing our results for (1.1), let us briefly review some conclusions obtained by Du and Ni [**10**, **11**]. The following notations and assumptions are necessary.

• Notations:

(i) $\mathbb{R}^m_+:=\{x=(x_1,\cdots,x_m)\in\mathbb{R}^m: x_i\geqslant 0,\ 1\leqslant i\leqslant m\}$.

(ii) For $x \in \mathbb{R}^m$, we simply write $x = (x_i)$ sometimes, and denote the transpose of x by x^T. For $x, y \in \mathbb{R}^m$, $x \preceq (\succeq)\, y$ means $x_i \leqslant (\geqslant)\, y_i$ for all $1 \leqslant i \leqslant m$; $x \prec (\succ)\, y$ means $x_i < (>)\, y_i$ for all $1 \leqslant i \leqslant m$.

(iii) If $x \preceq y$, we set $[x, y] = \{z \in \mathbb{R}^m : x \preceq z \preceq y\}$, the order interval.

(iv) Hadamard product: $x \circ y = (x_i y_i) \in \mathbb{R}^m$ for all $x, y \in \mathbb{R}^m$.

(v) For any given functions $s(t)$ and $\gamma(t)$, we say $s(t) \approx \gamma(t)$ if there exist two positive constants c_1, C_1 such that $c_1\gamma(t) \leqslant s(t) \leqslant C_1\gamma(t)$ for $t \gg 1$; we say $s(t) = o(\gamma(t))$ if $\lim_{t\to\infty} \frac{s(t)}{\gamma(t)} = 0$.

- Assumptions on reaction term f_i:

(**f1**) (i) Let $f = (f_1, \cdots, f_m) \in [C^1(\mathbb{R}^m_+)]^m$. System $f(u) = \mathbf{0}$ has only two roots in $\mathbb{R}^m_+$: $\mathbf{0} = (0, \cdots, 0)$ and $\tilde{u} = (\tilde{u}_1, \cdots, \tilde{u}_m) \succ \mathbf{0}$.
(ii) $\partial_j f_i(u) := \frac{\partial f_i(u)}{\partial u_j} \geqslant 0$ for $i \neq j$ and $u \in [\mathbf{0}, \hat{u}]$, where either $\hat{u} = \infty$ meaning $[\mathbf{0}, \hat{u}] = \mathbb{R}^m_+$, or $\tilde{u} \prec \hat{u} \in \mathbb{R}^m_+$. This implies that system (1.1) is cooperative in $[\mathbf{0}, \hat{u}]$.
(iii) The matrix $\nabla f(\mathbf{0}) = (\partial_j f_i(\mathbf{0}))_{m\times m}$ is irreducible with positive principal eigenvalue.
(iv) If $m_0 < m$, then $\partial_j f_i(u) > 0$ for $1 \leqslant j \leqslant m_0 < i \leqslant m$ and $u \in [\mathbf{0}, \tilde{u}]$.

(**f2**) $f(ku) \succeq kf(u)$ for any $0 \leqslant k \leqslant 1$ and $u \in [\mathbf{0}, \hat{u}]$.

(**f3**) The matrix $\nabla f(\tilde{u})$ is invertible, $\nabla f(\tilde{u})\tilde{u}^T \preceq \mathbf{0}$ and for every $1 \leqslant i \leqslant m$, either $\sum_{j=1}^m \partial_j f_i(\tilde{u})\tilde{u}_j < 0$, or $\sum_{j=1}^m \partial_j f_i(\tilde{u})\tilde{u}_j = 0$ and $f_i(u)$ is linear in $[\tilde{u} - \varepsilon_0\mathbf{1}, \tilde{u}]$ for some small $\varepsilon_0 > 0$, where $\mathbf{1} = (1, \cdots, 1) \in \mathbb{R}^m$.

(**f4**) Define $d_i = 0$ and $J_i \equiv 0$ for $i \in \{m_0 + 1, \cdots, m\}$. Denote $D = (d_i)$ and $J = (J_i)$. Problem

$$U_t = D \circ \int_{\mathbb{R}} J(x-y) \circ U(t,y)\mathrm{d}y - D \circ U + f(U) \text{ for } t > 0,\ x \in \mathbb{R} \tag{1.3}$$

has an invariant set $[\mathbf{0}, \hat{u}]$ and its every nontrivial solution is attracted by the equilibrium $\tilde{u}$. That is, if the initial value $U(0, x) \in [\mathbf{0}, \hat{u}]$, then $U(t, x) \in [\mathbf{0}, \hat{u}]$ for all $t > 0$ and $x \in \mathbb{R}$; if further $U(0, x) \not\equiv \mathbf{0}$, then $\lim_{t\to\infty} U(t, x) = \tilde{u}$ locally uniformly in $\mathbb{R}$.

In this paper we always assume that the conditions (**J**) and (**f1**)–(**f4**) hold, and the initial function $u_0 \in [\mathbf{0}, \hat{u}]$.

Under the above assumptions, one easily proves that (1.1) has a unique global solution (u, g, h) by using similar methods in [**12**, **38**]. Here, we suppose that its longtime behaviours are governed by a spreading–vanishing dichotomy, namely, one of the following alternatives must happen for (1.1)

(i) Spreading: $\lim_{t\to\infty} h(t) = -\lim_{t\to\infty} g(t) = \infty$ and $\lim_{t\to\infty} u(t, x) = \tilde{u}$ locally uniformly in $\mathbb{R}$.

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(ii) Vanishing: $\lim_{t\to\infty}[h(t) - g(t)] < \infty$ and $\lim_{t\to\infty}\|u(t,\cdot)\|_{C([g(t),h(t)])} = 0$.

As in [9] and [8], the understanding of spreading speed for free boundary problem highly relies on the associated semi-wave problem. The semi-wave problem corresponding to (1.1) is made up of the following two equations:

$$\begin{cases} D\circ\int_{-\infty}^{0} J(x-y)\circ\Phi(y)\mathrm{d}y - D\circ\Phi + c\Phi'(x) + f(\Phi) = 0, & -\infty < x < 0,\\ \Phi(-\infty) = \tilde{u}, \Phi(0) = \mathbf{0}, & \Phi(x) = (\phi_i(x)), \end{cases} \tag{1.4}$$

and

$$c = \sum_{i=1}^{m_0}\mu_i\int_{-\infty}^{0}\int_{0}^{\infty} J_i(x-y)\phi_i(x)\mathrm{d}y\mathrm{d}x. \tag{1.5}$$

In order not to cause confusion, as in [10] we say that (c, Φ) is a semi-wave solution of (1.1) if (c, Φ) satisfies (1.4)–(1.5). And if (c, Φ) solves (1.4), we say that Φ is a semi-wave solution for (1.3) with speed c. Moreover, we also call the solution of the problem

$$\begin{cases} D\circ\int_{\mathbb{R}} J(x-y)\circ\Psi(y)\mathrm{d}y - D\circ\Psi + c\Psi'(x) + f(\Psi) = 0, & -\infty < x < \infty,\\ \Psi(-\infty) = \tilde{u}, \Psi(\infty) = \mathbf{0}, & \Psi(x) = (\psi_i(x)). \end{cases} \tag{1.6}$$

the travelling wave solution of (1.3) with speed c. Du and Ni [10] obtained a complete understanding for the semi-wave solutions of (1.1), (1.3) and the travelling wave solution of (1.3). To state their conclusion, two following threshold conditions on J_i are important and necessary, namely,

(**J1**) $\int_0^\infty xJ_i(x)\mathrm{d}x < \infty$ if $\mu_i > 0$, $i \in \{1, \cdots, m_0\}$,

(**J2**) $\int_0^\infty e^{\lambda x}J_i(x)\mathrm{d}x < \infty$ for some $\lambda > 0$ and any $i \in \{1, \cdots, m_0\}$.

Clearly, the condition (**J2**) implies (**J1**) but not the other way around.

Theorem A. [10, Theorems 1.1 and 1.2] *The following conclusions hold:*

(i) *There exists a $C_* \in (0, \infty]$ such that the semi-wave problem (1.4) has a unique monotone solution if and only if $c \in (0, C_*)$, and the travelling wave problem (1.6) has a monotone solution if and only if $c \geqslant C_*$.*

(ii) *$C_* < \infty$ if and only if* (**J2**) *holds.*

(iii) *System (1.4)–(1.5) has a unique solution pair (c_0, Φ_0) with $c_0 > 0$ and Φ_0 non-increasing in $(-\infty, 0]$ if and only if* (**J1**) *holds.*

With the help of Theorem A and some comparison principles, Du and Ni [10] discussed the spreading speeds of $g(t)$ and $h(t)$ when spreading happens for (1.1),

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and proved that there is a finite spreading speed for (1.1) if and only if (**J1**) holds. Exactly, they obtained the following conclusion.

Theorem B ([**10**, Theorem 1.3]). *Let* $(u,\,g,\,h)$ *be a solution of* (1.1) *and spreading happens. Then*

$$\lim_{t\to\infty}\frac{-g(t)}{t}=\lim_{t\to\infty}\frac{h(t)}{t}=\begin{cases}c_0 \text{ if } (\mathbf{J1}) \text{ holds,}\\ \infty \text{ if } (\mathbf{J1}) \text{ does not hold,}\end{cases}$$

where c_0 *is uniquely determined by the semi-wave problem* (1.4)–(1.5).

When (**J1**) does not hold, we usually call the phenomenon the accelerated spreading. Additionally, some more accurate estimates on free boundaries were also derived in [**11**] if J_i satisfy

(J$^\gamma$) $J_i(x)\approx|x|^{-\gamma}$ for all $i\in\{1,\,\cdots,\,m_0\}$ and $m_0=m$.

Theorem C ([**11**, Theorem 1.5]). *Suppose that* $(\mathbf{J}^\gamma)$ *holds with* $\gamma\in(1,\,2]$. *Let* $(u,\,g,\,h)$ *be a solution of* (1.1) *and spreading happens. Then*

$$-g(t),\ h(t)\approx t^{1/(\gamma-1)}\ \textit{if}\ \gamma\in(1,2),\quad -g(t),\ h(t)\approx t\ln t\ \textit{if}\ \gamma=2.$$

Inspired by the above interesting results, attention is paid to the following four aspects:

(i) When spreading happens for (1.1), we give more accurate longtime behaviours of solution component u rather than that of spreading case mentioned above. Particularly, if $(\mathbf{J}^\gamma)$ holds with $\gamma\in(1,\,2]$, then some sharp estimates on solution component u, which are closely related to the behaviours of kernel function near infinity, are obtained.

(ii) Assume that (**J1**) holds. Choose a $\mu_j>0$ as the parameter and fix other μ_i. The limiting profile of solution pair $(c_0,\,\Phi_0)$ of system (1.4)–(1.5) as $\mu_j\to\infty$ is derived.

(iii) We obtain the dynamical properties of (1.3) with initial data $U(0,\,x)$, namely, if (**J2**) holds, then C_* is the asymptotic spreading speed of (1.3); if (**J2**) does not hold, then accelerated spreading happens for (1.3). Moreover, if $(\mathbf{J}^\gamma)$ holds with $\gamma\in(1,\,2]$, which implies that the accelerated spreading occurs, then more accurate longtime behaviours are obtained.

(iv) Choose a $\mu_j>0$ as the parameter and fix other μ_i. It is proved that the limiting problem of (1.1) is problem (1.3) as $\mu_j\to\infty$.

Now let's introduce our first main result.

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THEOREM 1.1. *Let $(u,\, g,\, h)$ be the unique solution of (1.1) and spreading happens. Then*

$$\begin{cases} \lim\limits_{t\to\infty} \max\limits_{|x|\leqslant ct} |u(t,x)-\tilde{u}| = 0 \quad \text{for any } c\in(0,c_0) \quad \text{if } (\mathbf{J1}) \text{ holds}, \\ \lim\limits_{t\to\infty} \max\limits_{|x|\leqslant ct} |u(t,x)-\tilde{u}| = 0 \quad \text{for any } c>0 \quad \text{if } (\mathbf{J1}) \text{ does not hold}, \end{cases}$$

where c_0 is uniquely determined by system (1.4)–(1.5).

REMARK 1.2. From Theorems B and 1.1, one easily obtains that for any $\lambda \in (0,\, 1)$ and $i = 1,\, 2,\, \cdots,\, m$,

$$\begin{cases} \lim\limits_{t\to\infty} \dfrac{\min\{x>0: u_i(t,x)=\lambda\tilde{u}_i\}}{t} = \lim\limits_{t\to\infty} \dfrac{\max\{x<0: u_i(t,x)=\lambda\tilde{u}_i\}}{t} \\ \quad = c_0 \ \text{ if } (\mathbf{J1}) \text{ holds}, \\ \lim\limits_{t\to\infty} \dfrac{\min\{x>0: u_i(t,x)=\lambda\tilde{u}_i\}}{t} = \lim\limits_{t\to\infty} \dfrac{\max\{x<0: u_i(t,x)=\lambda\tilde{u}_i\}}{t} \\ \quad = \infty \ \text{ if } (\mathbf{J1}) \text{ does not hold}, \end{cases}$$

where c_0 is the same as in Theorem 1.1.

REMARK 1.3. By Theorems B and 1.1 we know that if one of J_i with $\mu_i > 0$ violates

$$\int_0^\infty xJ_i(x)\mathrm{d}x < \infty,$$

then the accelerated spreading happens, which means that the species u_i will accelerate the propagation of other species. This phenomenon is also captured by Xu *et al.* [**33**] for the Cauchy problem, and is called the transferability of acceleration propagation.

Before giving our next main result, we need an additional assumption on f, i.e.,
(**f5**) For each $1 \leqslant i \leqslant m$, $\sum_{j=1}^m \partial_j f_i(\mathbf{0})\tilde{u}_j > 0$, $\sum_{j=1}^m \partial_j f_i(\tilde{u})\tilde{u}_j < 0$ and $f_i(\eta\tilde{u}) > 0$ for $\eta \in (0,\, 1)$.

