Can. J. Math. Vol. **50** (2), 1998 pp. 290–311.

NONCOMMUTATIVE DISC ALGEBRAS FOR SEMIGROUPS

KENNETH R. DAVIDSON AND GELU POPESCU

ABSTRACT. We study noncommutative disc algebras associated to the free product of discrete subsemigroups of $\mathbb{R}^+$. These algebras are associated to generalized Cuntz algebras, which are shown to be simple and purely infinite. The nonself-adjoint subalgebras determine the semigroup up to isomorphism. Moreover, we establish a dilation theorem for contractive representations of these semigroups which yields a variant of the von Neumann inequality. These methods are applied to establish a solution to the truncated moment problem in this context.

The starting point of this work is an old result of Douglas [15] establishing that any properly isometric representation of the cone G^+ of a discrete subgroup of the real line generates the same C*-algebra, and that the ordered semigroup may be recovered from this algebra. We establish that the nonself-adjoint algebra generated by such a representation is a function algebra, and that two such algebras are isomorphic if and only if the semigroups are order isomorphic. Mlak [19] establishes a dilation theorem for contractive representations of these semigroups. This yields a variant of the von Neumann inequality. These results suggest thinking of these algebras as generalized disc algebras, and in fact the computation of their maximal ideal space and Shilov boundary make this connection even more compelling.

Motivated by earlier work of the authors [22, 23, 24, 25, 28, 11, 12], we are led to consider non-commutative disc algebras associated to the free product of ordered semigroups. We need to first establish some facts about generalized Cuntz C*-algebras associated to these semigroups. The original Cuntz algebras [8] are associated to the free product of n copies of $\mathbb{N}$. A larger class based on the free product of n copies of a countable dense subsemigroup of $\mathbb{R}^+$ were investigated by Dinh [13, 14] motivated by the work on semigroups of endomorphisms of $B(H)$, especially [4]. The fact that both of these authors use n copies of a common semigroup allows them to make use of the homogeneity. This is not possible in our case, but nevertheless the proof that these algebras are simple and purely infinite follows the lines of Cuntz's original proof closely. This leads us to the conclusion that the C*-algebra generated by any isometric representation of the free product of ordered semigroups satisfying certain orthogonality relations generates the reduced C*-algebra of the semigroup.

Laca and Raeburn [17] have considered a class of C*-algebras associated to subsemigroups of groups with a semilattice structure introduced by Nica [20]. Their methods

Received by the editors April 18, 1997.

The first author was partially supported by an NSERC grant and a Killam Research Fellowship. The second author was partially supported by an NSF grant.

AMS subject classification: 47D25.

also yield the simplicity of the generalized Cuntz algebras that we consider (see [17, Cor. 5.3]). Their methods are different and do not yield directly that the C*-algebras are purely infinite. Apparently this fact has been observed independently by Laca [18]. However there are significant parallels with our methods, as there are with the original argument of Cuntz.

This allows us to establish the uniqueness of the corresponding nonself-adjoint algebra for such a representation. We call these algebras noncommutative disc algebras. We compute the set of characters, and this is an essential tool in establishing a complete isomorphism invariant for these algebras which allows the recovery of the semigroup.

Frahzo [16], Bunce [6] and the second author [22] establish an isometric dilation theorem for an n-tuple of (non-commuting) operators $(T_1, \dots, T_n)$ such that $\sum_{i=1}^n T_i T_i^* \leq I$. Here we establish the analogous result for representations of n ordered semigroups satisfying the corresponding norm condition. This yields an analogue of the von Neumann inequality for our noncommutative disk algebras. This extends related results such as [34, 1, 23, 24, 25, 28].

In the last section, we consider the truncated moment problem for operator-valued functions on these semigroups. The dilation theorem is the necessary tool to give a complete answer in terms of the positivity of an associated Toeplitz kernel. These results are a direct analogy to well known results for the positive integers for measures on the circle [2].

1**. Disc algebras for ordered semigroups.** Let G be any discrete subgroup of the real line $\mathbb{R}$, and let G^+ denote the positive cone $G^+ = G \cap [0, \infty)$. We will be concerned with the disc algebras associated to semigroups of this form. Consider the norm closed algebra $\mathcal{A}(G^+)$ generated by the left regular representation of G^+. This represents G^+ as a semigroup of isometries $\lambda(g)$ on $\ell^2(G^+)$ with orthonormal basis $\{\xi_g : g \in G^+\}$ given by

$$\lambda(g)\xi_h = \xi_{g+h} \quad \text{for } g, h \in G^+.$$

We also consider the reduced C*-algebra $\mathrm{C}^*_r(G^+)$ of this semigroup given by the C*-algebra generated by the left regular representation.

When $G = \mathbb{Z}$ and $G^+ = \mathbb{N}$, it is easy to see that $\lambda(1)$ is the unilateral shift. Thus $\mathcal{A}(\mathbb{N})$ is completely isometrically isomorphic to the disc algebra $\mathrm{A}(\mathbb{D})$ of all continuous functions on the closed unit disc $\mathbb{D}$ which are analytic on the interior. And the enveloping C*-algebra is the Toeplitz algebra on the unit circle $\mathrm{C}^*_r(\mathbb{N}) = \mathcal{T}\big(\mathrm{C}(\mathbb{T})\big)$. This algebra has a unique minimal ideal, namely the compact operators $\mathcal{K}$, which is the commutator ideal and the quotient $\mathrm{C}^*_r(\mathbb{N})/\mathcal{K} \simeq \mathrm{C}(\mathbb{T})$. The reason that the circle appears is that $\mathbb{T} = \hat{\mathbb{Z}}$ is the dual group of $\mathbb{Z}$. Moreover, the disc algebra is a function algebra with maximal ideal space $\mathbb{D}$ and Shilov boundary $\mathbb{T}$. By the Wold decomposition and Coburn's Theorem [7], every isometric representation of $\mathbb{N}$ generates a C*-algebra which is a quotient of $\mathcal{T}\big(\mathrm{C}(\mathbb{T})\big)$. We shall see that these results have direct parallels for all subgroups of $\mathbb{R}$.

The main step in this direction was obtained by Douglas [15].

https://doi.org/10.4153/CJM-1998-015-5 Published online by Cambridge University Press

THEOREM 1.1 (DOUGLAS). *Let G be any discrete subgroup of $\mathbb{R}$. The C*-algebra generated by any representation of G^+ as proper isometries is canonically isomorphic to $\mathrm{C}^*_r(G^+)$. Moreover, the commutator ideal $\mathcal{C}_G$ of this algebra is simple and the quotient $\mathrm{C}^*_r(G^+)/\mathcal{C}_G \simeq \mathrm{C}(\hat{G})$, where $\hat{G}$ is the compact dual group of the discrete group G. The C*-algebra generated by an arbitrary isometric representation of G^+ is either isomorphic to $\mathrm{C}^*_r(G^+)$ or to a quotient of $\mathrm{C}(\hat{G})$.*

It follows that the norm-closed algebra generated by any representation of G^+ as proper isometries is completely isometrically isomorphic to $\mathcal{A}(G^+)$ via its identification with the nonself-adjoint subalgebra of $\mathrm{C}^*_r(G^+)$ determined by the generators.

Douglas also proved that the semigroup can be recovered from its C*-algebra:

THEOREM 1.2 (DOUGLAS). *Let G_1 and G_2 be two discrete subgroups of $\mathbb{R}$. The C*-algebras $\mathrm{C}^*_r(G_1^+)$ and $\mathrm{C}^*_r(G_2^+)$ are isomorphic if and only if G_1 and G_2 are isomorphic as ordered groups.*

We also need to know that a basic result from dilation theory generalizes from $\mathbb{N}$ to arbitrary cones G^+. The famous isometric dilation theorem of Sz. Nagy [31] states that if T is a contraction on a Hilbert space $\mathcal{H}$, then there is an unitary operator U acting on a space $\mathcal{K} = \mathcal{K}^- \oplus \mathcal{H} \oplus \mathcal{K}^+$ such that $P_{\mathcal{H}} U^n|_{\mathcal{H}} \simeq T^n$ for $n \geq 0$. There is an obvious bijective correspondence between contractions T and the contractive representations of $\mathbb{N}$ given by $\rho(n) = T^n$ for $n \geq 0$. The dilation theorem states that every contractive representation of $\mathbb{Z}^+$ dilates to a unitary representation of $\mathbb{Z}$ on a larger space. Mlak [19] established the corresponding result for arbitrary subgroups of $\mathbb{R}$.

THEOREM 1.3 (MLAK). *Let G be any discrete subgroup of $\mathbb{R}$. Then every contractive representation ρ of G^+ can be dilated to a unitary representation σ of G on a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ such that*

$$\rho(g) \simeq P_{\mathcal{H}}\, \sigma(g)|_{\mathcal{H}} \quad \text{for } g \in G^+.$$

Mlak's elegant proof bears repeating here. By a Theorem of Sz. Nagy [32] (see [33, Theorem I.7.1]), it suffices to show that the function defined on G by $T(g) = \rho(g)$ and $T(-g) = \rho(g)^*$ for $g \in G^+$ is positive definite in the sense that the matrix $[T(g_j - g_i)] \geq 0$ for every finite subset $\{g_i\}$ of G. Since this only concerns differences, we may assume the list is given by $0 = g_0 < g_1 < \cdots < g_n$. Mlak observes that if W is the upper triangular operator with entries

$$W_{ij} = T(g_j - g_i) \quad \text{for } 0 \leq i \leq j \leq n$$

and $D = \mathrm{diag}(D_0, \ldots, D_n)$ is the diagonal operator with entries

$$D_0 = I \quad \text{and} \quad D_i = I - T(g_i - g_{i-1})^* T(g_i - g_{i-1}) \quad \text{for } 1 \leq i \leq n.$$

then $[T(g_j - g_i)] = W^* D W$ is positive as needed.

Our first result shows that $\mathcal{A}(G^+)$ is a function algebra, and identifies the maximal ideal space and Shilov boundary.

https://doi.org/10.4153/CJM-1998-015-5 Published online by Cambridge University Press

THEOREM 1.4. *The algebra $\mathcal{A}(G^+)$ associated to the positive cone G^+ of a discrete subgroup of $\mathbb{R}$ is* (*completely isometrically isomorphic to*) *a function algebra. The Shilov boundary is $\hat{G}$ and the maximal ideal space is the generalized disc $C\hat{G}$ given by the cone on $\hat{G}$. The Shilov boundary is connected.*

PROOF. Notice that if ψ is any semicharacter on G^+ (*i.e.*, a contractive homomorphism of G^+ into $\mathbb{C}$), we may consider this as a contractive representation of G^+ and dilate it by Mlak's Theorem to a unitary representation σ of G. This induces a $*$-representation, which we also denote by σ, of $C^*(G) \simeq C(\hat{G})$ which agrees with σ on the generators g (where we considered G imbedded as characters of $\hat{G}$ by sending g to $\hat{g}(\gamma) = \langle g, \gamma\rangle$). Thus we may define a contractive functional on $C^*_r(G^+)$ by factoring through the quotient algebra $C(\hat{G})$, applying σ and then compressing to the range of ψ. Since this map is multiplicative on the generators $\lambda(g)$ for $g \in G^+$, it follows that this functional is multiplicative on the norm-closed algebra $\mathcal{A}(G^+)$ that they generate. Hence ψ is an element of the maximal ideal space of $\mathcal{A}(G^+)$. Conversely, it is clear that the restriction of any multiplicative linear functional of $\mathcal{A}(G^+)$ to the generators is a contractive homomorphism into $\mathbb{C}$. Since the action of ψ on the generators uniquely determines ψ, all multiplicative linear functionals arise in this way.

Certain semicharacters are readily obtained. First if $\gamma \in \hat{G}$ is any character of G, then its restriction to G^+ is a homomorphism into the circle $\mathbb{T}$. Moreover, for any $r \in [0,1]$, we may define a functional

$$\gamma_r(g) = r^g\gamma(g).$$

When $r = 0$, this determines the trivial functional

$$\rho_0(g) = \delta_{0g} = \begin{cases} 1 & \text{if } g = 0 \\ 0 & \text{otherwise} \end{cases}$$

independent of γ. Thus we have identified a set of multiplicative functionals on $\mathcal{A}(G^+)$ corresponding to the cone on $\hat{G}$, namely the set

$$C\hat{G} := \hat{G} \times [0,1]/\hat{G} \times \{0\}.$$

We will show that these are all the semicharacters on G^+, and that the natural topology corresponds to the topology on the maximal ideal space of $\mathcal{A}(G^+)$.

Let ψ be a semicharacter of G^+ other than ρ_0. Then ψ does not vanish on G^+. Hence it factors as $\psi = |\psi|\gamma$ where $\gamma = \psi/|\psi|$. Clearly γ is a character, and extends by setting $\gamma(-g) = \overline{\gamma(g)}$ to a character on G. So consider the absolute value $|\psi|$ which is a semicharacter of G^+ into $[0,1]$. Fix a non-zero element g_0 of G^+ and define $r = |\psi(g_0)|^{1/g_0}$. Notice that $|\psi|$ is monotone decreasing; for otherwise, there would be $g < h$ in G^+ such that $|\psi(g)| < |\psi(h)|$, whence $|\psi(h-g)| > 1$ contrary to fact. If g is any other non-zero element of G^+, then for each integer $n > 0$, there is a unique integer m so that $mg_0 \le ng < (m+1)g_0$. Consequently, by the monotonicity of $|\psi|$,

$$r^{mg_0/n} \ge |\psi(g)| \ge r^{(m+1)g_0/n}.$$

Taking limits as n tends to infinity yields $|\psi(g)| = r^g$. Hence $\psi = \gamma_r$. Thus the maximal ideal space of $\mathcal{A}(G^+)$ equals $C\hat{G}$ at least as a set.

The topology on the maximal ideal space is a compact Hausdorff topology which is the weakest topology such that the functions $\gamma_r \longrightarrow \gamma_r(g)$ are continuous for each $g \in G^+$. It is evident that these functions are continuous on $C\hat{G}$ with the product topology because they are continuous on $\hat{G}$ by definition of the weak topology on $\hat{G}$ and this extends to the cone because the function $f(r) = r^g$ is continuous for each $g \in \mathbb{R}$. Consequently, the map from the maximal ideal space to the cone with the product topology is a continuous bijection of one compact Hausdorff space onto another, and thus is a homeomorphism.

Next, consider the map π from $\mathcal{A}(G^+)$ into $C(\hat{G})$ obtained by restricting the quotient map of $C^*_r(G^+)$ onto $C(\hat{G})$ to $\mathcal{A}(G^+)$. This map is a $*$-homomorphism, and thus is completely contractive. We will show that this map is completely isometric on $\mathcal{A}(G^+)$. The proof is analogous to the integer case for Toeplitz operators. First notice that

$$\lambda(g)^*\lambda(h) = \lambda(g)^*\lambda(g)\lambda(h-g) = \lambda(h-g) \quad \text{for } g \leq h \in G^+,$$

and similarly $\lambda(g)^*\lambda(h) = \lambda(g-h)^*$ when $g > h$. Thus every word consisting of a product of terms $\lambda(g_i)^*$ and $\lambda(h_i)$ reduces to one of the form $\lambda(h)\lambda(g)^*$. Therefore a calculation shows that every element of the commutator ideal can be approximated by something in the span of terms of the form

$$X = \lambda(g_1)\lambda(g_2)^*\big(\lambda(g)\lambda(g)^* - I\big)\lambda(g_3)\lambda(g_4)^*.$$

But it is easy to verify that $X\lambda(h) = 0$ if $h > \max\{g_3, g + g_4\}$. Thus for any X in the commutator ideal and any $\varepsilon > 0$, there is an element $h \in G^+$ so that $\|X\lambda(h)\| < \varepsilon$. It is also clear that the restriction of $\mathcal{A}(G^+)$ to $\lambda(h)\ell^2(G^+)$ is unitarily equivalent to the identity representation, and thus is completely isometric.

Consequently, if $[A_{ij}]$ is a matrix with coefficients in $\mathcal{A}(G^+)$, choose a matrix $[X_{ij}]$ with coefficients in the commutator ideal so that

$$\|[A_{ij} - X_{ij}]\| < \big\|\big[\pi(A_{ij})\big]\big\| + \varepsilon.$$

Then choose $h \in G^+$ so that $\|[X_{ij}\lambda(h)]\| < \varepsilon$. Then

$$\big\|\big[\pi(A_{ij})\big]\big\| > \big\|[A_{ij} - X_{ij}]\,\mathrm{diag}\big(\lambda(h)\big)\big\| - \varepsilon \geq \|[A_{ij}]\| - 2\varepsilon.$$

It follows that π is completely isometric on $\mathcal{A}(G^+)$. Since $\pi\big(\mathcal{A}(G^+)\big)$ is a unital subalgebra of $C(\hat{G})$, it follows that $\mathcal{A}(G^+)$ is a function algebra.

It also follows from this that the Shilov boundary of $\mathcal{A}(G^+)$ is contained in $\hat{G}$. On the other hand, if $\mathcal{O}$ is any open subset of $\hat{G}$, there is a polynomial $X = \sum_{i=1}^n a_i\lambda(g_i)\lambda(h_i)^*$ in the dense subalgebra of $C^*_r(G^+)$ such that $\pi(X)$ has norm one, but $\pi(X)$ has norm less than $1/2$ off of $\mathcal{O}$. Let $h = \max\{h_i\}$, and notice that $A = X\lambda(h)$ belongs to $\mathcal{A}(G^+)$. Then

$$|\pi(A)| = |\pi(X)\hat{h}| = |\pi(X)|,$$

https://doi.org/10.4153/CJM-1998-015-5 Published online by Cambridge University Press

where $\hat{h}(\gamma) = \langle h, \gamma \rangle$ is a character on $\hat{G}$. So $\pi(A)$ also peaks in O. Thus the Shilov boundary of $\mathcal{A}(G^+)$ is all of $\hat{G}$.