THEOREM 1.4. *Assume that* (**f5**) *holds and* ($\mathbf{J}^\gamma$) *holds with $\gamma \in (1,\, 2]$. Let $(u,\, g,\, h)$ be a solution of (1.1) and spreading happens. Then*

$$\begin{cases} \lim\limits_{t\to\infty} \max\limits_{|x|\leqslant s(t)} |u(t,x)-\tilde{u}| = 0 \ \text{for any } s(t)=o(t^{\frac{1}{\gamma-1}}) \ \text{if } \gamma\in(1,2), \\ \lim\limits_{t\to\infty} \max\limits_{|x|\leqslant s(t)} |u(t,x)-\tilde{u}| = 0 \ \text{for any } \ s(t)=o(t\ln t) \ \text{if } \gamma=2. \end{cases}$$

REMARK 1.5. We mention that in contrast to Theorem C that deals with the estimates of free boundaries $g(t)$ and $h(t)$, Theorem 1.4 focuses on the estimates of solution component $u(t,\, x)$. Therefore, the lower solutions in the proof of Theorem 1.4 are different from those in the proof of Theorem C, which leads to that (**f5**) is crucial to estimate (2.7) and necessary for our arguments, while not needed for Theorem C.

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REMARK 1.6. By Theorem C and the construction of lower solutions in the proof of Theorem 1.4, we know that the level set of the solution component u of (1.1) has a similar longtime behaviour with the free boundaries $g(t)$ and $h(t)$. More precisely, for every $\lambda \in (0, 1)$ and $i = 1, 2, \cdots, m$, we have

$$\begin{cases} -\max\{x<0: u_i(t,x)=\lambda\tilde{u}_i\}, \ \ \min\{x>0: u_i(t,x)=\lambda\tilde{u}_i\} \\ \quad \approx t^{\frac{1}{\gamma-1}} \ \text{ if } \ (\mathbf{J}^\gamma) \text{ holds} \quad \text{with} \ \ \gamma\in(1,2), \\ -\max\{x<0: u_i(t,x)=\lambda\tilde{u}_i\}, \ \ \min\{x>0: u_i(t,x)=\lambda\tilde{u}_i\} \\ \quad \approx t\ln t \ \text{ if } \ (\mathbf{J}^\gamma) \text{ holds} \quad \text{with} \ \ \gamma=2. \end{cases}$$

REMARK 1.7. It can be seen from Theorems B, 1.1 and 1.4, [**31**, Theorem 3.15] and [**36**, Theorem 1.2] that free boundary problem with nonlocal diffusion has richer dynamics than its counterpart with random diffusion. This phenomenon also appears for the corresponding Cauchy problem. The reason is that the kernel function plays an important role in studying the dynamics of nonlocal diffusion problem, and the accelerated spreading may happen if kernel function violates the so-called 'thin-tailed' condition, please see [**3**, **16**, **33**] and the references therein.

Now we assume that (**J1**) holds, and choose a $\mu_j > 0$ as the parameter and fix other μ_i. Denote the unique solution pair of (1.4)–(1.5) by $(c_{\mu_j}, \Phi^{c_{\mu_j}})$ with $\Phi^{c_{\mu_j}} = (\phi_i^{c_{\mu_j}})$. By the monotonicity of $\Phi^{c_{\mu_j}}$, there is a unique $l_{\mu_j} > 0$ such that $\phi_j^{c_{\mu_j}}(-l_{\mu_j}) = \frac{1}{2}\tilde{u}_j$. Define $\hat{\Phi}^{c_{\mu_j}}(x) = \Phi^{c_{\mu_j}}(x - l_{\mu_j})$. Our next result concerns the limit of $(c_{\mu_j}, l_{\mu_j}, \Phi^{c_{\mu_j}}, \hat{\Phi}^{c_{\mu_j}})$ as $\mu_j \to \infty$.

THEOREM 1.8. *If* (**J2**) *holds, then* $c_{\mu_j} \to C_*$, $l_{\mu_j} \to \infty$, $\Phi^{c_{\mu_j}}(x) \to \mathbf{0}$ *and* $\hat{\Phi}^{c_{\mu_j}}(x) \to \Psi(x)$ *as* $\mu_j \to \infty$, *where* (C_*, Ψ) *is the minimal speed solution pair of travelling wave problem* (1.6) *with* $\psi_j(0) = \frac{1}{2}\tilde{u}_j$. *If* (**J2**) *does not hold, then* $c_{\mu_j} \to \infty$ *as* $\mu_j \to \infty$.

For convenience, we define a new function $\hat{u}_0(x)$ by

$$\hat{u}_0(x) = u_0(x) \text{ for } |x| \leqslant h_0, \quad \hat{u}_0(x) = \mathbf{0} \text{ for } |x| > h_0.$$

THEOREM 1.9. *Let* $\hat{u} = \infty$ *in* (ii) *of* (**f1**) *and* U *be a solution of* (1.3) *with* $U(0, x) = \hat{u}_0(x)$. *For* $\lambda \in (0, 1)$, *denote the level set of the component* U_i *by* $E_\lambda^i = \{x \in \mathbb{R} : U_i(t, x) = \lambda\tilde{u}_i\}$, *and define* $x_{i,\lambda}^+ = \sup E_\lambda^i$ *and* $x_{i,\lambda}^- = \inf E_\lambda^i$, $i = 1, \cdots, m$.

(i) *If the condition* (**J2**) *holds,*

$$\lim_{t\to\infty}\frac{|x_{i,\lambda}^\pm|}{t} = C_*, \qquad \lim_{|x|\to\infty} U(t,x) = \mathbf{0} \quad \textit{for any } t \geqslant 0, \tag{1.7}$$

$$\begin{cases} \lim\limits_{t\to\infty}\max\limits_{|x|\leqslant ct}|U(t,x)-\tilde{u}| = 0 & \textit{for any } c\in(0,C_*), \\ \lim\limits_{t\to\infty}\max\limits_{|x|\geqslant ct}|U(t,x)| = 0 & \textit{for any } c > C_*. \end{cases} \tag{1.8}$$

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(ii) *If the condition* (**J2**) *does not hold,*

$$\lim_{t\to\infty}\frac{|x^{\pm}_{i,\lambda}|}{t}=\infty,\quad \lim_{t\to\infty}\max_{|x|\leqslant ct}|U(t,x)-\tilde{u}|=0 \;\; \textit{for any } c>0. \tag{1.9}$$

(iii) *If the conditions* (**f5**) *and* ($\mathbf{J}^{\gamma}$) *hold with* $\gamma\in(1,\,2]$,

$$\begin{cases}\lim\limits_{t\to\infty}\max\limits_{|x|\leqslant s(t)}|U(t,x)-\tilde{u}|=0 \;\textit{for any } s(t)=o(t^{\frac{1}{\gamma-1}}) \;\text{ if }\; \gamma\in(1,2),\\ \lim\limits_{t\to\infty}\max\limits_{|x|\leqslant s(t)}|U(t,x)-\tilde{u}|=0 \;\textit{for any } s(t)=o(t\ln t) \;\text{ if }\; \gamma=2.\end{cases}$$

As before, choose a $\mu_j>0$ as the parameter and fix other μ_i. Our last main result concerns the limiting problem of (1.1) as $\mu_j\to\infty$.

Theorem 1.10. *Problem* (1.3), *with* $U(0,x)=\hat{u}_0(x)$, *is the limiting problem of* (1.1) *as* $\mu_j\to\infty$. *More precisely, denoting the unique solution of* (1.1) *by* $(u_{\mu_j},g_{\mu_j},h_{\mu_j})$ *and letting* $\mu_j\to\infty$, *we have* $u_{\mu_j}(t,x)\to U(t,x)$ *locally uniformly in* $[0,\infty)\times\mathbb{R}$ *and* $-g_{\mu_j}(t),\;h_{\mu_j}(t)\to\infty$ *locally uniformly in* $(0,\infty)$.

This paper is as follows. In § 2, we prove Theorems 1.1 and 1.4. Section 3 is devoted to the proofs of Theorems 1.8, 1.9 and 1.10. In § 4, two epidemic models are taken as examples to illustrate our previous results.

2. Proofs of theorems 1.1 and 1.4

In this section, we will prove Theorems 1.1 and 1.4 by constructing some properly upper and lower solutions.

Proof of Theorem 1.1. Firstly, consider the following ODE system

$$\bar{u}_t=f(\bar{u}),\quad \bar{u}(0)=(\|u_{i0}(x)\|_{C([-h_0,h_0])})\in[\mathbf{0},\hat{u}].$$

It follows from condition (**f4**) and a comparison argument that

$$\limsup_{t\to\infty}u(t,x)\preceq\tilde{u}\text{ uniformly in }\mathbb{R}. \tag{2.1}$$

(i) Assume that (**J1**) holds. Let (c_0,Φ_0) be the unique solution pair of (1.4)–(1.5). For small $\varepsilon>0$ and $\sigma>0$, we define

$$\underline{h}(t)=c_0(1-2\varepsilon)t+\sigma,\;\; \underline{u}(t,x)=(1-\varepsilon)\left[\Phi_0(x-\underline{h}(t))+\Phi_0(-x-\underline{h}(t))-\tilde{u}\right]$$

for $t\geqslant0$ and $|x|\leqslant\underline{h}(t)$. By [10, Lemma 3.4], for small $\varepsilon>0$ there exist suitable T, $\sigma>0$ such that

$$g(t+T)\leqslant-\underline{h}(t),\;\; h(t+T)\geqslant\underline{h}(t),\;\; u(t+T,x)\succeq\underline{u}(t,x)\;\text{ for }\; t\geqslant0,\;|x|\leqslant\underline{h}(t).$$

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On the other hand, direct calculations show that, with $c_0^\varepsilon = c_0(1-3\varepsilon)$,

$$\begin{aligned}\max_{|x|\leqslant c_0^\varepsilon t} |\underline{u}(t,x) - (1-\varepsilon)\tilde{u}| &= (1-\varepsilon)\max_{|x|\leqslant c_0^\varepsilon t} |\Phi_0(x-\underline{h}(t)) + \Phi_0(-x-\underline{h}(t)) - 2\tilde{u}| \\ &\leqslant (1-\varepsilon)\max_{|x|\leqslant c_0^\varepsilon t}\big(|\Phi_0(x-\underline{h}(t)) - \tilde{u}| \\ &\quad + |\Phi_0(-x-\underline{h}(t)) - \tilde{u}|\big) \\ &= 2(1-\varepsilon)|\Phi_0(-c_0\varepsilon t - \sigma) - \tilde{u}| \to 0 \quad \text{as } t\to\infty.\end{aligned}$$

Therefore, $\liminf_{t\to\infty} u(t, x) \succeq (1-\varepsilon)\tilde{u}$ uniformly in $|x| \leqslant c_0(1-3\varepsilon)t$. Then for any $c \in (0, c_0)$, by letting $\varepsilon > 0$ sufficiently small such that $c < c_0(1-3\varepsilon)$, we have $\liminf_{t\to\infty} u(t, x) \succeq (1-\varepsilon)\tilde{u}$ uniformly in $|x| \leqslant ct$. In view of the arbitrariness of $\varepsilon > 0$, $\liminf_{t\to\infty} u(t, x) \succeq \tilde{u}$ uniformly in $|x| \leqslant ct$. This, combined with (2.1), gives our desired result.

(ii) Assume that (**J1**) does not hold. As in the proof of [**10**, Theorem 1.3], for any integer $n \geqslant 1$ and $1 \leqslant i \leqslant m_0$, we define

$$J_i^n(x) = \begin{cases} J_i(x) & \text{if } |x| \leqslant n, \\ \dfrac{n}{|x|}J_i(x) & \text{if } |x| \geqslant n, \end{cases} \qquad \tilde{J}_i^n = \frac{J_i^n(x)}{\|J_i^n\|_{L^1(\mathbb{R})}}, \quad J^n = (J_i^n), \quad \text{and } \tilde{J}^n = (\tilde{J}_i^n)$$

with $J_i^n(x) \equiv \tilde{J}_i^n \equiv 0$ for $m_0 + 1 \leqslant i \leqslant m$. Clearly, the following results about J_i^n and $\tilde{J}_i^n$ hold:

(1) $J_i^n(x) \leqslant J_i(x)$, $|x|J_i^n(x) \leqslant nJ_i(x)$, and for any $\alpha > 0$, there is $c > 0$ depending only on n, α, J_i such that $e^{\alpha|x|}J_i^n(x) \geqslant ce^{\frac{\alpha}{2}|x|}J_i(x)$ for $x \in \mathbb{R}$, which directly implies that $\tilde{J}^n$ satisfies (**J**) and (**J1**), but not (**J2**).

(2) J^n is non-decreasing in n, $\lim_{n\to\infty} J^n = \lim_{n\to\infty} \tilde{J}^n = J$ in $[L^1(\mathbb{R})]^m$ and locally uniformly in $\mathbb{R}$.

Let (u^n, g_n, h_n) be the unique solution of the following problem

$$\begin{cases} u_t^n = D\circ\displaystyle\int_{g_n(t)}^{h^n(t)} J^n(x-y)\circ u^n(t,y)\mathrm{d}y - D\circ u^n + f(u^n), & t>0,\ x\in(g_n(t),h_n(t)),\\ u^n(t,x) = 0, \quad t>0,\ x\notin(g_n(t),h_n(t)), \\ g_n'(t) = -\displaystyle\sum_{i=1}^{m_0}\mu_i\int_{g_n(t)}^{h_n(t)}\int_{-\infty}^{g_n(t)} J_i^n(x-y)u_i^n(t,x)\mathrm{d}y\mathrm{d}x, \quad t>0,\\ h_n'(t) = \displaystyle\sum_{i=1}^{m_0}\mu_i\int_{g_n(t)}^{h_n(t)}\int_{h_n(t)}^{\infty} J_i^n(x-y)u_i^n(t,x)\mathrm{d}y\mathrm{d}x, \quad t>0,\\ u^n(0,x) = u(T,x),\ g_n(T) = g(T), \quad h_n(0) = h(T), \quad x\in[g(T),\ h(T)], \end{cases}$$

where $T > 0$. For any integer $n \geqslant 1$, it follows from [10, Lemma 3.5] that there is a proper $T > 0$ such that

$$g_n(t) \geqslant g(t+T), \quad h_n(t) \leqslant h(t+T), u^n(t,x) \\ \preceq u(t+T,x) \text{ for } t \geqslant 0,\ g_n(t) \leqslant x \leqslant h_n(t).$$

Since f satisfies (**f1**)–(**f3**), the function $f(w) + (D^n - D) \circ w$ still satisfies (**f1**)–(**f3**) with $D^n = (d_i \|J_i^n\|_{L^1(\mathbb{R})})$ and $n \gg 1$. Denote the unique positive root of $f(w) + (D^n - D) \circ w = 0$ by $\tilde{u}^n$. Clearly, $\lim_{n\to\infty} \tilde{u}^n = \tilde{u}$. By [10, Lemmas 3.6 and 3.8], the following problem

$$\begin{cases} D \circ \int_{-\infty}^{0} J^n(x-y) \circ \Phi(y)\mathrm{d}y - D \circ \Phi + c\Phi'(x) + f(\Phi) = 0, \quad -\infty < x < 0, \\ \Phi(-\infty) = \tilde{u}^n, \quad \Phi(0) = \mathbf{0}, \quad c = \sum_{i=1}^{m_0} \mu_i \int_{-\infty}^{0}\int_{0}^{\infty} J_i^n(x-y)\phi_i(x)\mathrm{d}y\mathrm{d}x \end{cases}$$

has a unique solution pair $(c^n,\ \Phi^n)$ and $\lim_{n\to\infty} c^n = \infty$.