The Shilov boundary $\hat{G}$ is connected because G contains no elements of finite order. See [29, Theorem 2.5.6(c)]. ∎

Note that in the case of $G = \mathbb{Z}$, the maximal ideal space is $\mathrm{C}\mathbb{T}$ which is homeomorphic to the unit disc. So the cones $\mathrm{C}\hat{G}$ should be thought of as generalized discs.

An immediate consequence in conjunction with Mlak's Dilation Theorem 1.3 is an analogue of the von Neumann inequality for semigroups of contractions.

COROLLARY 1.5. *Let G^+ be the positive cone of a subgroup of $\mathbb{R}$. If T is a contractive homomorphism from G^+ into $\mathcal{B}(\mathcal{H})$, then for every polynomial $\sum_i a_i \hat{h}_i$ in $\mathbb{C}G^+$, we have*

$$\Big\|\sum_i a_i T(h_i)\Big\| \leq \Big\|\sum_i a_i \hat{h}_i\Big\|_{\mathrm{C}(\hat{G})}.$$

When T is a representation as proper isometries, this is an equality.

Blecher and Paulsen [5] define a universal operator algebra $\mathrm{OA}(S)$ for any semigroup S as the algebra generated by a contractive homomorphism of S with the property that given any contractive representation T of S into $\mathcal{B}(\mathcal{H})$, there is a completely contractive homomorphism of $\mathrm{OA}(S)$ onto this algebra extending the representation T. So an alternative formulation of this corollary is:

COROLLARY 1.6. *Let G^+ be the positive cone of a subgroup of $\mathbb{R}$. Then $\mathrm{OA}(G^+) = \mathcal{A}(G^+)$. Thus if T is a contractive homomorphism from G^+ into $\mathcal{B}(\mathcal{H})$, then there is a (unique) completely contractive homomorphism ρ of $\mathcal{A}(G^+)$ into the algebra $\mathrm{Alg}\big(\{T(g)\}\big)$ such that $\rho(\hat{h}) = T(h)$.*

It is possible to distinguish the ordered group up to isomorphism from $\mathcal{A}(G^+)$ as in the C*-algebra case.

COROLLARY 1.7. *Let G_1 and G_2 be two discrete subgroups of $\mathbb{R}$. The algebras $\mathcal{A}(G_1^+)$ and $\mathcal{A}(G_2^+)$ are isomorphic if and only if G_1 and G_2 are isomorphic as ordered groups.*

PROOF. Let G^+ be the positive cone of an ordered group, and consider $\mathcal{A}(G^+)$. By Theorem 1.4, the Shilov boundary of $\mathcal{A}(G^+)$ is the compact group $\hat{G}$. A result of Bohr and van Kampen (see [15]) shows that $\pi^1(\hat{G})$, the group of connected components of $\mathrm{C}(\hat{G})^{-1}$, is isomorphic to G as an abelian group. Those components which intersect $\mathcal{A}(G^+)$ correspond to the positive cone G^+. Thus G^+ is recovered as an ordered group from $\mathcal{A}(G^+)$. ∎

EXAMPLE 1.8. Consider n positive real numbers α_i which are linearly independent over the rationals. Let $G = \sum_{i=1}^n \mathbb{Z}\alpha_i$. Then $G \simeq \mathbb{Z}^n$ and thus $\hat{G} \simeq \mathbb{T}^n$. However, the spectrum $\mathrm{C}\hat{G}$ imbeds naturally into $\mathbb{C}^n$ differently depending on the α_i chosen. Indeed, suppose that $\gamma = (z_1, \ldots, z_n)$ is a point in the n-torus in $\mathbb{C}^n$ and $0 < r < 1$. The functional γ_r is determined by its action on the generators $\gamma_r(\alpha_i) = r^{\alpha_i} z_i$. Moreover it is easy to see that the map from $C\hat{G}$ to $\mathbb{C}^n$ which takes γ_r to the n-tuple $\big(\gamma_r(\alpha_1),\dots,\gamma_r(\alpha_n)\big)$ is a homeomorphism. Thus we see that the maximal ideal space of $A(G^+)$ is identified with

$$\mathbb{D}(\alpha_1,\dots,\alpha_n) := \big\{(z_1,\dots,z_n) : \alpha_1^{-1}\log|z_1| = \cdots = \alpha_n^{-1}\log|z_n| \le 0\big\}.$$

The function algebra is the subalgebra of $C\big(\mathbb{D}(\alpha_1,\dots,\alpha_n)\big)$ spanned by the monomials $\{z_1^{k_1}z_2^{k_2}\cdots z_n^{k_n} : \sum_{i=1}^n k_i\alpha_i \ge 0\}$.

EXAMPLE 1.9. Say that a subsemigroup G of $\mathbb{R}$ is *commensurable* provided that whenever elements $g_1,\dots,g_n$ of G are given, there is an element $g_0 \in G$ such that each g_i is a non-negative integer multiple of g_0. It is an easy exercise to show that such semigroups are the positive cones of groups G which are order isomorphic to a subgroup of the rationals $\mathbb{Q}$. They are determined by divisibility as follows. Fix a non-zero element g_0 and consider the supernatural number $\mathbf{n} = \prod_{p \text{ prime}} p^{k_p}$ where $k_p \in \mathbb{N}_0 \cup \{\aleph_0\}$ is the supremum of those integers k such that $p^{-k}g_0 \in G$. While this formal product depends on the choice of g_0, it is determined up to a finite change in finitely many exponents.

The dual group of G is the solenoid $S(\mathbf{n})$ which is the projective limit of circles associated to the supernatural number $\mathbf{n}$. Namely, write $\mathbf{n}$ as an infinite product of finite integers q_i, and let ρ_i be the mapping of a circle $\mathbb{T}_i = \mathbb{T}$ onto $\mathbb{T}_{i-1}$ obtained by sending z to z^{q_i}. Then $S(\mathbf{n})$ is the projective limit of this system. It is easy to see that a finite change in the definition of $\mathbf{n}$ does not affect $S(\mathbf{n})$ up to homeomorphism. This is a connected compact space which is some sort of idealized circle. So the maximal ideal space is by analogy some sort of strange disc.

Since the semigroup is the inductive limit of subsemigroups, each of which is isomorphic to $\mathbb{N}$, we are able to see that the algebra $A(G^+)$ is also the inductive limit of subalgebras, each of which is isometrically isomorphic to the disc algebra. Many of the results in this paper can be established more easily for this class of semigroups by invoking the well-known results for $\mathbb{N}$ and taking limits appropriately.

2. **Generalized Cuntz Algebras.** Let G_i^+ for $1 \le i \le n$ be n subsemigroups of $\mathbb{R}$. Consider representations V_i of G_i^+ as semigroups of isometries on a common Hilbert space H such that

$$(\dagger) \qquad \sum_{i=1}^n V_i(g_i)V_i(g_i)^* < I \quad \text{for all } g_i \in G_i^+ \setminus \{0\}.$$

Such a representation (uniquely) determines a representation of the free product semigroup $*_{i=1}^n G_i^+$ by sending an element $\sigma = g_1g_2\cdots g_k$, where $g_j \in G_{i_j}^+$, to the isometry

$$V(\sigma) := V_{i_1}(g_1)V_{i_2}(g_2)\cdots V_{i_k}(g_k).$$

These representations of the free product are characterized by the property that if σ and τ are two elements such that neither is a multiple of the other, then $V(\sigma)$ and $V(\tau)$ are isometries with pairwise orthogonal ranges. Such representations exist, for example the

https://doi.org/10.4153/CJM-1998-015-5 Published online by Cambridge University Press

left regular representation of $*_{i=1}^n G_i^+$ satisfies (†). This will not be separably acting unless each G_i is countable.

We consider the generalized Cuntz algebra $\mathcal{O}(*_{i=1}^n G_i^+)$ to be the universal C*-algebra generated by isometric representations of the semigroups G_i^+ satisfying (†). This is the unique C*-algebra generated by n such isometric representations such that there is a canonical $*$-homomorphism onto the C*-algebra generated by any particular instance of (†). We will show that this C*-algebra is simple when $n \geq 2$ and at least one of the G_i^+ is dense in $\mathbb{R}^+$, and in fact does not depend on the choice of the representations V_i. In particular, it is the C*-algebra generated by the left regular representation of $*_{i=1}^n G_i^+$ and thus is the reduced semigroup C*-algebra $\mathrm{C}_r^*(*_{i=1}^n G_i^+)$.

When $G_i = \mathbb{N}$ for all $1 \leq i \leq n$, condition (†) simplifies to the single condition on the generators $S_i = V_i(1)$:

$$\sum_{i=1}^n S_i S_i^* < I.$$

Such an n-tuple determines a unique C*-algebra known as the Cuntz-Toeplitz algebra [8], which is an extension of the compact operators by the Cuntz algebra $\mathcal{O}_n$. In this case, one obtains the Cuntz algebra by requiring equality in the above relation. When at least one of the semigroups G_i^+ is dense, equality can never occur because the range projections $V_i(g)V_i(g)^*$ are a decreasing function of g.