As before, for small $\varepsilon > 0$ and $\sigma > 0$, define

$$\underline{h}_n(t) = c^n(1-2\varepsilon)t + \sigma, \quad \underline{u}^n(t,x) = (1-\varepsilon)\left[\Phi^n(x - \underline{h}_n(t)) + \Phi^n(-x - \underline{h}_n(t)) - \tilde{u}^n\right],$$

with $t \geqslant 0$ and $|x| \leqslant \underline{h}_n(t)$. By [10, Lemma 3.7], for small $\varepsilon > 0$ and large n, there exist $\sigma > 0$ and $T > 0$ such that

$$g(t+T) \leqslant -\underline{h}_n(t), \quad h(t+T) \geqslant \underline{h}_n(t), \quad u(t+T,x) \succeq \underline{u}^n(t,x) \text{ for } t \geqslant 0,\ |x| \leqslant \underline{h}_n(t).$$

Similarly, $\liminf_{t\to\infty} u(t,\ x) \succeq \liminf_{t\to\infty} \underline{u}^n(t,\ x) \succeq (1-\varepsilon)\tilde{u}^n$ uniformly in $|x| \leqslant c^n(1-3\varepsilon)t$. Since $\lim_{n\to\infty} c^n = \infty$, for any fixed $c > 0$ there are large $N \gg 1$ and small $\varepsilon_0 > 0$ such that $c < c^n(1-3\varepsilon)$ for $n \geqslant N$ and $\varepsilon \in (0,\ \varepsilon_0)$. Thus $\liminf_{t\to\infty} u(t,\ x) \succeq (1-\varepsilon)\tilde{u}^n$ uniformly in $|x| \leqslant ct$. Letting $n \to \infty$ and $\varepsilon \to 0$, we derive $\liminf_{t\to\infty} u(t,\ x) \succeq \tilde{u}$ uniformly in $|x| \leqslant ct$. Together with (2.1), the desired result is immediately obtained. The proof is ended. □

To prove Theorem 1.4, the following two technical lemmas are crucial, and their proofs can be found in [11] and [12].

LEMMA 2.1 [12, (2.11)]. *Let $P(x)$ satisfy* (**J**) *and $\varphi_l(x) = l - |x|$ with $l > 0$. Then for any $\varepsilon > 0$, there exists $L_\varepsilon > 0$ such that for any $l > L_\varepsilon$,*

$$\int_{-l}^{l} P(x-y)\varphi_l(y)\mathrm{d}y \geqslant (1-\varepsilon)\varphi_l(x) \quad \text{in} \quad [-l, l].$$

LEMMA 2.2 [11, Lemma 6.5]. *Let $P(x)$ satisfy* (**J**) *and $\varphi(x) = \min\left\{1,\ \frac{l_2 - |x|}{l_1}\right\}$ with $l_2 > l_1 > 0$. Then for any $\varepsilon > 0$, there is $L_\varepsilon > 0$ such that for any $l_2 > l_1 > L_\varepsilon$ and $l_2 - l_1 > L_\varepsilon$,*

$$\int_{-l_2}^{l_2} P(x-y)\varphi(y)\mathrm{d}y \geqslant (1-\varepsilon)\varphi(x) \quad \text{in} \quad [-l_2, l_2].$$

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Proof of Theorem 1.4. Clearly, (2.1) still holds. Thus it remains to show the lower limits of u. The discussion will be divided into two steps.

Step 1: In this step, we deal with the case $1<\gamma<2$, and prove

$$\liminf_{t\to\infty} u(t,x)\succeq \tilde{u} \text{ uniformly in } [-s(t),s(t)] \text{ for any } s(t)=o(t^{\frac{1}{\gamma-1}}). \tag{2.2}$$

For small $\varepsilon>0$, we define

$$\underline{h}(t)=(\sigma t+\theta)^{\frac{1}{\gamma-1}},\quad \underline{u}(t,x)=\tilde{u}_\varepsilon\left(1-\frac{|x|}{\underline{h}(t)}\right)\qquad \text{for } t\geqslant 0,\ |x|\leqslant \underline{h}(t),$$

where $\tilde{u}_\varepsilon=(1-\varepsilon)\tilde{u}$ and σ, $\theta>0$ are to be determined later. Then we are going to verify that there exist proper σ, T and $\theta>0$ such that

$$\begin{cases} \underline{u}_t\preceq D\circ\displaystyle\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\circ\underline{u}(t,y)\mathrm{d}y-D\circ\underline{u}+f(\underline{u}), & t>0,\ |x|<\underline{h}(t),\\ \underline{u}(t,\pm\underline{h}(t))\preceq \mathbf{0}, & t>0,\\ \underline{h}'(t)\leqslant \displaystyle\sum_{i=1}^{m_0}\mu_i\int_{-\underline{h}(t)}^{\underline{h}(t)}\int_{\underline{h}(t)}^{\infty} J_i(x-y)\underline{u}_i(t,x)\mathrm{d}y\mathrm{d}x, & t>0,\\ -\underline{h}'(t)\geqslant -\displaystyle\sum_{i=1}^{m_0}\mu_i\int_{-\underline{h}(t)}^{\underline{h}(t)}\int_{-\infty}^{-\underline{h}(t)} J_i(x-y)\underline{u}_i(t,x)\mathrm{d}y\mathrm{d}x, & t>0,\\ \underline{h}(0)\leqslant h(T),\ \ \underline{u}(0,x)\preceq u(T,x), & |x|\leqslant \underline{h}(0). \end{cases} \tag{2.3}$$

Once it is done, by the comparison method we have

$$g(t+T)\leqslant -\underline{h}(t),\quad h(t+T)\geqslant \underline{h}(t),\quad u(t+T,x)\succeq \underline{u}(t,x)\quad \text{for}\ \ t\geqslant 0,\ |x|\leqslant \underline{h}(t). \tag{2.4}$$

Moreover, for any $s(t)=o(t^{\frac{1}{\gamma-1}})$, direct computations show

$$\lim_{t\to\infty}\max_{|x|\leqslant s(t)}|\underline{u}(t,x)-(1-\varepsilon)\tilde{u}|=(1-\varepsilon)\lim_{t\to\infty}|\tilde{u}|\frac{s(t)}{\underline{h}(t)}=0,$$

which, together with (2.4) and the arbitrariness of ε, yields (2.2).

Now let's verify (2.3). To prove the first inequality in (2.3), we firstly show that there is a constant $\hat{c}>0$ depending only on J such that

$$\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\circ\underline{u}(t,y)\mathrm{d}y\succeq \hat{c}\tilde{u}_\varepsilon \underline{h}^{1-\gamma}(t)\quad \text{for}\ \ t>0,\ |x|\leqslant \underline{h}(t). \tag{2.5}$$

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In fact, for $x \in [0,\, \underline{h}(t)/4]$, we have

$$\begin{aligned}\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\circ \underline{u}(t,y)\mathrm{d}y &= \int_{-\underline{h}(t)-x}^{\underline{h}(t)-x} J(y)\circ \underline{u}(t,x+y)\mathrm{d}y\\ &\succeq \tilde{u}_\varepsilon \circ \int_{\underline{h}(t)/8}^{\underline{h}(t)/4} J(y)\left(1-\frac{x+y}{\underline{h}(t)}\right)\mathrm{d}y\\ &\succeq \tilde{u}_\varepsilon \circ \int_{\underline{h}(t)/8}^{\underline{h}(t)/4} J(y)\frac{y}{\underline{h}(t)}\mathrm{d}y\\ &\succeq \frac{\tilde{u}_\varepsilon c_1}{\underline{h}(t)}\int_{\underline{h}(t)/8}^{\underline{h}(t)/4} y^{1-\gamma}\mathrm{d}y = \tilde{u}_\varepsilon \hat{c}_1 \underline{h}^{1-\gamma}(t).\end{aligned}$$

Similarly, for $x \in [\underline{h}(t)/4,\, \underline{h}(t)]$, we have

$$\begin{aligned}\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\circ \underline{u}(t,y)\mathrm{d}y &\succeq \tilde{u}_\varepsilon \circ \int_{-\underline{h}(t)/4}^{-\underline{h}(t)/8} J(y)\left(1-\frac{x+y}{\underline{h}(t)}\right)\mathrm{d}y\\ &\succeq \tilde{u}_\varepsilon \circ \int_{-\underline{h}(t)/4}^{-\underline{h}(t)/8} J(y)\frac{-y}{\underline{h}(t)}\mathrm{d}y\\ &\succeq \tilde{u}_\varepsilon \hat{c}_1 \underline{h}^{1-\gamma}(t).\end{aligned}$$

Since J_i and $\underline{u}$ are both even in x, estimate (2.5) is obtained.

On the other hand, by lemma 2.1, one can let θ sufficiently large such that

$$\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\circ \underline{u}(t,y)\mathrm{d}y \succeq (1-\varepsilon^2)\underline{u}(t,x) \quad \text{for} \ \ t>0,\ |x|\leqslant \underline{h}(t). \tag{2.6}$$

By the assumptions on f, one easily shows that there is a $\boldsymbol{C}\succ 0$ such that $f(\eta\tilde{u})\succeq \min\{\eta,\, 1-\eta\}\boldsymbol{C}$ for any $\eta\in[0,\,1]$. Hence there is a positive constant $\bar{c}$ depending only on f such that

$$f\left((1-\varepsilon)\left(1-\frac{|x|}{\underline{h}(t)}\right)\tilde{u}\right)\succeq \left(1-\frac{|x|}{\underline{h}(t)}\right)f((1-\varepsilon)\tilde{u})\succeq \left(1-\frac{|x|}{\underline{h}(t)}\right)\varepsilon\boldsymbol{C}\succeq \bar{c}\varepsilon\underline{u}. \tag{2.7}$$

Applying (2.5)–(2.7) we arrive at

$$\begin{aligned}&D\circ\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\circ \underline{u}(t,y)\mathrm{d}y - D\circ\underline{u} + f(\underline{u})\\ &= \frac{\bar{c}\varepsilon\mathbf{1}}{2}\circ\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\circ \underline{u}(t,y)\mathrm{d}y\\ &+\left(D-\frac{\bar{c}\varepsilon\mathbf{1}}{2}\right)\circ\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\circ \underline{u}(t,y)\mathrm{d}y - D\circ\underline{u}+f(\underline{u})\end{aligned}$$

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$$\begin{aligned}
&\succeq \frac{\bar c\varepsilon\mathbf{1}}{2}\circ\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\circ\underline{u}(t,y)\mathrm{d}y+(1-\varepsilon^2)\left(D-\frac{\bar c\varepsilon\mathbf{1}}{2}\right)\circ\underline{u}(t,x)-D\circ\underline{u}+\bar c\varepsilon\underline{u}\\
&\succeq \frac{\bar c\varepsilon\mathbf{1}}{2}\circ\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y)\circ\underline{u}(t,y)\mathrm{d}y\\
&\succeq \frac{\bar c\varepsilon}{2}\tilde u_\varepsilon\hat c_1\underline{h}^{1-\gamma}(t)\succeq\frac{\sigma\underline{h}^{1-\gamma}(t)}{\gamma-1}\tilde u_\varepsilon\succeq\underline{u}_t
\end{aligned}$$

provided that ε and σ are suitably small. So the first inequality in (2.3) holds.

The second inequality in (2.3) is obvious. Now we show the third one in (2.3). Simple calculations yield, with $\tilde u_{i\varepsilon}=(1-\varepsilon)\tilde u_i$,

$$\begin{aligned}
\int_{-\underline{h}(t)}^{\underline{h}(t)}\int_{\underline{h}(t)}^{\infty} J_i(x-y)\underline{u}_i(t,x)\mathrm{d}y\mathrm{d}x &= \tilde u_{i\varepsilon}\int_{-\underline{h}(t)}^{\underline{h}(t)}\int_{\underline{h}(t)}^{\infty} J_i(x-y)\left(1-\frac{|x|}{\underline{h}(t)}\right)\mathrm{d}y\mathrm{d}x\\
&\geqslant \tilde u_{i\varepsilon}\int_{0}^{\underline{h}(t)}\int_{\underline{h}(t)}^{\infty} J_i(x-y)\left(1-\frac{x}{\underline{h}(t)}\right)\mathrm{d}y\mathrm{d}x\\
&= \frac{\tilde u_{i\varepsilon}}{\underline{h}(t)}\int_{-\underline{h}(t)}^{0}\int_{0}^{\infty} J_i(x-y)(-x)\mathrm{d}y\mathrm{d}x\\
&= \frac{\tilde u_{i\varepsilon}}{\underline{h}(t)}\int_{0}^{\underline{h}(t)}\int_{x}^{\infty} J_i(y)x\mathrm{d}y\mathrm{d}x\\
&= \frac{\tilde u_{i\varepsilon}}{\underline{h}(t)}\left(\int_{0}^{\underline{h}(t)}\int_{0}^{y}+\int_{\underline{h}(t)}^{\infty}\int_{0}^{\underline{h}(t)}\right) J_i(y)x\mathrm{d}x\mathrm{d}y\\
&\geqslant \frac{\tilde u_{i\varepsilon}}{2\underline{h}(t)}\int_{0}^{\underline{h}(t)} J_i(y)y^2\mathrm{d}y\\
&\geqslant \frac{c_1\tilde u_{i\varepsilon}}{2\underline{h}(t)}\int_{\underline{h}(t)/2}^{\underline{h}(t)} y^{2-\gamma}\mathrm{d}y\\
&\geqslant \tilde c_1(\sigma t+\theta)^{(2-\gamma)/(\gamma-1)},
\end{aligned}$$

which indicates the third inequality in (2.3). Since J_i and $\underline{u}$ are both symmetric about x, the fourth inequality of (2.3) also holds.