When $G_i^+ = G^+$ is a common countable, dense subgroup of $\mathbb{R}$, condition (†) simplifies to the conditions

$$\sum_{i=1}^n V_i(g)V_i(g)^* < I \quad \text{for all } g \in G^+.$$

We obtain a C*-algebra considered by Dinh [13]. He shows that this C*-algebra is simple, and thus is independent of the choice of representation of (†). Our proofs will parallel the arguments of Cuntz and Dinh. However, they both make essential use of the homogeneity resulting from taking all the G_i^+ to be equal. We will use similar methods, obtaining an expectation onto an AF subalgebra. However, even when the G_i^+ are equal, our expectation will be onto a smaller algebra as we average over a larger group of automorphisms.

LEMMA 2.1. *Let V_i be isometric representations of semigroups G_i^+ satisfying* (†). *Then* $\mathrm{C}^*\big(\{V_i(g_i) : g_i \in G_i^+,\ 1 \leq i \leq n\}\big)$ *is the closed span of the set* $\{V(\sigma)V(\tau)^* : \sigma, \tau \in *_{i=1}^n G_i^+\}$. *Moreover this set is closed under multiplication.*

The proof is easy and exactly parallels the case of the Cuntz algebra. The details are left to the reader.

The following result should be compared with the Laca-Raeburn approach [17, Section 4]. They also make use of the natural map from the free product onto the (abelian) direct product of the groups in order to construct an expectation. Because our methods are specific to this example, we obtain an explicit description of the image algebra, and thereby can establish that it is AF. This then allows us to use the original argument of Cuntz to show directly that our algebras are purely infinite.

THEOREM 2.2. *There is a faithful expectation* Φ *of the C*-algebra* $\mathcal{O}(*_{i=1}^n G_i^+)$ *associated to the free product of the positive cones of subgroups of* $\mathbb{R}$ *onto an AF C*-subalgebra* $\mathcal{F}(*_{i=1}^n G_i^+)$.

PROOF. Let $V_i(g)$ for $g \in G_i^+$ be the isometric representations of G_i^+ which generate the universal algebra $\mathcal{O}(*_{i=1}^n G_i^+)$. Let V denote the induced representation of $*_{i=1}^n G_i^+$.

For each $\gamma = (\gamma_1, \dots, \gamma_n)$ in $\prod_{i=1}^n \hat{G}_i$, we may define an automorphism α_γ of $\mathcal{O}(*_{i=1}^n G_i^+)$ such that

$$\alpha_\gamma\big(V_i(g_i)\big) = \langle g_i, \gamma_i \rangle V_i(g_i) \quad \text{for } g_i \in G_i^+.$$

Indeed, the maps sending g_i to $\langle g_i, \gamma_i \rangle V_i(g_i)$ are isometric representations of G_i^+ which satisfy (†) and generate the C*-algebra $\mathcal{O}(*_{i=1}^n G_i^+)$. By the universal property of the C*-algebra, there is an automorphism α_γ with the desired property.

The product character γ determines a unique character on $*_{i=1}^n G_i^+$ by sending $\sigma = g_1 g_2 \cdots g_k$, where $g_j \in G_{i_j}^+$, to

$$\gamma(\sigma) = \prod_{j=1}^k \langle g_j, \gamma_{i_j} \rangle.$$

It follows easily that

$$\alpha_\gamma\big(V(\sigma)V(\tau)^*\big) = \gamma(\sigma)\overline{\gamma(\tau)}V(\sigma)V(\tau)^* \quad \text{for } \sigma \in *_{i=1}^n G_i^+.$$

Notice that the map taking γ to $\alpha_\gamma\big(V(\sigma)V(\tau)^*\big)$ is norm continuous. Since these terms span $\mathcal{O}(*_{i=1}^n G_i^+)$, it follows that the map taking γ to $\alpha_\gamma(X)$ is norm continuous for each $X \in \mathcal{O}(*_{i=1}^n G_i^+)$.

Define a map Φ from $\mathcal{O}(*_{i=1}^n G_i^+)$ into itself by

$$\Phi(X) = \int_{\prod_{i=1}^n \hat{G}_i} \alpha_\gamma(X)\, d\gamma$$

where the measure $d\gamma$ is Haar measure on the compact group $\prod_{i=1}^n \hat{G}_i$. It is immediately evident that this map is positive, completely contractive and faithful. A simple calculation using the translation invariance of $d\gamma$ yields the fact that Φ is idempotent. Thus it is an expectation.

We wish to show that the range $\mathcal{F}(*_{i=1}^n G_i^+)$ of Φ is an AF subalgebra of $\mathcal{O}(*_{i=1}^n G_i^+)$. There is a canonical homomorphism of $*_{i=1}^n G_i^+$ onto $\prod_{i=1}^n G_i^+$ which is the identity on each G_i^+. Let the image of an element σ be denoted by $|\sigma|$, which we will call the *length* of σ. A routine calculation shows that

$$\Phi\big(V(\sigma)V(\tau)^*\big) = \begin{cases} V(\sigma)V(\tau)^* & \text{if } |\sigma| = |\tau| \\ 0 & \text{if } |\sigma| \neq |\tau|. \end{cases}$$

For each point $\mathbf{g} = (g_1, \dots, g_n) \in \prod_{i=1}^n G_i^+$, let $\mathcal{K}_{\mathbf{g}}$ denote the space spanned by the set

$$\big\{V(\sigma)V(\tau)^* : |\sigma| = |\tau| = \mathbf{g}\big\}.$$

https://doi.org/10.4153/CJM-1998-015-5 Published online by Cambridge University Press

It is easy to verify that if $V(\sigma_i)V(\tau_i)^* \in \mathcal{K}_{\mathbf{g}_i}$, then

$$V(\sigma_1)V(\tau_1)^*V(\sigma_2)V(\tau_2)^* = \begin{cases} V(\sigma_1\sigma')V(\tau_2)^* & \text{if } \sigma_2 = \tau_1\sigma' \\ V(\sigma_1)V(\tau_2\tau')^* & \text{if } \sigma_2\tau' = \tau_1 \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

In particular, for each $\mathbf{g}$ this spanning set forms a complete set of matrix units for $\mathcal{K}_{\mathbf{g}}$, and hence this forms a C*-algebra isomorphic to the space of compact operators on some Hilbert space (which is separable precisely when each G_i is countable). Moreover, if $\mathbf{g}$ and $\mathbf{h}$ are points in $\prod_{i=1}^n G_i^+$ which is endowed with the product order $\ll$, then

$$\mathcal{K}_{\mathbf{g}}\mathcal{K}_{\mathbf{h}} \subset \begin{cases} \mathcal{K}_{\mathbf{h}} & \text{if } \mathbf{g} \ll \mathbf{h} \\ \mathcal{K}_{\mathbf{g}} & \text{if } \mathbf{g} \gg \mathbf{h} \\ \{0\} & \text{otherwise.} \end{cases}$$

In particular, it follows that $\mathcal{F}(*_{i=1}^n G_i^+)$ is a C*-algebra. Moreover, it is the direct limit of the subalgebras $\mathcal{K}_G := \sum_{\mathbf{g}\in G} \mathcal{K}_{\mathbf{g}}$ as G runs over all finite subsets of $\prod_{i=1}^n G_i^+$. To show that the range is AF, it suffices to show that each of these subalgebras is AF. This may be done by explicitly exhibiting each of these algebras as an inductive limit. Consider finite dimensional subalgebras of the form

$$\sum_{\mathbf{g}\in G} \operatorname{span}\{V(\sigma)V(\tau)^* : \sigma, \tau \in S_{\mathbf{g}}\} \tag{2}$$

where $S_{\mathbf{g}}$ are finite subsets of words of length $\mathbf{g}$. In order for this to be an algebra, it suffices by equation (1) to check that whenever $\mathbf{g} \ll \mathbf{h}$ in G and for some $\sigma \in S_{\mathbf{g}}$ and $\tau \in S_{\mathbf{h}}$ there is an element ρ such that $\tau = \sigma\rho$, then $S_{\mathbf{h}}$ must contain $\sigma'\rho$ for every $\sigma' \in S_{\mathbf{g}}$. It is evident that starting with arbitrary finite subsets $S_{\mathbf{g}}$, they may be enlarged to finite sets $S'_{\mathbf{g}}$ to have this property. So the range of Φ is an AF algebra as required. ∎

We remark that the universal property of $\mathcal{O}(*_{i=1}^n G_i^+)$ is not needed to construct the automorphisms α_γ, and that this can be done for the C*-algebra $\mathrm{C}^*(\{V_i(g)\})$ generated by any representations satisfying (†). The expectation can be constructed in the same manner, and it has the nice property that the expectations commute with the natural homomorphism of $\mathcal{O}(*_{i=1}^n G_i^+)$ onto $\mathrm{C}^*(\{V_i(g)\})$. Moreover the analysis shows that the restriction of this homomorphism to the AF subalgebra is an isomorphism. Since the expectation is faithful, the homomorphism itself is also an isomorphism. Thus the algebra is simple. This approach was used by Dinh. Instead we follow the original approach of Cuntz.

LEMMA 2.3. *Assume that at least one of the semigroups G_i^+ is dense in $\mathbb{R}^+$. For each non-zero positive element X in the dense $*$-subalgebra*

$$\mathcal{A}_\infty = \operatorname{alg\,span}\{V(\sigma)V(\tau)^* : \sigma, \tau \in *_{i=1}^n G_i^+\},$$

there is an element $A \in \mathcal{A}_\infty$ such that

$$A^*XA = I \quad \textit{and} \quad \|A\| = \|\Phi(X)\|^{-1/2}.$$

https://doi.org/10.4153/CJM-1998-015-5 Published online by Cambridge University Press

PROOF. Since Φ is faithful, $X_0 = \Phi(X)$ is strictly positive. For convenience, we may normalize so that $\|X_0\| = 1$. The element X_0 lies in a finite dimensional subalgebra $\mathcal{B}$ of $\mathcal{F}(*_{i=1}^n G_i^+)$ of the type described in equation (2). Thus by the spectral theorem for Hermitian matrices, there is a minimal idempotent P in this subalgebra such that $PX_0P = P$.

The next step is to determine the structure of P. We may write $P = \sum_{\mathbf{g}\in\mathcal{G}} P_g$, where each $P_{\mathbf{g}} \in \operatorname{span}\{V(\sigma)V(\tau)^* : \sigma, \tau \in \mathcal{S}_{\mathbf{g}}\}$. Let $\mathcal{G}_0$ be the subset of $\{\mathbf{g} \in \mathcal{G} : P_{\mathbf{g}} \neq 0\}$ which are minimal in this set with respect to the product order on $\prod_{i=1}^n G_i^+$. Then let $\mathcal{H}$ be the subset of $\mathcal{G}$ consisting of all elements which are not less than or equal to some element of $\mathcal{G}_0$. Then $\mathcal{B}_{\mathcal{H}} := \mathcal{B} \cap \mathcal{K}_{\mathcal{H}}$ is an ideal of $\mathcal{B}$. The image of P under the quotient map will be a minimal projection which is unitarily equivalent to $\sum_{\mathbf{g}\in\mathcal{G}_0} \oplus P_{\mathbf{g}}$. It follows that $\mathcal{G}_0$ is a singleton $\{\mathbf{g}_0\}$. A minimal projection in $\mathcal{B} \cap \mathcal{K}_{\mathbf{g}_0}$ is a rank one projection of the form

$$\sum_{\sigma,\tau\in\mathcal{S}_{\mathbf{g}_0}} a_\sigma \bar{a}_\tau V(\sigma)V(\tau)^* = WW^*$$

where

$$\sum_{\sigma\in\mathcal{S}_{\mathbf{g}_0}} |a_\sigma|^2 = 1 \quad \text{and} \quad W = \sum_{\sigma\in\mathcal{S}_{\mathbf{g}_0}} a_\sigma V(\sigma).$$

Note that W is an isometry in $\mathcal{A}_\infty$. It follows that $P - WW^*$ lies in the ideal $\mathcal{B}_{\mathcal{H}}$. So $W^*PW - I$ lies in the span of terms $V(\sigma)V(\tau)^*$ where $|\sigma| = |\tau| \gg \mathbf{0}$.

Now observe that $W^*PXPW - I$ must lie in the span of terms of the form $V(\sigma)V(\tau)^*$ where $|\sigma| \vee |\tau| \gg \mathbf{0}$. But given any finite set of such terms $Y_i = V(\sigma_i)V(\tau_i)^*$, we shall show that there is an isometry $V \in \mathcal{A}_\infty$ such that $V^*Y_iV = 0$ for all i. Let us write $\sigma_i = s_i\sigma_i'$ and $\tau = t_i\tau_i'$ where $s_i \in G_{m_i}^+ \setminus \{0\}$ and $t_i \in G_{n_i}^+ \setminus \{0\}$. We may suppose that G_1 is dense. Choose an element $g_1 \in G_1^+ \setminus \{0\}$ smaller than $\min\{s_i, t_j : m_i = 1 = n_j\}$ and any $g_2 \in G_2^+ \setminus \{0\}$. Then define $V = V(g_1g_2)$. It is immediate from equation (1) that $V^*Y_iV = 0$ as desired. It now follows that $V^*W^*PXPWV = I$. ∎

THEOREM 2.4. *The C*-algebra $\mathcal{O}(*_{i=1}^n G_i^+)$ associated to the free product of $n \geq 2$ positive cones of subgroups of $\mathbb{R}$ is simple and purely infinite provided that at least one of the semigroups G_i^+ is dense in $\mathbb{R}^+$. In all cases, this C*-algebra is determined by any solution of the relations* (†)*.*

PROOF. We will establish that for each non-zero element X in $\mathcal{O}(*_{i=1}^n G_i^+)$, there are elements A, B in $\mathcal{O}(*_{i=1}^n G_i^+)$ so that $AXB = I$. From this it is immediate that $\mathcal{O}(*_{i=1}^n G_i^+)$ is simple. Moreover, a result of Cuntz [9] (*e.g.* [10, V.5.5]) implies that this condition is equivalent to being purely infinite.

Clearly we may replace X by X^*X, and thus may suppose that X is positive. Since Φ is faithful, it follows that $X_0 = \Phi(X) \neq 0$. We may normalize so that $\|X_0\| = 1$. Now X may be approximated within $1/2$ by an element Y in $\mathcal{A}_\infty$. Thus $\|\Phi(Y)\| > 1/2$. By the previous lemma, there is an element B such that $B^*YB = I$ and $\|B\|^2 < 2$. Consequently $\|B^*XB - I\| < 1$. It follows that $A = B(B^*XB)^{-1/2}$ is defined and satisfies $A^*XA = I$. ∎

The following corollary is immediate provided that at least one of the groups G_i is dense in $\mathbb{R}$. When all are discrete, this is a result of Cuntz.

COROLLARY 2.5. *If V_i are isometric representations of G_i^+ satisfying* (†)*, then* $\mathrm{C}^*\big(\{V_i(g_i) : g_i \in G_i^+,\ 1 \le i \le n\}\big)$ *is canonically ∗-isomorphic to* $\mathcal{O}(*_{i=1}^n G_i^+)$.

Since, in particular, such a representation is the left regular representation of $*_{i=1}^n G_i^+$, we obtain:

COROLLARY 2.6. *If G_i^+ are the positive cones of subgroups of $\mathbb{R}$ for $1 \le i \le n$ and $n \ge 2$, then* $\mathcal{O}(*_{i=1}^n G_i^+) = \mathrm{C}_r^*(*_{i=1}^n G_i^+)$.

3**. Noncommutative Disc Algebras** In [23], the second author introduced the noncommutative disc algebra $\mathcal{A}_n$ to be the norm-closed algebra generated by n isometries with pairwise orthogonal ranges. The C*-algebra generated by these isometries is either the Cuntz algebra $\mathcal{O}_n$ or the Cuntz-Toeplitz algebra $\mathcal{E}_n$. However the quotient map from $\mathcal{E}_n$ onto $\mathcal{O}_n$ is completely isometric on the nonself-adjoint algebra generated by the generators of $\mathcal{E}_n$. Thus $\mathcal{A}_n$ is uniquely determined up to complete isomorphism [25]. Frahzo [16], Bunce [6] and the second author [22] showed that any n-tuple $T = [T_1, \dots, T_n]$ which is a contraction dilates to an n-tuple of isometries with orthogonal ranges. This yields an important analogue of von Neumann's inequality [23] for such n-tuples. This algebra is naturally associated to the semigroup $*_{i=1}^n \mathbb{N}$, and as we have seen, an isometric representation of $\mathcal{A}_n$ is obtained from the left regular representation of this semigroup.

So by analogy, we define a noncommutative disc algebra associated to the free product $*_{i=1}^n G_i^+$ by $\mathcal{A}(*_{i=1}^n G_i^+)$ to be the norm-closed algebra generated by the left regular representation of the semigroup. The results of the last section yield the following immediate consequence:

COROLLARY 3.1. *Let G_i^+ be the positive cones of subgroups of $\mathbb{R}$ for $1 \le i \le n$ and $n \ge 2$. Let V_i be any isometric representations of G_i^+ on a common Hilbert space $\mathcal{H}$ satisfying* (†)*. Then the norm-closed* (*nonself-adjoint*) *algebra* $\mathrm{Alg}\big(\{V_i(g_i) : g_i \in G_i^+, 1 \le i \le n\}\big)$ *is completely isometrically isomorphic to* $\mathcal{A}(*_{i=1}^n G_i^+)$.

PROOF. Corollary 2.5 shows that the C*-algebras determined by V_i and the left regular representation are ∗-isomorphic via a map that intertwines the two representations. Thus it carries the algebra $\mathrm{Alg}\big(\{V_i(g_i) : g_i \in G_i^+, 1 \le i \le n\}\big)$ onto $\mathcal{A}(*_{i=1}^n G_i^+)$. Since ∗-isomorphisms are completely isometric, the result follows. ∎

We wish to show that one can recover the semigroup $*_{i=1}^n G_i^+$ from the algebra $\mathcal{A}(*_{i=1}^n G_i^+)$. To this end, we compute the characters (multiplicative linear functionals) of $\mathcal{A}(*_{i=1}^n G_i^+)$.