For σ, θ and ε chosen as above, since spreading happens, there exists $T>0$ such that

$$-\underline{h}(0)\geqslant g(T),\quad \underline{h}(0)\leqslant h(T),\quad \underline{u}(0,x)\preceq\tilde u_\varepsilon\preceq u(T,x)\quad \text{for } |x|\leqslant\underline{h}(0).$$

Therefore, (2.3) holds. Step 1 is finished.

Step 2: We now deal with the case $\gamma=2$ and prove

$$\liminf_{t\to\infty} u(t,x)\succeq\tilde u \text{ uniformly in } [-s(t),s(t)]\ \text{ for any } s(t)=o(t\ln t). \tag{2.8}$$

For small $\varepsilon>0$, define

$$\underline{h}(t)=\sigma(t+\theta)\ln(t+\theta),\quad \underline{u}(t,x)=\tilde u_\varepsilon\min\left\{1,\ \frac{\underline{h}(t)-|x|}{(t+\theta)^{\frac{1}{2}}}\right\}\qquad \text{for } t\geqslant 0,\ |x|\leqslant\underline{h}(t),$$

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where $\tilde{u}_\varepsilon = (1-\varepsilon)\tilde{u} := (\tilde{u}_{i\varepsilon})$ and σ, $\theta > 0$ to be determined later. Now we are ready to show

$$\begin{cases} \underline{u}_t \preceq D \circ \int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y) \circ \underline{u}(t,y)\mathrm{d}y - D \circ \underline{u} + f(\underline{u}), & t>0,\ |x|<\underline{h}(t),\ x \neq \underline{h}(t)-(t+\theta)^{\frac{1}{2}},\\ \underline{u}(t, \pm\underline{h}(t)) \preceq \mathbf{0}, & t>0,\\ \underline{h}'(t) \leqslant \sum_{i=1}^{m_0} \mu_i \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{\underline{h}(t)}^{\infty} J_i(x-y)\underline{u}_i(t,x)\mathrm{d}y\mathrm{d}x, & t>0,\\ -\underline{h}'(t) \geqslant -\sum_{i=1}^{m_0} \mu_i \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{-\infty}^{-\underline{h}(t)} J_i(x-y)\underline{u}_i(t,x)\mathrm{d}y\mathrm{d}x, & t>0,\\ \underline{h}(0) \leqslant h(T),\ \ \underline{u}(0,x) \preceq u(T,x), & |x| \leqslant \underline{h}(0). \end{cases} \tag{2.9}$$

Once this is done, the comparison argument yields

$$g(t+T) \leqslant -\underline{h}(t), \quad h(t+T) \geqslant \underline{h}(t), \quad u(t+T,x) \succeq \underline{u}(t,x) \quad \text{for} \ \ t \geqslant 0,\ |x| \leqslant \underline{h}(t),$$

which, similarly to Step 1, implies that (2.8) holds.

Now we verify the first inequality of (2.9). As in Step 1, we first show that there is a positive constant $\tilde{c}_1$, relying only on J, such that

$$\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y) \circ \underline{u}(t,y)\mathrm{d}y \succeq \frac{\tilde{c}_1 \ln(t+\theta)}{4(t+\theta)^{\frac{1}{2}}} \tilde{u}_\varepsilon \text{ for } t>0,\ \underline{h}(t)-(t+\theta)^{\frac{1}{2}} \leqslant |x| \leqslant \underline{h}(t). \tag{2.10}$$

In fact, it is clear that

$$\begin{aligned} \int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y) \circ \underline{u}(t,y)\mathrm{d}y &\succeq \tilde{u}_\varepsilon \circ \int_{\underline{h}(t)-(t+\theta)^{\frac{1}{2}}}^{\underline{h}(t)} J(x-y) \frac{\underline{h}(t)-y}{(t+\theta)^{\frac{1}{2}}}\mathrm{d}y\\ &= \frac{\tilde{u}_\varepsilon}{(t+\theta)^{\frac{1}{2}}} \circ \int_{\underline{h}(t)-(t+\theta)^{\frac{1}{2}}-x}^{\underline{h}(t)-x} J(y)(\underline{h}(t)-x-y)\mathrm{d}y. \end{aligned}$$

And, when $x \in [\underline{h}(t)-(t+\theta)^{\frac{1}{2}},\ \underline{h}(t)-\frac{3}{4}(t+\theta)^{\frac{1}{2}}]$, we have

$$\begin{aligned} \tilde{u}_\varepsilon \circ \int_{\underline{h}(t)-(t+\theta)^{\frac{1}{2}}-x}^{\underline{h}(t)-x} J(y)(\underline{h}(t)-x-y)\mathrm{d}y &\succeq \tilde{u}_\varepsilon \circ \int_{\frac{1}{4}(t+\theta)^{\frac{1}{4}}}^{\frac{1}{4}(t+\theta)^{\frac{1}{2}}} J(y)(\underline{h}(t)-x-y)\mathrm{d}y\\ &\succeq c_1 \tilde{u}_\varepsilon \int_{\frac{1}{4}(t+\theta)^{\frac{1}{4}}}^{\frac{1}{4}(t+\theta)^{\frac{1}{2}}} \frac{1}{y}\mathrm{d}y = \frac{c_1 \tilde{u}_\varepsilon \ln(t+\theta)}{4}. \end{aligned} \tag{2.11}$$

When $x \in [\underline{h}(t)-\frac{3}{4}(t+\theta)^{\frac{1}{2}},\ \underline{h}(t)]$,

$$\begin{aligned} \tilde{u}_\varepsilon \circ \int_{\underline{h}(t)-(t+\theta)^{\frac{1}{2}}-x}^{\underline{h}(t)-x} J(y)(\underline{h}(t)-x-y)\mathrm{d}y &\succeq \tilde{u}_\varepsilon \circ \int_{-\frac{1}{4}(t+\theta)^{\frac{1}{2}}}^{-\frac{1}{4}(t+\theta)^{\frac{1}{4}}} J(y)(-y)\mathrm{d}y\\ &\succeq c_1 \tilde{u}_\varepsilon \int_{-\frac{1}{4}(t+\theta)^{\frac{1}{2}}}^{-\frac{1}{4}(t+\theta)^{\frac{1}{4}}} \frac{-1}{y}\mathrm{d}y, \end{aligned}$$

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and thus (2.11) holds. By the symmetry of J_i and $\underline{u}$, (2.11) also holds for $x \in [-\underline{h}(t), -\underline{h}(t) + (t+\theta)^{\frac{1}{2}}]$. So (2.10) is derived.

Making use of lemma 2.2 with $l_2 = \underline{h}(t)$ and $l_1 = (t+\theta)^{\frac{1}{2}}$ one has

$$\int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y) \circ \underline{u}(t,y)\mathrm{d}y \succeq (1-\varepsilon^2)\underline{u}(t,x) \text{ for } t>0,\ |x| \leqslant \underline{h}(t). \tag{2.12}$$

Similarly to Step 1, there exists a positive constant $\bar{c}$ such that

$$f(\underline{u}) \succeq \bar{c}\varepsilon\underline{u} \text{ for } t>0,\ |x| \leqslant \underline{h}(t). \tag{2.13}$$

From (2.10), (2.12) and (2.13), it follows that, when $\underline{h}(t) - (t+\theta)^{\frac{1}{2}} \leqslant |x| \leqslant \underline{h}(t)$,

$$\begin{aligned} D \circ \int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y) \circ \underline{u}(t,y)\mathrm{d}y - D \circ \underline{u} + f(\underline{u}) &\succeq \frac{\bar{c}\varepsilon\mathbf{1}}{2} \circ \int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y) \circ \underline{u}(t,y)\mathrm{d}y \\ &\succeq \frac{\tilde{C}_1\bar{c}\varepsilon\ln(t+\theta)}{8(t+\theta)^{\frac{1}{2}}}\tilde{u}_\varepsilon \\ &\succeq \frac{2\sigma\ln(t+\theta)}{(t+\theta)^{\frac{1}{2}}}\tilde{u}_\varepsilon \succeq \underline{u}_t \end{aligned}$$

provided that ε and σ are small, and θ is large. Moreover, when $|x| \leqslant \underline{h}(t) - (t+\theta)^{\frac{1}{2}}$,

$$\begin{aligned} D \circ \int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y) \circ \underline{u}(t,y)\mathrm{d}y - D \circ \underline{u} + f(\underline{u}) &\succeq \frac{\bar{c}\varepsilon\mathbf{1}}{2} \circ \int_{-\underline{h}(t)}^{\underline{h}(t)} J(x-y) \circ \underline{u}(t,y)\mathrm{d}y \\ &\succeq \mathbf{0} = \underline{u}_t. \end{aligned}$$

The first inequality of (2.9) is proved. The second inequality of (2.9) is obvious.

Now we deal with the third inequality in (2.9). Careful computations show

$$\begin{aligned} \int_{-\underline{h}(t)}^{\underline{h}(t)} \int_{\underline{h}(t)}^{\infty} J_i(x-y)\underline{u}_i(t,x)\mathrm{d}y\mathrm{d}x &\geqslant \tilde{u}_{i\varepsilon} \int_0^{\underline{h}(t)-(t+\theta)^{\frac{1}{2}}} \int_{\underline{h}(t)}^{\infty} J_i(x-y)\mathrm{d}y\mathrm{d}x \\ &= \tilde{u}_{i\varepsilon} \int_{(t+\theta)^{\frac{1}{2}}}^{\underline{h}(t)} \int_x^{\infty} J_i(y)\mathrm{d}y\mathrm{d}x \\ &\geqslant \tilde{u}_{i\varepsilon} \int_{(t+\theta)^{\frac{1}{2}}}^{\underline{h}(t)} \int_{(t+\theta)^{\frac{1}{2}}}^{y} J_i(y)\mathrm{d}x\mathrm{d}y \\ &\geqslant \tilde{u}_{i\varepsilon} c_1 \int_{(t+\theta)^{\frac{1}{2}}}^{\underline{h}(t)} \frac{y-(t+\theta)^{\frac{1}{2}}}{y^2}\mathrm{d}y \\ &\geqslant c_1\tilde{u}_{i\varepsilon}\left(\ln \underline{h}(t) - \frac{\ln(t+\theta)}{2} + \frac{(t+\theta)^{\frac{1}{2}}}{\underline{h}(t)} - 1\right) \\ &\geqslant c_1\tilde{u}_{i\varepsilon}\left(\ln\sigma + \frac{\ln(t+\theta)}{2} + \ln(\ln(t+\theta)) - 1\right), \end{aligned}$$

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which implies the third inequality in (2.9) provided that θ is large and σ is small. From the symmetry of J_i and $\underline{u}$ on x, it follows that the fourth one in (2.9) also holds.

Since spreading happens for (1.1), for ε, θ and σ chosen as above, we can choose $T > 0$ properly such that $-\underline{h}(0) \geqslant g(T)$, $\underline{h}(0) \leqslant h(T)$ and $\underline{u}(0, x) \preceq \tilde{u}(1-\varepsilon) \preceq u(T, x)$ for $|x| \leqslant \underline{h}(0)$. So (2.9) is proved, and Step 2 is complete. Theorem 1.4 directly follows from (2.1), (2.2) and (2.8). □

3. Proofs of theorems 1.8, 1.9 and 1.10

In this section, we first show the limits of solution of semi-wave problem (1.4)–(1.5), namely, to prove Theorem 1.8.

Proof of Theorem 1.8. We first prove the result when (**J2**) holds. By some comparison considerations, c_{μ_j} is non-decreasing in $\mu_j > 0$. Thanks to $c_{\mu_j} < C_*$, we have $C_\infty = \lim_{\mu_j\to\infty} c_{\mu_j} \leqslant C_*$. We shall show that $\lim_{\mu_j\to\infty} l_{\mu_j} = \infty$. Clearly,

$$0 \leqslant \int_{-\infty}^{0}\int_{0}^{\infty} J_j(x-y)\phi_j^{c_{\mu_j}}(x)\mathrm{d}y\mathrm{d}x \leqslant \frac{c_{\mu_j}}{\mu_j} \leqslant \frac{C_*}{\mu_j}. \tag{3.1}$$

Case 1: J_j does not have compact support. Then for every $n > 0$, by (1.5) one sees

$$\begin{aligned}\frac{C_*}{\mu_j} &\geqslant \int_{-\infty}^{0}\int_{0}^{\infty} J_j(x-y)\phi_j^{c_{\mu_j}}(x)\mathrm{d}y\mathrm{d}x \geqslant \int_{-n-1}^{-n}\phi_j^{c_{\mu_j}}(x)\int_{n+1}^{\infty}J_j(y)\mathrm{d}y\mathrm{d}x\\ &\geqslant \phi_j^{c_{\mu_j}}(-n)\int_{n+1}^{\infty}J_j(y)\mathrm{d}y \geqslant 0,\end{aligned}$$

which implies $\lim_{\mu_j\to\infty}\phi_j^{c_{\mu_j}}(-n) = 0$. Noting that $\phi_j^{c_{\mu_j}}(x)$ is decreasing in $x \leqslant 0$, we have that $\lim_{\mu_j\to\infty}\phi_j^{c_{\mu_j}}(x) = 0$ locally uniformly in $(-\infty, 0]$, which yields $\lim_{\mu_j\to\infty} l_{\mu_j} = \infty$.

Case 2: J_j is compactly supported. Let $[-L, L]$ be the smallest set which contains the support of J_j. Combining (3.1) with the uniform boundedness of ${\phi_j^{c_{\mu_j}}}'(x)$, one easily has that $\lim_{\mu_j\to\infty}\phi_j^{c_{\mu_j}}(x) = 0$ locally uniformly in $[-L, 0]$. Since ${\Phi^{c_{\mu_j}}}'$ is uniformly bounded about $\mu_j > 1$, it follows from a compact argument that there are a sequence $\{\mu_j^n\}$ with $\mu_j^n \to \infty$ and a non-increasing function $\Phi_\infty = (\phi_i^\infty) \in [C((-\infty, 0])]^m$ such that $\Phi^{c_{\mu_j^n}} \to \Phi_\infty$ locally uniformly in $(-\infty, 0]$ as $n \to \infty$. Clearly, $\Phi_\infty \in [\mathbf{0}, \tilde{u}]$. By the dominated convergence theorem,

$$D\circ\int_{-\infty}^{0} J(x-y)\circ\Phi_\infty(t,y)\mathrm{d}y - D\circ\Phi_\infty + c\Phi_\infty'(x) + f(\Phi_\infty) = 0, \quad -\infty < x < 0.$$

Thus,

$$d_j\int_{-\infty}^{0} J_j(x-y)\phi_j^\infty(y)\mathrm{d}y - d_j\phi_j^\infty + c_{\mu_j}{\phi_j^\infty}' + f_j(\phi_1^\infty, \phi_2^\infty, \cdots, \phi_m^\infty) = 0, \quad -\infty < x < 0. \tag{3.2}$$

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Moreover, $\phi_j^\infty(x) = 0$ in $[-L, 0]$. If $\phi_j^\infty(x) \not\equiv 0$ for $x \leqslant 0$, there exists $L_1 \leqslant -L$ such that $\phi_j^\infty(L_1) = 0 < \phi_j^\infty(x)$ in $(-\infty, L_1)$. By (3.2), (**J**) and the assumptions on f, we have

$$0 = d_j \int_{-\infty}^{0} J_j(L_1 - y)\phi_j^\infty(y)\mathrm{d}y + f_j(\underbrace{\phi_1^\infty(L_1), \cdots, 0}_{j}, \cdots, \phi_m^\infty(L_1)) > 0,$$

which implies that $\phi_j^\infty(x) \equiv 0$ for $x \leqslant 0$. Hence, $\lim_{\mu_j \to \infty} l_{\mu_j} = \infty$.

Notice that $\hat{\Phi}^{c_{\mu_j}}$ and $(\hat{\Phi}^{c_{\mu_j}})'$ are uniformly bounded for $\mu_j > 1$ and $x \leqslant -l_{\mu_j}$. By a compact consideration again, for any sequence $\{\mu_j^n\}$ with $\mu_j^n \to \infty$, there exists a subsequence, denoted by itself, such that $\lim_{n\to\infty} \hat{\Phi}^{c_{\mu_j^n}} = \hat{\Phi}^\infty (= (\hat{\phi}_i^\infty))$ locally uniformly in $\mathbb{R}$ for some non-increasing and continuous function $\hat{\Phi}^\infty \in [\mathbf{0}, \tilde{u}]$. Moreover, $\hat{\Phi}^\infty(0) = (\hat{\phi}_1^\infty(0), \cdots, \tilde{u}_j/2, \cdots, \hat{\phi}_m^\infty(0))$. Again using the dominated convergence theorem yields

$$D \circ \int_{\mathbb{R}} J(x-y) \circ \hat{\Phi}^\infty(y)\mathrm{d}y - D \circ \hat{\Phi}^\infty + C_\infty(\hat{\Phi}^\infty)' + f(\hat{\Phi}^\infty) = 0, \quad -\infty < x < \infty.$$

Together with the properties of $\hat{\Phi}^\infty$ and the assumptions on f, we easily derive that $\hat{\Phi}^\infty(-\infty) = \tilde{u}$ and $\hat{\Phi}^\infty(\infty) = \mathbf{0}$. Thus, $(C_\infty, \hat{\Phi}^\infty)$ is a solution pair of (1.6). By Theorem A, C_* is the minimal speed of (1.6). Noticing that $C_\infty \leqslant C_*$, we derive that $C_\infty = C_*$ and $\hat{\Phi}^\infty = \Psi$. Due to the arbitrariness of sequence $\{\mu_j^n\}$, $\hat{\Phi}^{c_{\mu_j}}(x) \to \Psi(x)$ locally uniformly in $\mathbb{R}$ as $\mu_j \to \infty$.

We now show that if (**J2**) does not hold, then $c_{\mu_j} \to \infty$ as $\mu_j \to \infty$. Since c_{μ_j} is non-decreasing in $\mu_j > 0$, $\lim_{\mu_j\to\infty} c_{\mu_j} := C_\infty \in (0, \infty]$. Arguing indirectly, assume $C_\infty \in (0, \infty)$. Then following the similar lines in previous arguments, one can prove that (1.6) has a solution pair (C_∞, Φ_∞) with Φ_∞ non-increasing, $\Phi_\infty(-\infty) = \tilde{u}$ and $\Phi_\infty(\infty) = \mathbf{0}$. This is a contradiction to Theorem A. So $C_\infty = \infty$ and the proof is complete. □

Then we give the proof of Theorem 1.9.

Proof of Theorem 1.9. (i) Since (**J2**) holds, problem (1.6) has a solution pair (C_*, Ψ_{C_*}) with $C_* > 0$ and Ψ_{C_*} non-increasing in $\mathbb{R}$. We first claim that $\Psi_{C_*} = (\psi_i) \succ \mathbf{0}$ and Ψ_{C_*} is monotonically decreasing in $\mathbb{R}$. For $1 \leqslant i \leqslant m_0$ and $l > 0$, define $\tilde{\psi}_i(x) = \psi_i(x - l)$. Applying [**10**, Lemma 2.2] to $\tilde{\psi}$ yields $\psi_i(x) > 0$ for $x < l$. By the arbitrariness of $l > 0$, we have $\psi_i > 0$ in $\mathbb{R}$. For $m_0 + 1 \leqslant i \leqslant m$, it follows from the assumptions on f that $\psi_i' < 0$ in $\mathbb{R}$, which implies $\psi_i > 0$ in $\mathbb{R}$.

To show the monotonicity of Ψ_{C_*}, it remains to verify that ψ_i is decreasing in $\mathbb{R}$ for every $1 \leqslant i \leqslant m_0$. For $\delta > 0$, we define $w(x) = \psi_i(x - \delta) - \psi_i(x)$. Clearly, $w(x) \geqslant 0$ in $\mathbb{R}$ and $w(x) \not\equiv 0$ for $x < 0$. By (1.6), $w(x)$ satisfies

$$d_i \int_{-\infty}^{\infty} J_i(x-y)w(y)\mathrm{d}y - d_i w(x) + C_* w'(x) + q(x)w(x) \leqslant 0, \quad x \in \mathbb{R}.$$

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By [8, Lemma 2.5], $w(x) > 0$ in $x < 0$, and so $\psi_i(x)$ is decreasing in $x < 0$. As before, for any $l > 0$, define $\tilde{\psi}_i(x) = \psi_i(x - l)$. Similarly, we can show that $\psi_i(x)$ is decreasing in $x < l$. Thus, our claim is verified.

Define $\bar{U} = \sigma\Psi_{C_*}(x - C_*t)$ with $\sigma \gg 1$. We then show that $\bar{U}$ is an upper solution of (1.3). In view of the assumptions on $U(0, x)$ and our above analysis, there is $\sigma \gg 1$ such that $\bar{U}(0, x) = \sigma\Psi_{C_*}(x) \succeq U(0, x)$ in $\mathbb{R}$. Moreover, by (**f2**), we have $\sigma f(\Psi_{C_*}(x - C_*t)) \succeq f(\sigma\Psi_{C_*}(x - C_*t))$, and thus

$$\bar{U}_t = -C_*\sigma\Psi'_{C_*}(x - C_*t) \succeq D \circ \int_{-\infty}^{\infty} J(x - y) \circ \bar{U}(t, y)\mathrm{d}y - D \circ \bar{U} + f(\bar{U}).$$

It follows from a comparison argument that $U(t, x) \preceq \bar{U}(t, x)$ for $t \geqslant 0$ and $x \in \mathbb{R}$. Noticing the properties of ψ_i, for any $\lambda \in (0, 1)$ there is a unique $y_* \in \mathbb{R}$ such that $\sigma\psi_i(y_*) = \lambda\tilde{u}_i$. Therefore,

$$x^-_{i,\lambda}(t) \leqslant x^+_{i,\lambda}(t) \leqslant y_* + C_*t. \tag{3.3}$$

Similarly, we can prove that for suitable $\sigma_1 \gg 1$, the function $\sigma_1\Psi_{C_*}(-x - C_*t)$ is also an upper solution of (1.3), and there is a unique $\tilde{y}_* \in \mathbb{R}$ such that $\sigma_1\psi_i(\tilde{y}_*) = \lambda\tilde{u}_i$. Then one easily derives $-\tilde{y}_* - C_*t \leqslant x^-_{i,\lambda}(t) \leqslant x^+_{i,\lambda}(t)$. This, together with (3.3), leads to

$$\limsup_{t\to\infty} \frac{|x^-_{i,\lambda}(t)|}{t} \leqslant \limsup_{t\to\infty} \frac{|x^+_{i,\lambda}(t)|}{t} \leqslant C_*.$$

To prove the first limit of (1.7), it remains to show

$$\liminf_{t\to\infty} \frac{|x^+_{i,\lambda}(t)|}{t} \geqslant \liminf_{t\to\infty} \frac{|x^-_{i,\lambda}(t)|}{t} \geqslant C_*. \tag{3.4}$$

Assume $\mu_1 > 0$, and fix other μ_i. Denote the unique solution of (1.1) by $(u_{\mu_1}, g_{\mu_1}, h_{\mu_1})$ with $u_{\mu_1} = (u^i_{\mu_1})$. By a comparison consideration, $U(t, x) \succeq u_{\mu_1}$ in $[0, \infty) \times [g_{\mu_1}(t), h_{\mu_1}(t)]$ for any $\mu_1 > 0$. Moreover, we can choose μ_1 sufficiently large, say $\mu_1 > \tilde{\mu} > 0$, so that spreading happens for $(u_{\mu_1}, g_{\mu_1}, h_{\mu_1})$ (Similarly to the criteria for spreading and vanishing in [**4**, **12**, **38**], we here assume that spreading happens for (1.1) if μ_1 is large enough). Moreover, from Theorem B, it follows that

$$\lim_{t\to\infty} \frac{-g_{\mu_1}}{t} = \lim_{t\to\infty} \frac{h_{\mu_1}}{t} = c_0.$$

To stress the dependence of c_0 on μ_1, we rewrite c_0 as c_{μ_1}. By Theorem 1.8, $\lim_{\mu_1\to\infty} c_{\mu_1} = C_*$. As $\lambda \in (0, 1)$, we can choose δ small enough such that $\lambda\tilde{u}_i < \tilde{u}_i - \delta$. By virtue of Theorem 1.1, for any $0 < \varepsilon \ll 1$, there is $T > 0$ such that

$$\lambda\tilde{u}_i < \tilde{u}_i - \delta \leqslant u^i_{\mu_1} \leqslant \tilde{u}_i + \delta \text{ for } t \geqslant T, \quad |x| \leqslant (c_{\mu_1} - \varepsilon)t,$$

which obviously implies $x^-_{i,\lambda}(t) \leqslant -(c_{\mu_1} - \varepsilon)t$ and $x^+_{i,\lambda}(t) \geqslant (c_{\mu_1} - \varepsilon)t$. The arbitrariness of ε and μ_1 implies (3.4).

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Additionally, since both $\sigma\Psi_{C_*}(x - C_*t)$ and $\sigma_1\Psi_{C_*}(-x - C_*t)$ are the upper solutions of (1.3), it is easy to prove the second limit of (1.7).

Now we prove (1.8). Let $\bar{u}$ be the solution of

$$\bar{u}_t = f(\bar{u}), \quad \bar{u}(0) = (\|u_{i0}(x)\|_{C([-h_0,h_0])}).$$

By **(f4)** and comparison principle, we derive

$$\limsup_{t\to\infty} U(t,x) \preceq \tilde{u} \text{ uniformly in } \mathbb{R}. \tag{3.5}$$

As before, for the fixed $c \in (0, C_*)$, let $\mu_1 > \tilde{\mu}$ large enough such that $c < c_{\mu_1}$. Using Theorem 1.1 and comparison principle, we see $\liminf_{t\to\infty} U(t, x) \succeq \tilde{u}$ uniformly in $[-ct, ct]$ which, combined with (3.5), yields the desired result.

Moreover, since $\sigma\Psi_{C_*}(x - C_*t) \succeq U(t, x)$ and $\sigma_1\Psi_{C_*}(-x - C_*t) \succeq U(t, x)$ for $t \geqslant 0$ and $x \in \mathbb{R}$, we have that, for any fixed $c > C_*$,

$$\begin{aligned} 0 \leqslant \sup_{|x|\geqslant ct} U_i(t,x) &\leqslant \sup_{x\geqslant ct} U_i(t,x) + \sup_{x\leqslant -ct} U_i(t,x) \\ &\leqslant \sup_{x\geqslant ct} \sigma\psi_i(x - C_*t) + \sup_{x\leqslant -ct} \sigma_1\psi_i(-x - C_*t) \\ &= (\sigma + \sigma_1)\psi_i(ct - C_*t) \to 0 \text{ as } t \to \infty. \end{aligned}$$

Therefore, conclusion (i) is proved.

(ii) We now assume that (**J2**) does not hold, but (**J1**) is true. By Theorem 1.8, $\lim_{\mu_1\to\infty} c_{\mu_1} = \infty$. Thanks to the above arguments, $x^-_{i,\lambda}(t) \leqslant -(c_{\mu_1} - \varepsilon)t$ and $x^+_{i,\lambda}(t) \geqslant (c_{\mu_1} - \varepsilon)t$. Letting $\mu_1 \to \infty$ and $\varepsilon \to 0$, we have $\lim_{t\to\infty} |x^\pm_{i,\lambda}|/t = \infty$, and thus the first limit of (1.9) holds. We then prove the second limit of (1.9). For any $c > 0$, let μ_1 be large enough such that $c_{\mu_1} > c$ and spreading happens for $(u_{\mu_1}, g_{\mu_1}, h_{\mu_1})$. By a comparison argument and Theorem 1.1, we see $\liminf_{t\to\infty} U(t, x) \succeq \tilde{u}$ uniformly in $|x| \leqslant ct$. Together with (3.5), the second limit of (1.9) is obtained.