THEOREM 3.2. *Let G_i^+ be the positive cones of subgroups of $\mathbb{R}$ for $1 \le i \le n$ and $n \ge 2$. By rearranging if necessary, we may suppose that the first m of these semigroups are isomorphic to $\mathbb{N}$ and the others are dense. There is a canonical homeomorphism between the space of characters of $\mathcal{A}(*_{i=1}^n G_i^+)$ and the space consisting of those semicharacters ψ of $*_{i=1}^n G_i^+$ which satisfy*

$$(\ddagger) \qquad \sum_{i=1}^n |\psi(g_i)|^2 \le 1 \quad \text{for all } g_i \in G_i^+ \setminus \{0\}.$$

https://doi.org/10.4153/CJM-1998-015-5 Published online by Cambridge University Press

This space may be identified with the union of the unit ball $\mathbb{B}_m$ *of* $\mathbb{C}^m$ *and the cones* $\mathrm{C}\hat{G}_i$ *for* $m+1 \le i \le n$, *where the trivial semicharacters* ρ_0 *in each set are identified to a single point* (*which we denote by* ρ_0).

PROOF. There is a natural map from the space of characters of $A(*_{i=1}^n G_i^+)$ into the space of semicharacters of $*_{i=1}^n G_i^+$ given by restriction to the set of generators $\{\lambda(g) : g \in *_{i=1}^n G_i^+\}$. Every linear functional on an operator algebra is completely contractive [21, Prop. 3.7]. The generators of $A(*_{i=1}^n G_i^+)$ satisfy (†), which can be interpreted as the conditions

$$\big\|\big[\lambda(g_1)\dots\lambda(g_n)\big]\big\| \le 1 \quad \text{for all } g_i \in G_i^+ \setminus \{0\}.$$

Given a multiplicative functional Ψ, define the semicharacter $\psi(g) = \Psi\big(\lambda(g)\big)$. Applying this functional to the given row matrix must yield a contraction, which is to say

$$\sum_{i=1}^n |\psi(g_i)|^2 = \big\|\big[\psi(g_1)\cdots\psi(g_n)\big]\big\|^2 \le 1 \quad \text{for all } g_i \in G_i^+ \setminus \{0\}.$$

Since the set $\{\lambda(g) : g \in *_{i=1}^n G_i^+\}$ generates $A(*_{i=1}^n G_i^+)$ as a norm-closed algebra, this mapping from characters of the algebra to the semicharacters of the semigroup is injective. Moreover since the topologies on the two spaces are the weak topologies, it is evident that the map is continuous. Once it is established that this map is surjective, this will be a continuous bijection between compact Hausdorff spaces; and thus is a homeomorphism.

We first show that if a semicharacter ψ satisfying (‡) is non-zero on one of the dense semigroups G_j^+ for some $j > m$, then ψ vanishes on $G_i^+ \setminus \{0\}$ for all $i \ne j$. Thus there is a semicharacter ψ_j of G_j^+ such that

$$\psi(g) = \begin{cases} \psi_j(g) & \text{if } g \in G_j^+ \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, let ψ_j denote the restriction of ψ to G_j^+, and suppose that $\psi \ne \rho_0$. Then by Theorem 1.4, there is a real number $r \in (0,1]$ such that $|\psi_j(g)| = r^g$ for all $g \in G_j^+$. It follows that

$$\sup_{g \in G_j^+ \setminus \{0\}} |\psi(g)|^2 = 1.$$

Hence condition (‡) implies that the restriction of ψ to each G_i^+ for $i \ne j$ is ρ_0.

It follows that the space of semicharacters of $*_{i=1}^n G_i^+$ satisfying (‡) splits as the union of the semicharacters of $*_{i=1}^m \mathbb{N}$ and the semicharacters of each G_i^+ for $m < i \le n$ with ρ_0 identified. By Theorem 1.4, these latter spaces are equal to the cones $\mathrm{C}\hat{G}_i$. The semicharacters of $*_{i=1}^m \mathbb{N}$ are determined by the action on the generators z_i of $G_i^+ \simeq \mathbb{N}$ for $1 \le i \le m$ and the necessary and sufficient condition

$$\sum_{i=1}^m |\psi(z_i)|^2 \le 1.$$

The map taking ψ to $\big(\psi(z_1), \ldots, \psi(z_m)\big)$ is a bijection which carries the space of semicharacters satisfying (‡) onto the ball $\mathbb{B}_m$.

Now it is possible to verify that the map from characters of the algebra to semicharacters of the semigroup satisfying (‡) is surjective. Suppose that $j > m$ and $\psi \in \mathrm{C}\hat{G}_j$ is given. The subspace $\ell^2(G_j^+)$ of $\ell^2(*_{i=1}^n G_i^+)$ spanned by the basis vectors $\{\xi_g : g \in G_j^+\}$ is co-invariant for $\mathcal{A}(*_{i=1}^n G_i^+)$. Thus the compression to this subspace is a completely contractive homomorphism. This map takes $\lambda(g)$ to 0 unless $g \in G_j^+$, in which case it is sent to the left regular representation of G_j^+. So this maps onto $\mathcal{A}(G_j^+)$. Compose this with the multiplicative linear functional associated to ψ to obtain a multiplicative functional Ψ on $\mathcal{A}(*_{i=1}^n G_i^+)$. It is evident that the image of Ψ in the space of semicharacters is ψ as desired.

Similarly, suppose that ψ is a semicharacter on $*_{i=1}^m \mathbb{N}$ corresponding to a point $\lambda \in \mathbb{B}_m$. The compression of $\mathcal{A}(*_{i=1}^n G_i^+)$ to the co-invariant subspace spanned by $\{\xi_g : g \in *_{i=1}^m G_i^+\}$ maps the algebra onto the algebra $\mathcal{A}_m$. The characters of $\mathcal{A}_m$ are known to correspond to $\mathbb{B}_m$ [25, 12]. So the surjectivity is established as above. This completes the proof. ∎

We can use this result to provide complete isomorphism invariants for these algebras.

THEOREM 3.3. *Let G_i^+ and H_j^+ be positive cones of subgroups of $\mathbb{R}$. The two algebras $\mathcal{A}(*_{i=1}^n G_i^+)$ and $\mathcal{A}(*_{j=1}^k H_j^+)$ are isomorphic if and only if the semigroups are isomorphic, which holds if and only if $k = n$ and there is a permutation π so that G_i^+ is isomorphic to $H_{\pi(i)}^+$ for $1 \le i \le n$.*

PROOF. The three conditions are successively stronger. So suppose that the two algebras are isomorphic. Then their character spaces are homeomorphic. After deleting the trivial character ρ_0, the character set of $\mathcal{A}(*_{i=1}^n G_i^+)$ splits into distinct components. Possibly one is homeomorphic to $\mathbb{B}_m \setminus \{0\}$, where m is the number of the G_i^+ which are isomorphic to $\mathbb{N}$, and the other components are $\Omega_j = \hat{G}_j \times (0, 1]$ for $m < j \le n$. By Theorem 1.4, these sets are indeed connected. The component homeomorphic to a punctured ball is recognized by the fact that the first cohomotopy group $\pi^1(\mathbb{B}_m \setminus \{0\}) = \pi^1(S^{2m-1})$, the group of connected components of the group of invertible functions on the sphere S^{2m-1}, is isomorphic to $\mathbb{Z}$ if $m = 1$ and to 0 for $m > 1$. The dimension m is determined by the invariance of domain theorem, or by considering higher cohomotopy groups. Of course, the semigroup G_j^+ is not determined as an *ordered* group by $\hat{G}_j$. So additional argument is needed. Consider the ideal $J_j = \cap_{\psi \in \Omega_j} \ker \psi$. It is evident that $\mathcal{A}(*_{i=1}^n G_i^+)/J_j$ is (completely isometrically) isomorphic to $\mathcal{A}(G_j^+)$. But by Corollary 1.7, this determines G_j^+ up to order isomorphism. Consequently we have shown that the integers m and n are determined from the algebra, and that the semigroups G_j^+ are determined up to order isomorphism as required. The only possible change is a permutation of the terms, as there is no order on the set of components. ∎

4. **A Dilation Theorem.** We wish to obtain the analogue of the von Neumann inequality for the algebras $\mathcal{A}(*_{i=1}^n G_i^+)$. To this end, we also require a dilation theorem for contractive representations of G_i^+ satisfying the norm condition (†). This simultaneously

https://doi.org/10.4153/CJM-1998-015-5 Published online by Cambridge University Press

generalizes Mlak's Dilation Theorem and the dilation theorems of Frahzo [16], Bunce [6] and the second author [22]. Moreover it establishes uniqueness of the minimal dilation.