We now suppose that (**J1**) does not hold. It then follows from Theorem 1.1 that for any $c > 0$, there is $T > 0$ such that

$$\lambda\tilde{u}_i < \tilde{u}_i - \delta \leqslant u^i_{\mu_1} \leqslant \tilde{u}_i + \delta \quad \text{for} \quad t \geqslant T, \quad |x| \leqslant ct,$$

which clearly indicates the first limit of (1.9). As for the second limit of (1.9), by use of Theorem 1.1 and (3.5), we immediately obtain it.

(iii) As above, $U(t, x) \succeq u_{\mu_1}(t, x)$ for $t \geqslant 0$ and $x \in \mathbb{R}$. By Theorem 1.4 and (3.5), we immediately derive the desired result. Thus, the proof is complete. □

Finally, we show the proof of Theorem 1.10.

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Proof of Theorem 1.10. Recall that μ_j is the parameter and the other μ_i are fixed. By the comparison principle, $(u_{\mu_j}, g_{\mu_j}, h_{\mu_j})$ is non-decreasing in $\mu_j > 0$. Hence,

$$\lim_{\mu_j\to\infty} g_{\mu_j}(t) = G(t) \in [-\infty, -h_0], \quad \lim_{\mu_j\to\infty} h_{\mu_j}(t) = H(t) \in [h_0, \infty],$$

and

$$\hat{U}(t,x) = \lim_{\mu_j\to\infty} u_{\mu_j}(t,x) \preceq U(t,x) \text{ for } t>0, \quad G(t)<x<H(t),$$

where $u_{\mu_j} = (u^i_{\mu_j})$, $\hat{U} = (\hat{U}_i)$, and U is the unique solution of (1.3) satisfying $U(0, x) = \hat{u}_0(x)$.

We now claim that $G(t) = -\infty$ and $H(t) = \infty$ for all $t > 0$. We only prove the former since the latter can be handled by similar arguments. Arguing indirectly, assume that there is $t_0 > 0$ such that $G(t_0) > -\infty$. Then $-h_0 \geqslant g_{\mu_j}(t) \geqslant G(t) \geqslant G(t_0) > -\infty$ for $t \in (0, t_0]$. By the condition (**J**), there are small ε_1, $\delta > 0$ such that $J_j(|x|) > \varepsilon_1$ for $|x| \leqslant 2\delta$. Therefore, for $t \in (0, t_0]$,

$$\begin{aligned} g'_{\mu_j}(t) &= -\sum_{i=1}^{m_0} \mu_i \int_{g_{\mu_j}(t)}^{h_{\mu_j}(t)} \int_{-\infty}^{g_{\mu_j}(t)} J_i(x-y) u^i_{\mu_j}(t,x)\mathrm{d}y\mathrm{d}x \\ &\leqslant -\mu_j \int_{g_{\mu_j}(t)}^{h_{\mu_j}(t)} \int_{-\infty}^{g_{\mu_j}(t)} J_j(x-y) u^j_{\mu_j}(t,x)\mathrm{d}y\mathrm{d}x \\ &\leqslant -\mu_j \int_{g_{\mu_j}(t)}^{g_{\mu_j}(t)+\delta} \int_{g_{\mu_j}(t)-\delta}^{g_{\mu_j}(t)} J_j(x-y) u^j_{\mu_j}(t,x)\mathrm{d}y\mathrm{d}x \\ &\leqslant -\mu_j \varepsilon_1 \delta \int_{g_{\mu_j}(t)}^{g_{\mu_j}(t)+\delta} u^j_{\mu_j}(t,x)\mathrm{d}x. \end{aligned}$$

Moreover, for any $(t, x) \in (0, t_0] \times (G(t), G(t)+\delta)$, we let μ_j large enough such that $x \in (g_{\mu_j}(t), 0)$, and thus $\hat{U}_j(t, x) \geqslant u_{\mu_j}(t, x) > 0$. Then by the dominated convergence theorem, we see

$$\lim_{\mu_j\to\infty} \int_{g_{\mu_j}(t)}^{g_{\mu_j}(t)+\delta} u^j_{\mu_j}(t,x)\mathrm{d}x = \int_{G(t)}^{G(t)+\delta} \hat{U}_j(t,x)\mathrm{d}x > 0 \text{ for } \ t \in (0, t_0].$$

Then, as $\mu_j \to \infty$,

$$-\frac{g_{\mu_j}(t_0)+h_0}{\mu_j} \geqslant \varepsilon_1\delta \int_0^{t_0} \int_{g_{\mu_j}(t)}^{g_{\mu_j}(t)+\delta} u^j_{\mu_j}(t,x)\mathrm{d}x\mathrm{d}t \to \varepsilon_1\delta \int_0^{t_0} \int_{G(t)}^{G(t)+\delta} \hat{U}_j(t,x)\mathrm{d}x\mathrm{d}t > 0,$$

which clearly implies $G(t_0) = -\infty$. We get a contradiction. So our claim is true. Combining with the monotonicity of $g_{\mu_j}(t)$ and $h_{\mu_j}(t)$ in t, one easily shows that $-\lim_{\mu_j\to\infty} g_{\mu_j}(t) = \lim_{\mu_j\to\infty} h_{\mu_j}(t) = \infty$ locally uniformly in $(0, \infty)$.

Now we prove that $\hat{U}$ satisfies (1.3). For any $(t, x) \in (0, \infty) \times \mathbb{R}$, there are large $\hat{\mu}_j > 0$ and $t_1 < t$ such that $x \in (g_{\mu_j}(s), h_{\mu_j}(s))$ for all $\mu_j \geqslant \hat{\mu}_j$ and $s \in [t_1, t]$.

Notice $d_i = 0$ and $J_i(x) \equiv 0$ in $\mathbb{R}$ for $i = m_0 + 1, \cdots, m$. Integrating the first m equations in (1.1) over t_1 to $s \in (t_1, t]$ yields

$$u^i_{\mu_j}(s,x) - u^i_{\mu_j}(t_1,x) = \int_{t_1}^{s} \left(d_j \mathcal{L}[u^i_{\mu_j}](\tau,x) + f_i(u^1_{\mu_j}(\tau,x), \cdots, u^m_{\mu_j}(\tau,x)) \right) \mathrm{d}\tau \quad \text{for} \;\; 1 \leqslant i \leqslant m.$$

Letting $\mu_j \to \infty$ and using the dominated convergence theorem, we have

$$\hat{U}_i(s,x) - \hat{U}_i(t_1,x) = \int_{t_1}^{s} \left(d_i \mathcal{L}[\hat{U}_i](\tau,x) + f_i(\hat{U}_1(\tau,x), \cdots, \hat{U}_m(\tau,x)) \right) \mathrm{d}\tau \quad \text{for} \;\; 1 \leqslant i \leqslant m.$$

Then differentiating the above equations by s, one knows that $\hat{U}$ solves (1.3) for any $(t, x) \in (0, \infty) \times \mathbb{R}$. Moreover, since $\mathbf{0} \preceq \hat{U}(t, x) \preceq U(t, x)$ in $(0, \infty) \times \mathbb{R}$, it is easy to see that $\lim_{t \to 0} \hat{U}(t, x) = \mathbf{0}$. By the uniqueness of solution to (1.3), $\hat{U}(t, x) \equiv U(t, x)$ in $[0, \infty) \times \mathbb{R}$. Using Dini's theorem, our desired results directly follow. □

4. Examples

In this section, we introduce two epidemic models to explain our previous conclusions.

Example 4.1. To investigate the spreading of some infectious diseases, such as cholera, Capasso and Maddalena [6] studied the following model:

$$\begin{cases} u_{1t} - d_1 \Delta u_1 = -a u_1 + c u_2 =: f_1(u), & t > 0, \;\; x \in \Omega, \\ u_{2t} - d_2 \Delta u_2 = -b u_2 + G(u_1) =: f_2(u), & t > 0, \;\; x \in \Omega \end{cases} \tag{4.1}$$

with $u = (u_1, u_2)$. Moreover, u_1 and u_2 represent the concentration of the infective agents, such as bacteria, and the infective human population, respectively. Both of them adopt the random diffusion (or called local diffusion) strategy. Positive constants d_1 and d_2 are their respective diffusion rates, $-a u_1$ is the death rate of the infection agents, $c u_2$ is the growth rate of the agents contributed by the infective humans, and $-b u_2$ is the death rate of the infective human population. The function $G(u_1)$ describes the infective rate of humans, and its assumptions will be given later.

Recently, much research for model (4.1) and its variations has been conducted. For example, one can refer to [2, 24] for the free boundary problem with local diffusion, and [36] for the spreading speed. Particularly, Zhao *et al.* [38] recently replaced the local diffusion term of u_1 with the nonlocal diffusion operator like (1.2), and assumed $d_2 = 0$. They found that the dynamics of their model is little different from that of [2], especially for the criteria of spreading and vanishing. Later on, Zhao *et al.* [37] further replaced the term $c u_2$ with $c \int_{g(t)}^{h(t)} K(x - y) u_2(t, y) \mathrm{d}y$, which results in some new difficulties. Very recently, Wang and Du [30] assumed that the infective agents and the infective human population both adopt nonlocal

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diffusion strategy and the term cu_2 is also replaced by $c\int_{g(t)}^{h(t)} K(x-y)u_2(t,\,y)\mathrm{d}y$. Some new techniques were introduced in [30] when they dealt with the related eigenvalue problem.

As in [30], we here assume that the dispersal of the infective human population is approximated by the nonlocal diffusion, and thus propose the following model, with $u = (u_1,\, u_2)$,

$$\begin{cases} u_{1t} = d_1\int_{g(t)}^{h(t)} J_1(x-y)u_1(t,y)\mathrm{d}y - d_1u_1 + f_1(u), & t>0,\ x\in(g(t),h(t)),\\ u_{2t} = d_2\int_{g(t)}^{h(t)} J_2(x-y)u_2(t,y)\mathrm{d}y - d_2u_2 + f_2(u), & t>0,\ x\in(g(t),h(t)),\\ u_i(t,g(t)) = u_i(t,h(t)) = 0, & t>0,\ i=1,2,\\ g'(t) = -\displaystyle\sum_{i=1}^{2}\mu_i\int_{g(t)}^{h(t)}\int_{-\infty}^{g(t)} J_i(x-y)u_i(t,x)\mathrm{d}y\mathrm{d}x, & t>0,\\ h'(t) = \displaystyle\sum_{i=1}^{2}\mu_i\int_{g(t)}^{h(t)}\int_{h(t)}^{\infty} J_i(x-y)u_i(t,x)\mathrm{d}y\mathrm{d}x, & t>0,\\ -g(0)=h(0)=h_0>0;\ u(0,x)=u_0(x)=(u_{10}(x),u_{20}(x)), & |x|\leqslant h_0, \end{cases} \tag{4.2}$$

where J_i satisfy (**J**), d_i, a, b, c are positive constants, $\mu_i \geqslant 0$ and $\mu_1+\mu_2>0$. Function $G(z)$ satisfies

(i) $G\in C^1([0,\,\infty))$, $G(0)=0$, $G'(z)>0$ for $z\geqslant 0$ and $G'(0)>\frac{ab}{c}$;

(ii) $(\frac{G(z)}{z})'<0$ for $z>0$ and $\lim\limits_{t\to\infty}\frac{G(z)}{z}<\frac{ab}{c}$;
Assumptions (i) and (ii) clearly imply that there is a unique positive constant $\tilde{u}_1$ such that $\frac{G(\tilde{u}_1)}{\tilde{u}_1}=\frac{ab}{c}$. Define $\tilde{u}_2=\frac{G(\tilde{u}_1)}{b}$.

(iii) $(\frac{G(\tilde{u}_1)}{\tilde{u}_1})'<-\frac{ab}{c}$.

An example for G is $G(z)=\frac{\alpha z}{1+\beta z}$ with $\alpha>\frac{ab}{c}$ and $\beta>\frac{\alpha c}{ab}$. By the similar methods in [12], we easily get the following spreading–vanishing dichotomy for (4.2): Either

(i) Spreading: $\lim\limits_{t\to\infty}h(t)=-\lim\limits_{t\to\infty}g(t)=\infty$ (necessarily $\mathcal{R}_0=\frac{G'(0)c}{ab}>1$) and $\lim\limits_{t\to\infty}u(t,\,x)=(\tilde{u}_1,\,\tilde{u}_2)=:\tilde{u}$ locally uniformly in $\mathbb{R}$, or

(ii) Vanishing: $\lim\limits_{t\to\infty}(h(t)-g(t))<\infty$ and $\lim\limits_{t\to\infty}\|u(t,\,\cdot)\|_{C([g(t),h(t)])}=0$.

It is easy to show that conditions (**f1**)–(**f4**) hold for f. For model (4.2), the corresponding $m_0=m=2$. Hence, Theorem 1.1 is valid for (4.2).

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THEOREM 4.2. *Let (u, g, h) be a solution of (4.2) and spreading happens. Then*

$$\begin{cases} \lim\limits_{t\to\infty} \max\limits_{|x|\leqslant ct} |u(t,x)-\tilde{u}| = 0 \text{ for any } c \in (0, c_0) \text{ if } (\mathbf{J1}) \text{ holds,} \\ \lim\limits_{t\to\infty} \max\limits_{|x|\leqslant ct} |u(t,x)-\tilde{u}| = 0 \text{ for any } c > 0 \text{ if } (\mathbf{J1}) \text{ does not hold,} \end{cases}$$

where c_0 is uniquely determined by the corresponding semi-wave problem (1.4)–(1.5).

However, one easily checks that f does not satisfy (**f5**). Thus, Theorem 1.4 cannot be directly applied to (4.2). But by using some new lower solution we still can prove that similar results in Theorem 1.4 hold for problem (4.2).

THEOREM 4.3. *Assume that J_i satisfy $(\mathbf{J}^\gamma)$ with $\gamma \in (1, 2]$. Let spreading happens for (4.2). Then*

$$\begin{cases} \lim\limits_{t\to\infty} \max\limits_{|x|\leqslant s(t)} |u(t,x)-\tilde{u}| = 0 \text{ for any } s(t) = o(t^{\frac{1}{\gamma-1}}) \text{ if } \gamma \in (1,2), \\ \lim\limits_{t\to\infty} \max\limits_{|x|\leqslant s(t)} |u(t,x)-\tilde{u}| = 0 \text{ for any } s(t) = o(t\ln t) \text{ if } \gamma = 2. \end{cases}$$

Proof. **Step 1:** Consider problem

$$\begin{cases} \bar{u}_{1t} = -a\bar{u}_1 + c\bar{u}_2, \ \ \bar{u}_{2t} = -b\bar{u}_2 + G(\bar{u}_1), \\ \bar{u}_1(0) = \|u_{10}(x)\|_{C([-h_0,h_0])}, \ \bar{u}_2(0) = \|u_{20}(x)\|_{C([-h_0,h_0])}. \end{cases}$$

It follows from simple phase-plane analysis that $\lim\limits_{t\to\infty} \bar{u}_1(t) = \tilde{u}_1$ and $\lim\limits_{t\to\infty} \bar{u}_2(t) = \tilde{u}_2$. By a comparison argument, $u(t, x) \preceq \bar{u}(t) = (\bar{u}_1(t), \bar{u}_2(t))$ for $t \geqslant 0$ and $x \in \mathbb{R}$. Thus,

$$\limsup_{t\to\infty} u(t,x) \preceq \tilde{u} \quad \text{uniformly in } \mathbb{R}. \tag{4.3}$$

It remains to show the lower limits of u. We will carry it out in two steps.

Step 2: This step concerns the case $\gamma \in (1, 2)$. We will construct a suitably lower solution, which is different from that of Step 2 in proof of Theorem 1.4, to show

$$\liminf_{t\to\infty} u(t,x) \succeq \tilde{u} \quad \text{uniformly in } |x| \leqslant s(t) \quad \text{for any } s(t) = o(t^{\frac{1}{\gamma-1}}). \tag{4.4}$$

For small $\varepsilon > 0$ and $0 < \frac{\alpha_2}{2} < \alpha_1 < \alpha_2 < 1$, define

$$\underline{h}(t) = (\sigma t + \theta)^{\frac{1}{\gamma-1}}, \quad \underline{u}(t,x) = (\tilde{u}_1(1-\varepsilon^{\alpha_1}), \tilde{u}_2(1-\varepsilon^{\alpha_2}))l(t,x) \text{ with } l(t,x) = 1 - \frac{|x|}{\underline{h}(t)}$$

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for $t \geqslant 0$ and $|x| \leqslant \underline{h}(t)$, where σ and $\theta > 0$ are to be determined later. We shall prove that for small $\varepsilon > 0$, there exist proper T, σ and $\theta > 0$ such that

$$\begin{cases} \underline{u}_{1t} \leqslant d_1 \int_{-\underline{h}(t)}^{\underline{h}(t)} J_1(x-y)\underline{u}_1(t,y)\mathrm{d}y - d_1\underline{u}_1 + f_1(\underline{u}), & t>0,\ |x|<\underline{h}(t), \\ \underline{u}_{2t} \leqslant d_2 \int_{-\underline{h}(t)}^{\underline{h}(t)} J_2(x-y)\underline{u}_2(t,y)\mathrm{d}y - d_2\underline{u}_2 + f_2(\underline{u}), & t>0,\ |x|<\underline{h}(t), \\ \underline{u}_i(t,\pm\underline{h}(t)) \leqslant 0, & t>0,\ i=1,2, \\ -\underline{h}(t) \geqslant -\sum_{i=1}^{2}\mu_i \int_{-\underline{h}(t)}^{h(t)}\int_{-\infty}^{-h(t)} J_i(x-y)\underline{u}_i(t,x)\mathrm{d}y\mathrm{d}x, & t>0, \\ \underline{h}'(t) \leqslant \sum_{i=1}^{2}\mu_i \int_{-\underline{h}(t)}^{h(t)}\int_{\underline{h}(t)}^{\infty} J_i(x-y)\underline{u}_i(t,x)\mathrm{d}y\mathrm{d}x, & t>0, \\ -\underline{h}(0) \geqslant g(T),\ \underline{h}(0) \leqslant h(T);\ \ \underline{u}(0,x) \preceq u(T,x), & |x| \leqslant \underline{h}(0). \end{cases} \tag{4.5}$$

As before, once (4.5) is proved, then by the comparison principle and our definition of the lower solution $(\underline{u}, -\underline{h}, \underline{h})$, we easily derive

$$\liminf_{t\to\infty} u(t,x) \succeq (\tilde{u}_1(1-\varepsilon^{\alpha_1}), \tilde{u}_2(1-\varepsilon^{\alpha_2})) \text{ uniformly } \quad \text{in } \ |x| \leqslant s(t),$$

which, combined with the arbitrariness of ε, yields (4.4).

Now we verify (4.5). To prove the first two inequalities in (4.5), similarly to Step 2 in the proof of Theorem 1.4, one can show that there exists $\hat{C} > 0$ such that

$$\int_{-\underline{h}(t)}^{\underline{h}(t)} J_i(x-y)\underline{u}_i(t,y)\mathrm{d}y \geqslant \hat{C}\tilde{u}_i(1-\varepsilon^{\alpha_i})\underline{h}^{1-\gamma}(t) \ \text{ for } \ t>0,\ |x| \leqslant \underline{h}(t). \tag{4.6}$$

The direct computation shows that, when ε is small,

$$c\tilde{u}_2(1-\varepsilon^{\alpha_2}) - (a+\varepsilon)\tilde{u}_1(1-\varepsilon^{\alpha_1}) = \varepsilon^{\alpha_1}\left[a\tilde{u}_1 - c\tilde{u}_2\varepsilon^{\alpha_2-\alpha_1} - \varepsilon^{1-\alpha_1}\tilde{u}_1(1-\varepsilon^{\alpha_1})\right] > 0,$$

which implies

$$c\underline{u}_2(t,x) \geqslant (a+\varepsilon)\underline{u}_1(t,x) \quad \text{ for } t \geqslant 0, \quad |x| \leqslant \underline{h}(t). \tag{4.7}$$

Furthermore, we claim that, for small $\varepsilon > 0$,

$$G(\underline{u}_1(t,x)) \geqslant (b+\varepsilon)\underline{u}_2(t,x) \quad \text{for } t \geqslant 0, \quad |x| \leqslant \underline{h}(t). \tag{4.8}$$

To this end, we first prove that if ε is suitably small,

$$\frac{G(\tilde{u}_1(1-\varepsilon^{\alpha_1})l(t,x))}{\tilde{u}_1(1-\varepsilon^{\alpha_1})l(t,x))} \geqslant \frac{ab}{c}(1+\varepsilon^{\alpha_1}) \quad \text{for } t \geqslant 0, \quad |x| \leqslant \underline{h}(t). \tag{4.9}$$

By the assumptions on G, one sees

$$\frac{G(\tilde{u}_1(1-\varepsilon^{\alpha_1})l(t,x))}{\tilde{u}_1(1-\varepsilon^{\alpha_1})l(t,x)} \geqslant \frac{G(\tilde{u}_1(1-\varepsilon^{\alpha_1}))}{\tilde{u}_1(1-\varepsilon^{\alpha_1})}.$$

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Thus it is sufficient to prove that, for $t \geqslant 0$ and $|x| < \underline{h}(t)$,

$$\frac{G(\tilde{u}_1(1-\varepsilon^{\alpha_1}))}{\tilde{u}_1(1-\varepsilon^{\alpha_1})} \geqslant \frac{ab}{c}(1+\varepsilon^{\alpha_1}).$$

Define

$$\Gamma(\varepsilon) = \frac{G(\tilde{u}_1(1-\varepsilon^{\alpha_1}))}{\tilde{u}_1(1-\varepsilon^{\alpha_1})} - \frac{ab}{c}(1+\varepsilon^{\alpha_1}) \quad \text{for } 0 < \varepsilon \ll 1.$$

Obviously, $\Gamma(0) = 0$. From our assumptions on G, it follows that for $0 < \varepsilon \ll 1$,

$$\begin{aligned}\Gamma'(\varepsilon) &= -\left(\frac{G(\tilde{u}_1(1-\varepsilon^{\alpha_1}))}{\tilde{u}_1(1-\varepsilon^{\alpha_1})}\right)' \alpha_1 \varepsilon^{\alpha_1 - 1} - \frac{ab}{c}\alpha_1\varepsilon^{\alpha_1-1} \\ &= \alpha_1 \varepsilon^{\alpha_1-1}\left[-\left(\frac{G(\tilde{u}_1(1-\varepsilon^{\alpha_1}))}{\tilde{u}_1(1-\varepsilon^{\alpha_1})}\right)' - \frac{ab}{c}\right] > 0.\end{aligned}$$

So (4.9) holds.

Now, we continue to prove (4.8). Obviously, it holds when $x = \pm\underline{h}(t)$. When $|x| < \underline{h}(t)$, we have

$$\begin{aligned}G(\underline{u}_1) - (b+\varepsilon)\underline{u}_2 &= l(t,x)\left(-b\tilde{u}_2(1-\varepsilon^{\alpha_2}) + \tilde{u}_1(1-\varepsilon^{\alpha_1})\frac{G(\tilde{u}_1(1-\varepsilon^{\alpha_1})l(t,x))}{\tilde{u}_1(1-\varepsilon^{\alpha_1})l(t,x)} - \varepsilon\tilde{u}_2(1-\varepsilon^{\alpha_2})\right) \\ &\geqslant l(t,x)\left(-b\tilde{u}_2(1-\varepsilon^{\alpha_2}) + \tilde{u}_1(1-\varepsilon^{\alpha_1})\frac{ab}{c}(1+\varepsilon^{\alpha_1}) - \varepsilon\tilde{u}_2(1-\varepsilon^{\alpha_2})\right) \\ &= l(t,x)\varepsilon^{\alpha_2}\left(b\tilde{u}_2 - \frac{ab\tilde{u}_1}{c}\varepsilon^{2\alpha_1-\alpha_2} - \varepsilon^{1-\alpha_2}\tilde{u}_2(1-\varepsilon^{\alpha_2})\right) > 0\end{aligned}$$

provided that ε is properly small. By (4.6), (4.7) and (4.8), we have that, for small $\sigma > 0$,

$$\begin{aligned}d_1\int_{-\underline{h}(t)}^{\underline{h}(t)} J_1(x-y)\underline{u}_1(t,y)\mathrm{d}y - d_1\underline{u}_1 - a\underline{u}_1 + c\underline{u}_2 &\geqslant \frac{\varepsilon}{2}\int_{-\underline{h}(t)}^{\underline{h}(t)} J_1(x-y)\underline{u}_1(t,y)\mathrm{d}y \\ &\geqslant \hat{C}\tilde{u}_1(1-\varepsilon^{\alpha_1})\underline{h}^{1-\gamma}(t) \\ &\geqslant \frac{\tilde{u}_1(1-\varepsilon^{\alpha_1})\sigma\underline{h}^{1-\gamma}}{\gamma-1} \geqslant \underline{u}_{1t}, \\ d_2\int_{-\underline{h}(t)}^{\underline{h}(t)} J_2(x-y)\underline{u}_2(t,y)\mathrm{d}y - d_2\underline{u}_2 - b\underline{u}_2 + G(\underline{u}_1) &\geqslant \frac{\varepsilon}{2}\int_{-\underline{h}(t)}^{\underline{h}(t)} J_2(x-y)\underline{u}_2(t,y)\mathrm{d}y \\ &\geqslant \hat{C}\tilde{u}_2(1-\varepsilon^{\alpha_2})\underline{h}^{1-\gamma}(t) \\ &\geqslant \frac{\tilde{u}_2(1-\varepsilon^{\alpha_2})\sigma\underline{h}^{1-\gamma}}{\gamma-1} \geqslant \underline{u}_{2t}.\end{aligned}$$

Therefore, the first two inequalities in (4.5) hold.

Clearly, $\underline{u}_i(t, \pm\underline{h}(t)) = 0$ for $t \geqslant 0$.

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In the following, we prove the fourth and fifth inequalities of (4.5). Similarly to the proof of Theorem 1.4, for large $\theta > 0$ and small $\sigma > 0$, one has

$$\begin{aligned}
&\sum_{i=1}^{2}\mu_i\int_{-\underline{h}(t)}^{\underline{h}(t)}\int_{\underline{h}(t)}^{\infty}J_i(x-y)\underline{u}_i(t,x)\mathrm{d}y\mathrm{d}x\\
&\quad=\sum_{i=1}^{2}(1-\varepsilon^{\alpha_i})\mu_i\tilde{u}_i\int_{-\underline{h}(t)}^{\underline{h}(t)}\int_{\underline{h}(t)}^{\infty}J_i(x-y)l(t,x)\mathrm{d}y\mathrm{d}x\\
&\quad\geqslant\sum_{i=1}^{2}\frac{(1-\varepsilon^{\alpha_i})\mu_i\tilde{u}_i}{\underline{h}(t)}\left(\int_0^{\underline{h}(t)}\int_0^{y}+\int_{\underline{h}(t)}^{\infty}\int_0^{\underline{h}(t)}\right)J_i(y)x\mathrm{d}x\mathrm{d}y\\
&\quad\geqslant\sum_{i=1}^{2}\frac{(1-\varepsilon^{\alpha_i})\mu_i\tilde{u}_i}{2\underline{h}(t)}\int_0^{\underline{h}(t)}J_i(y)y^2\mathrm{d}y\\
&\quad\geqslant\sum_{i=1}^{2}\frac{c_1(1-\varepsilon^{\alpha_i})\mu_i\tilde{u}_i}{2\underline{h}(t)}\int_{\underline{h}(t)/2}^{\underline{h}(t)}y^{2-\gamma}\mathrm{d}y\\
&\quad\geqslant\tilde{C}_1(\sigma t+\theta)^{\frac{2-\gamma}{\gamma-1}}\\
&\quad\geqslant\frac{\sigma(\sigma t+\theta)^{\frac{2-\gamma}{\gamma-1}}}{\gamma-1}=\underline{h}'(t).
\end{aligned}$$

So the fourth inequality in (4.5) holds. The fifth one follows from the symmetry of J and $\underline{u}$ on x.