THEOREM 4.1. *Let G_i^+ be positive cones of discrete subgroups of $\mathbb{R}$ for $1 \le i \le n$. Let T_i be contractive representations of G_i^+ on a common Hilbert space satisfying the norm condition*

$$(\dagger) \qquad \sum_{i=1}^{n} T_i(g_i)T_i(g_i)^* < I \quad \text{for all } g_i \in G_i^+ \setminus \{0\}.$$

Then there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and isometric representations V_i of G_i^+ on $\mathcal{K}$ such that

(i) $\sum_{i=1}^n V_i(g_i)V_i(g_i)^* < I$ for all $g_i \in G_i^+ \setminus \{0\}$.
(ii) $V_i^*(g_i)\,|_{\mathcal{H}} = T_i(g_i)^*$ for all $g_i \in G_i^+$, $1 \le i \le n$.
(iii) $\mathcal{K} = \bigvee_{\sigma \in *_{i=1}^n G_i^+} V(\sigma)\mathcal{H}$.

Moreover this dilation is unique up to unitary equivalence which fixes $\mathcal{H}$.

PROOF. Using Mlak's Dilation Theorem 1.3, dilate each T_i to an isometric representation V_i on a larger Hilbert space $\mathcal{K}_i$. Let λ_i be the restriction to G_i^+ of the left regular representation of $*_{i=1}^n G_i^+$. Then replace each V_i by $V_i \oplus \lambda_i^{(\alpha)}$ on a Hilbert space $\mathcal{K} = \mathcal{K}_i \oplus \ell^2(*_{i=1}^n G_i^+)^{(\alpha)}$, where α is a cardinal sufficiently large to ensure that the subspace

$$\mathcal{M}_i := \overline{\cup_{g \in G_i^+ \setminus \{0\}} \operatorname{Ran}\bigl(V_i(g)\bigr)}$$

has codimension equal to the dimension of $\mathcal{K}$.

Existence of a dilation will be established provided that there are unitaries U_i in $\mathcal{B}(\mathcal{K})$ such that $U_i|_{\mathcal{H}} = I$ and $U_i\mathcal{M}_i$ are pairwise orthogonal. Indeed, the isometric representations $V_i'(g) = U_iV_i(g)U_i^*$ are dilations of T_i with orthogonal ranges, and thus satisfy (i). Minimality and uniqueness will be dealt with later.

Let $P_i = P_{\mathcal{M}_i}$, and note that

$$P_i = \mathop{\text{SOT-lim}}_{\substack{g \downarrow 0 \\ g \in G_i^+ \setminus \{0\}}} V_i(g)V_i(g)^*.$$

Decomposing $\mathcal{K} = \mathcal{H} \oplus \mathcal{K}'$, we obtain a matrix of the form

$$P_i = \begin{bmatrix} A_i & B_i \\ B_i^* & C_i \end{bmatrix}.$$

Therefore

$$\begin{aligned} \sum_{i=1}^n A_i = P_{\mathcal{H}} \sum_{i=1}^n P_i\,|_{\mathcal{H}} &= \mathop{\text{SOT-lim}}_{\substack{g \downarrow 0 \\ g \in G_i^+ \setminus \{0\}}} \sum_{i=1}^n P_{\mathcal{H}}\, V_i(g)V_i(g)^*\,|_{\mathcal{H}} \\ &= \mathop{\text{SOT-lim}}_{\substack{g \downarrow 0 \\ g \in G_i^+ \setminus \{0\}}} \sum_{i=1}^n T_i(g)T_i(g)^* \le I. \end{aligned}$$

Consider the contraction

$$X = [A_1^{1/2} \dots A_n^{1/2}].$$

Let

$$D_X = (I_n - X^*X)^{1/2} = [\delta_{ij} - A_i^{1/2}A_j^{1/2}] =: [X_{ij}].$$

Then $\begin{bmatrix} X \\ D_X \end{bmatrix}$ is an isometry. Thus the columns

$$S_j = \begin{bmatrix} A_j^{1/2} \\ X_{1j} \\ \vdots \\ X_{nj} \\ 0 \end{bmatrix}$$

(where the 0 acts on a space isomorphic to $\mathcal{K}$) for $1 \le j \le n$ are isometries with pairwise orthogonal ranges. Identify the range space with $\mathcal{K} = \mathcal{H} \oplus \mathcal{K}'$ in such a way that the map is the identity from $\mathcal{H}$ as the first component to $\mathcal{H}$ as a summand of $\mathcal{K}$. Then we may define projections with pairwise orthogonal ranges by

$$Q_i = S_iS_i^* = \begin{bmatrix} A_i & D_i \\ D_i^* & E_i \end{bmatrix}.$$

By construction, $\left(\sum_{i=1}^n Q_i\right)^\perp$ has range dimension equal to the dimension of $\mathcal{K}$.

Now $B_iB_i^* = A_i - A_i^2 = D_iD_i^*$. Thus there are partial isometries W_i on $\mathcal{K}'$ so that $B_iW_i = D_i$. Since both P_i and Q_i have ranges with complements of dimension equal to the dimension of $\mathcal{K}$, both B_i and D_i have large kernels. So the partial isometry W_i may be chosen to be a unitary. Define

$$P_i' := \begin{bmatrix} I & 0 \\ 0 & W_i \end{bmatrix}^* \begin{bmatrix} A_i & B_i \\ B_i^* & C_i \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & W_i \end{bmatrix} = \begin{bmatrix} A_i & D_i \\ D_i^* & F_i \end{bmatrix}.$$

By construction, Q_i is the smallest positive operator with entries $\begin{bmatrix} A_i & D_i \\ D_i^* & * \end{bmatrix}$, and thus $P_i' \ge Q_i$. So

$$P_i' - Q_i = R_i = \begin{bmatrix} 0 & 0 \\ 0 & Y_i \end{bmatrix}$$

is a projection. Because of the condition on the ranges of the Q_i, there is sufficient room to move the ranges of the Y_i onto pairwise orthogonal subspaces which are also orthogonal to the ranges of all the Q_i's. That is, there are unitary operators W_i' on $\mathcal{K}'$ such that $I \oplus W_i'$ is the identity on the range of Q_i and $W_i'Y_iW_i'^*$ are pairwise orthogonal projections with range orthogonal to the range of $\sum_{i=1}^n Q_i$. Thus conjugating P_i' by $I \oplus W_i'$ yields the desired projections.

Now that a dilation has been constructed, a minimal dilation is obtained by replacing the Hilbert space $\mathcal{K}$ by the smallest possible space $\bigvee_{\sigma \in *_{i=1}^n G_i^+} V(\sigma)\mathcal{H}$. Suppose that V and

https://doi.org/10.4153/CJM-1998-015-5 Published online by Cambridge University Press

V' are two minimal dilations on spaces $\mathcal{K}$ and $\mathcal{K}'$ respectively. Define a linear map W on alg span$\{V(\sigma)\mathcal{H} : \sigma \in *_{i=1}^n G_i^+\}$ by

$$WV(\sigma)h = V'(\sigma)h \quad \text{for all } \sigma \in *_{i=1}^n G_i^+,\ h \in \mathcal{H}.$$

Compute

$$\big(V(\sigma)h, V(\tau)k\big) = \begin{cases} \big(V(\rho)h, k\big) = \big(T(\rho)h, k\big) & \text{if } \sigma = \tau\rho \\ \big(h, V(\rho)k\big) = \big(h, T(\rho)k\big) & \text{if } \tau = \sigma\rho \\ 0 & \text{otherwise.} \end{cases}$$

This depends only on the original representation, and thus the same identity holds for V'. Therefore we obtain

$$\big(WV(\sigma)h, WV(\tau)k\big) = \big(V(\sigma)h, V(\tau)k\big)$$

for all $\sigma, \tau \in *_{i=1}^n G_i^+$ and $h, k \in \mathcal{H}$. This shows that W is well defined and extends to an isometry from the closed span of $\{V(\sigma)\mathcal{H} : \sigma \in *_{i=1}^n G_i^+\}$ to the corresponding set for V'. In other words, W is a unitary that intertwines V and V'. By considering the identity element e, we obtain

$$Wh = WV(e)h = V'(e)h = h \quad \text{for all } h \in \mathcal{H}.$$

Hence W agrees with the identity operator on $\mathcal{H}$. This establishes the desired uniqueness. ∎

This provides a complete classification of the completely contractive representations of $\mathcal{A}(*_{i=1}^n G_i^+)$.

COROLLARY 4.2. *Let G_i^+ be positive cones of discrete subgroups of $\mathbb{R}$ for $1 \le i \le n$. Let T_i be contractive representations of G_i^+ on a common Hilbert space satisfying the norm condition*

$$(\dagger) \qquad \sum_{i=1}^n T_i(g_i)T_i(g_i)^* < I \quad \text{for all } g_i \in G_i^+ \setminus \{0\}.$$

Then there is a (*unique*) *completely contractive homomorphism of the algebra $\mathcal{A}(*_{i=1}^n G_i^+)$ into* $\mathrm{Alg}\big(\{T_i(g_i) : g_i \in G_i^+, 1 \le i \le n\}\big)$ *which takes $V_i(g_i)$ to $T_i(g_i)$ for all $g_i \in G_i^+$ and $1 \le i \le n$.*

Conversely, every completely contractive representation of $\mathcal{A}(*_{i=1}^n G_i^+)$ arises in this way.