Since spreading happens, for such σ, θ and ε as chosen above, there is a $T > 0$ such that $-\underline{h}(0) \geqslant g(T)$, $\underline{h}(0) \leqslant h(T)$ and $\underline{u}(0, x) \preceq (\tilde{u}_1(1-\varepsilon^{\alpha_1}), \tilde{u}_2(1-\varepsilon^{\alpha_2})) \preceq u(T, x)$ in $[-\underline{h}(0), \underline{h}(0)]$. Hence (4.5) hold, and this step is complete.

Step 3: We now handle the case $\gamma = 2$. That is, we will prove that for any $s(t) = o(t\ln t)$,

$$\liminf_{t\to\infty} u(t,x) \succeq \tilde{u} \text{ uniformly in } |x| \leqslant s(t). \tag{4.10}$$

For fixed $0 < \frac{\alpha_2}{2} < \alpha_1 < \alpha_2 < 1$ and small $\varepsilon > 0$, we define

$$\begin{aligned}
&\underline{h}(t)=\sigma(t+\theta)\ln(t+\theta),\quad \zeta(t,x)=\min\left\{1,\ \frac{\underline{h}(t)-|x|}{(t+\theta)^{1/2}}\right\},\\
&\underline{u}(t,x)=(\tilde{u}_1(1-\varepsilon^{\alpha_1}),\ \tilde{u}_2(1-\varepsilon^{\alpha_2}))\zeta(t,x)
\end{aligned}$$

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for $t \geqslant 0$ and $|x| \leqslant \underline{h}(t)$, with $\sigma,\ \theta > 0$ to be determined later. We will prove that there exist proper T, σ and $\theta > 0$ such that

$$\begin{cases} \underline{u}_{1t} \leqslant d_1 \int_{-\underline{h}(t)}^{\underline{h}(t)} J_1(x-y)\underline{u}_1(t,y)\mathrm{d}y - d_1\underline{u}_1 + f_1(\underline{u}), & t>0,\ |x|<\underline{h}(t),\ |x| \\ & \quad \neq \underline{h}(t)-(t+\theta)^{\frac{1}{2}}, \\ \underline{u}_{2t} \leqslant d_2 \int_{-\underline{h}(t)}^{\underline{h}(t)} J_2(x-y)\underline{u}_2(t,y)\mathrm{d}y - d_2\underline{u}_2 + f_2(\underline{u}), & t>0,\ |x|<\underline{h}(t),\ |x| \\ & \quad \neq \underline{h}(t)-(t+\theta)^{\frac{1}{2}}, \\ \underline{u}_i(t, \pm\underline{h}(t)) \leqslant 0, & t>0,\ i=1,2, \\ -\underline{h}(t) \geqslant -\sum_{i=1}^{2} \mu_i \int_{-\underline{h}(t)}^{h(t)} \int_{-\infty}^{-h(t)} J_i(x-y)\underline{u}_i(t,x)\mathrm{d}y\mathrm{d}x, & t>0, \\ \underline{h}'(t) \leqslant \sum_{i=1}^{2} \mu_i \int_{-\underline{h}(t)}^{h(t)} \int_{\underline{h}(t)}^{\infty} J_i(x-y)\underline{u}_i(t,x)\mathrm{d}y\mathrm{d}x, & t>0, \\ -\underline{h}(0) \geqslant g(T),\ \underline{h}(0) \leqslant h(T);\ \ \underline{u}(0,x) \preceq u(T,x), & |x| \leqslant \underline{h}(0). \end{cases} \tag{4.11}$$

Once (4.11) is derived, we similarly can complete this step. It is not hard to verify that (4.7) and (4.8) are still valid for small $\varepsilon > 0$. Then by following similar lines with the proof of Theorem 1.4, one can obtain (4.11). The details are omitted. Our desired results directly follow from (4.3), (4.4) and (4.10). The proof is complete. □

On the other hand, noticing that the growth rate of infectious agents may be of concave nonlinearity, Hsu and Yang [**17**] recently proposed the following variation of model (4.1)

$$\begin{cases} u_{1t} = d_1\Delta u_1 - au_1 + H(u_2), & t>0,\ x\in\Omega, \\ u_{2t} = d_2\Delta u_2 - bu_2 + G(u_1), & t>0,\ x\in\Omega, \end{cases} \tag{4.12}$$

where $H(u_2)$ and $G(u_1)$ satisfy that $H, G \in C^2([0,\infty))$, $H(0) = G(0) = 0$, $H', G' > 0$ in $[0,\infty)$, $H^{''}, G^{''} > 0$ in $(0,\infty)$, and $G(H(\hat{z})/a) < b\hat{z}$ for some $\hat{z}$. Examples for such H and G are $H(z) = \alpha z/(1+z)$ and $G(z) = \beta\ln(z+1)$ with $\alpha,\ \beta > 0$ and $\alpha\beta > ab$. Based on the above assumptions, it is easy to show that if $0 < H'(0)G'(0)/(ab) \leqslant 1$, the unique nonnegative constant equilibrium is $(0,\ 0)$, and if $H'(0)G'(0)/(ab) > 1$, there are only two nonnegative constant equilibria, i.e., $(0,\ 0)$ and $(\tilde{u}_1,\ \tilde{u}_2) \succ \mathbf{0}$. Some further results about (4.12) can be seen from [**17**] and [**32**].

Motivated by the above works, Nguyen and Vo [**26**] very recently incorporated nonlocal diffusion and free boundary into model (4.12), and thus obtained the

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following problem:

$$
\begin{cases}
u_{1t} = d_1\int_{g(t)}^{h(t)} J_1(x-y)u_1(t,y)\mathrm{d}y - d_1u_1 - au_1 + H(u_2), & t>0,\ x\in(g(t),h(t)),\\
u_{2t} = d_2\int_{g(t)}^{h(t)} J_2(x-y)u_2(t,y)\mathrm{d}y - d_2u_2 - bu_2 + G(u_1), & t>0,\ x\in(g(t),h(t)),\\
u_i(t,g(t)) = u_i(t,h(t)) = 0, & t>0,\ i=1,2,\\
g'(t) = -\sum_{i=1}^{2}\mu_i\int_{g(t)}^{h(t)}\int_{-\infty}^{g(t)} J_i(x-y)u_i(t,x)\mathrm{d}y\mathrm{d}x, & t>0,\\
h'(t) = \sum_{i=1}^{2}\mu_i\int_{g(t)}^{h(t)}\int_{h(t)}^{\infty} J_i(x-y)u_i(t,x)\mathrm{d}y\mathrm{d}x, & t>0,\\
-g(0)=h(0)=h_0>0;\ u_1(0,x)=u_{10}(x),\ u_2(0,x)=u_{20}(x), & |x|\leqslant h_0.
\end{cases}
\tag{4.13}
$$

They proved that problem (4.13) has a unique global solution, and its dynamics are also governed by a spreading–vanishing dichotomy. Now, we give more accurate estimates on longtime behaviours of the solution to (4.13). Assume $H'(0)G'(0)/(ab) > 1$. One can easily check that (**f1**)–(**f5**) hold with $\tilde{u} = (\tilde{u}_1,\, \tilde{u}_2)$ and $\hat{u} = \infty$. Thus, Theorems 1.1 and 1.4 are valid for the solution of (4.13). For convenience of readers, the results are listed as below.

THEOREM 4.4. *Let $(u,\, g,\, h)$, with $u = (u_1,\, u_2)$, be a solution of (4.13) and $m_0 = m = 2$ in conditions (**J1**) and ($\mathbf{J}^\gamma$). If spreading happens, then*

$$
\begin{cases}
\lim\limits_{t\to\infty}\max\limits_{|x|\leqslant ct}|u(t,x)-\tilde{u}| = 0 \quad \textit{for any } c\in(0,c_0) \quad \textit{if } (\mathbf{J1}) \textit{ holds},\\
\lim\limits_{t\to\infty}\max\limits_{|x|\leqslant ct}|u(t,x)-\tilde{u}| = 0 \quad \textit{for any } c>0 \quad \textit{if } (\mathbf{J1}) \textit{ does not hold},\\
\lim\limits_{t\to\infty}\max\limits_{|x|\leqslant s(t)}|u(t,x)-\tilde{u}| = 0 \quad \textit{for any } \ s(t)\\
\quad = o(t^{\frac{1}{\gamma-1}}) \textit{ if } (\mathbf{J}^\gamma) \textit{ holds for } \ \gamma\in(1,2),\\
\lim\limits_{t\to\infty}\max\limits_{|x|\leqslant s(t)}|u(t,x)-\tilde{u}| = 0 \quad \textit{for any } s(t)=o(t\ln t) \textit{ if } (\mathbf{J}^\gamma) \textit{ holds for } \gamma=2,
\end{cases}
$$

where c_0 is uniquely determined by the corresponding semi-wave problem (1.4)–(1.5).

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EXAMPLE 4.5. Our second example is the following West Nile virus model with nonlocal diffusion and free boundaries

$$\begin{cases} H_t = d_1 \displaystyle\int_{g(t)}^{h(t)} J_1(x-y)H(t,y)\mathrm{d}y - d_1 H + a_1(e_1 - H)V - b_1 H, & t>0, \quad x \in (g(t), h(t)), \\ V_t = d_2 \displaystyle\int_{g(t)}^{h(t)} J_2(x-y)V(t,y)\mathrm{d}y - d_2 V + a_2(e_2 - V)H - b_2 V, & t>0, \quad x \in (g(t), h(t)), \\ H(t,x) = V(t,x) = 0, & t>0, \quad x \in \{g(t), h(t)\}, \\ g'(t) = -\mu \displaystyle\int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J_2(x-y)H(t,x)\mathrm{d}y\mathrm{d}x, & t>0, \\ h'(t) = \mu \displaystyle\int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J_2(x-y)H(t,x)\mathrm{d}y\mathrm{d}x, & t>0, \\ -g(0) = h(0) = h_0 > 0; \ (H,\, V)|_{t=0} = (u_{10}(x),\, u_{20}(x)), & |x| \leqslant h_0, \end{cases} \tag{4.14}$$

where J_i satisfy (**J**). Constants d_i, a_i, b_i, e_i and μ are positive, $H(t,\, x)$ and $V(t,\, x)$ are the densities of the infected bird (host) and mosquito (vector) populations, respectively. The biological interpretation of the West Nile virus model can be referred to the literatures [**1**, **18**, **25**, **31**]. Set

$$f_1(H,V) = a_1(e_1 - H)V - b_1 H, \quad f_2(H,V) = a_2(e_2 - V)H - b_2 V.$$

Then the system $f_1(H,\, V) = f_2(H,\, V) = 0$ has a unique positive solution $(\tilde{H},\, \tilde{V})$ with

$$(\tilde{H}, \tilde{V}) = \left(\frac{a_1 a_2 e_1 e_2 - b_1 b_2}{a_1 a_2 e_2 + b_1 a_2}, \quad \frac{a_1 a_2 e_1 e_2 - b_1 b_2}{a_1 a_2 e_1 + a_1 b_2}\right)$$

if and only if $a_1 a_2 e_1 e_2 > b_1 b_2$.

The authors of [**12**] proved that the dynamics of (4.14) are governed by the spreading vanishing dichotomy: Either

(i) Spreading: $\lim_{t\to\infty} h(t) = -\lim_{t\to\infty} g(t) = \infty$ (necessarily $\frac{a_1 a_2 e_1 e_2}{b_1 b_2} > 1$) and $\lim_{t\to\infty} (H(t,\, x),\, V(t,\, x)) = (\tilde{H},\, \tilde{V})$ locally uniformly in $\mathbb{R}$, or

(ii) Vanishing: $\lim_{t\to\infty} (h(t) - g(t)) < \infty$ and $\lim_{t\to\infty} [\|H(t,\, \cdot)\|_{C([g(t),h(t)])} + \|V(t,\, \cdot)\|_{C([g(t),h(t)])}] = 0$.

If $a_1 a_2 e_1 e_2 > b_1 b_2$, then the conditions (**f1**)–(**f5**) hold with $\hat{u} = (e_1,\, e_2)$. The more accurate longtime behaviours of solution to (4.14) can be summarized as follows.

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THEOREM 4.6. *Let* (H, V, g, h) *be a solution of* (4.14) *and* $m_0 = m = 2$ *in conditions* $(\mathbf{J1})$ *and* $(\mathbf{J}^{\gamma})$. *If spreading happens, then*

$$\begin{cases} \lim_{t\to\infty} \max_{|x|\leqslant ct} \left(|H(t,x)-\tilde{H}|+|V(t,x)-\tilde{V}|\right)=0 & \text{for any } c\in(0,c_0) \quad \text{if } (\mathbf{J1}) \text{ holds,} \\ \lim_{t\to\infty} \max_{|x|\leqslant ct} \left(|H(t,x)-\tilde{H}|+|V(t,x)-\tilde{V}|\right)=0 & \text{for any } c>0 \quad \text{if } (\mathbf{J1}) \text{ does not hold,} \\ \lim_{t\to\infty} \max_{|x|\leqslant s(t)} \left(|H(t,x)-\tilde{H}|+|V(t,x)-\tilde{V}|\right)=0 & \text{for any } s(t) \\ \quad = o(t^{\frac{1}{\gamma-1}}) \text{ if } (\mathbf{J}^{\gamma}) \text{ holds with } \gamma\in(1,2), & \\ \lim_{t\to\infty} \max_{|x|\leqslant s(t)} \left(|H(t,x)-\tilde{H}|+|V(t,x)-\tilde{V}|\right)=0 & \text{for any } s(t) \\ \quad = o(t\ln t) \text{ if } (\mathbf{J}^{\gamma}) \text{ holds with } \gamma=2, & \end{cases}$$

where c_0 *is uniquely determined by the corresponding semi-wave problem* (1.4)–(1.5).

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