PROOF. By the preceding theorem, the representations T_i may be dilated to isometric representations V_i on a larger Hilbert space $\mathcal{K}$ which also satisfy (†). By Corollary 1.7,

$$\mathcal{A} = \mathrm{Alg}\big(\{V_i(g_i) : g_i \in G_i^+,\ 1 \le i \le n\}\big)$$

https://doi.org/10.4153/CJM-1998-015-5 Published online by Cambridge University Press

is completely isometrically isomorphic to $\mathcal{A}(*_{i=1}^n G_i^+)$. By the dilation condition (ii) of the preceding theorem, the compression of $\mathcal{A}$ to $\mathcal{H}$ is a completely contractive map which is a homomorphism because $\mathcal{H}$ is co-invariant. By identifying $\mathcal{A}$ with $\mathcal{A}(*_{i=1}^n G_i^+)$, we obtain the desired map.

Uniqueness follows since the $V_i(g_i)$'s generate $\mathcal{A}(*_{i=1}^n G_i^+)$. Moreover, given a completely contractive representation of $\mathcal{A}(*_{i=1}^n G_i^+)$, the restriction to the group elements satisfies (†) as in the first paragraph of the proof of Theorem 3.2. So every representation is determined by one of these semigroup representations. ∎

We obtain a von Neumann inequality for families of contractive representations by specializing the result above to elements of the group algebra ("polynomials").

COROLLARY 4.3. *Let G_i^+ be positive cones of discrete subgroups of $\mathbb{R}$ for $1 \leq i \leq n$. Let T_i be contractive representations of G_i^+ on a common Hilbert space satisfying the norm condition* (†)*. Then*

$$\left\|\sum a_\sigma T(\sigma)\right\| \leq \left\|\sum a_\sigma \lambda(\sigma)\right\|_{\mathcal{A}(*_{i=1}^n G_i^+)}$$

for all elements of $\mathbb{C} *_{i=1}^n G_i^+$.

We obtain an explicit expression for the Arveson Dilation [3] (see [21, Corollary 6.7]), and thus obtain a variant of our corollary extending results in [1, 25].

COROLLARY 4.4. *Let G_i^+ be positive cones of discrete subgroups of $\mathbb{R}$ for $1 \leq i \leq n$. Let T_i be contractive representations of G_i^+ on a common Hilbert space $\mathcal{H}$ satisfying the norm condition* (†)*. Then there is a completely positive unital map Ψ from $\mathcal{O}(*_{i=1}^n G_i^+)$ into $\mathcal{B}(\mathcal{H})$ such that*

$$\Psi\big(\lambda(\sigma)\lambda(\tau)^*\big) = T_\sigma T_\tau^*$$

for all $\sigma, \tau \in *_{i=1}^n G_i^+$.

PROOF. Let V be the isometric dilation of T provided by Theorem 4.1. Then by Corollary 2.5, there is a $*$-isomorphism π of $\mathcal{O}(*_{i=1}^n G_i^+)$ into $\mathcal{B}(\mathcal{K})$ such that $\pi\big(\lambda(\sigma)\big) = V_\sigma$ for all $\sigma \in *_{i=1}^n G_i^+$. The compression Ψ of π to $\mathcal{H}$ agrees with T on the elements of the semigroup. Moreover, $\mathcal{H}$ is co-invariant for each V_σ. Hence

$$\Psi\big(\lambda(\sigma)\lambda(\tau)^*\big) = P_{\mathcal{H}} V_\sigma V_\tau^* P_{\mathcal{H}}|_{\mathcal{H}} = P_{\mathcal{H}} V_\sigma P_{\mathcal{H}} V_\tau^* P_{\mathcal{H}}|_{\mathcal{H}} = T_\sigma T_\tau^*.$$ ∎

5**. Moment Problems.** A classical moment problem for the circle asks if there is a (positive) regular Borel measure μ on the circle $\mathbb{T}$ with certain prescribed Fourier coefficients. For example the full moment problem gives the full sequence $\hat{\mu}(k) = a_k$ for $k \geq 0$ and the truncated moment problem provides a_k for $0 \leq k \leq n$. The answer is provided in terms of the positivity of an associated formal Toeplitz operator. The Riesz representation theorem states that measures on $\mathbb{T}$ correspond to positive linear functionals on $C(\mathbb{T})$. Moreover, they are automatically completely positive, and the moment problem

https://doi.org/10.4153/CJM-1998-015-5 Published online by Cambridge University Press

has an operator-valued analogue that determines the existence of a completely positive map on $\mathrm{C}(\mathbb{T})$ with prescribed values on certain powers of z. The results of the previous two sections will be applied here to analyze certain operator-valued moment problems for the semigroups $*_{i=1}^n G_i^+$. See [2] for more information on the classical moment problem. In [26, 27, 28], the second author has done other work in this direction.

We will replace the space $\mathrm{C}(\mathbb{T})$ with our generalized Cuntz algebra $\mathcal{O}(*_{i=1}^n G_i^+)$. The Fourier coefficients of a map Φ from $\mathcal{O}(*_{i=1}^n G_i^+)$ into $\mathcal{B}(\mathcal{H})$ are given by their evaluation on $\lambda(\sigma)$ for σ in $*_{i=1}^n G_i^+$, say $A_\sigma = \Phi\big(\lambda(\sigma)\big)$. We consider the moment problem associated to any subset Σ of $*_{i=1}^n G_i^+$ which is

(i) *generating* in the sense that

$$*_{i=1}^n G_i^+ = \bigcup_{n\geq 1} \Sigma^n,$$

where Σ^n denotes the set of all products of n elements of Σ, and

(ii) *hereditary* in the sense that if $\rho\tau \in \Sigma$, then $\tau \in \Sigma$.

Such sets will be called *admissible*. A few examples that are of interest are

(1) $\Sigma = *_{i=1}^n G_i^+$ is the full moment problem.

(2) $\Sigma = \{g \in *_{i=1}^n G_i^+ : |g| \in S\}$ where S is a downward directed subset of $\prod_{i=1}^n G_i^+$ containing a non-zero element of each G_i^+. This yields several natural truncated moment problems. As special cases, one might take $S = \{\mathbf{g} : \mathbf{0} \ll \mathbf{g} \ll (1,\dots,1)\}$. Another natural example takes $S = (\bigcup_{i=1}^n G_i^+)^k$. This is the set of all words which have a minimal expression as the product of at most k terms.

(3) By specifying $A_0 = I$, we are limiting consideration to completely positive contractions. In general, the case of arbitrary A_0 can be deduced from this special case via a simple normalization trick.

A Toeplitz form K on $\Sigma\times\Sigma$ is an operator valued function such that $K(\rho\sigma,\rho\tau) = K(\sigma,\tau)$ whenever $\rho\sigma$ and $\rho\tau$ belong to Σ. The form is Hermitian if $K(\tau,\sigma) = K(\sigma,\tau)^*$. It is said to be *positive semidefinite* provided that

$$\sum_{\sigma,\tau\in\Sigma} \big(K(\sigma,\tau)h(\tau), h(\sigma)\big) \geq 0$$

for all finitely supported functions h from Σ into $\mathcal{H}$.

THEOREM 5.1. *Let Σ be an admissible subset of $*_{i=1}^n G_i^+$. A family of operators $\{A_\sigma : \sigma \in \Sigma\}$ in $\mathcal{B}(\mathcal{H})$ are the moments of a completely positive map Φ from $\mathcal{O}(*_{i=1}^n G_i^+)$ into $\mathcal{B}(\mathcal{H})$ if and only if the Toeplitz form K on $\Sigma\times\Sigma$ given by*

$$K(\sigma,\tau) = \begin{cases} A_\rho & \text{if } \tau = \sigma\rho \\ A_\rho^* & \text{if } \sigma = \tau\rho \\ 0 & \text{otherwise} \end{cases}$$

is positive semidefinite.

https://doi.org/10.4153/CJM-1998-015-5 Published online by Cambridge University Press

PROOF. We will assume that $A_0 = I$. The general case may be obtained by replacing A_0 by I and A_σ by $(A_0 + \varepsilon I)^{-1/2} A_\sigma (A_0 + \varepsilon I)^{-1/2}$ for small ε and taking limits. The details are left to the reader.

One direction is straightforward. If Φ is a completely positive map with the desired Fourier coefficients, then by Stinespring's Theorem [30] (see [21, Theorem 4.1]) there is a larger Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a $*$-representation of $\mathcal{O}(*_{i=1}^n G_i^+)$ on $\mathcal{K}$ such that

$$\Phi(A) = P_{\mathcal{H}}\, \pi(A)\, |_{\mathcal{H}} \qquad \text{for all } A \in \mathcal{O}(*_{i=1}^n G_i^+).$$

Let $V_\sigma = \pi\big(\lambda(\sigma)\big)$. These are isometries satisfying (†). It follows from the above equation that

$$P_{\mathcal{H}}\, V_\sigma^* V_\tau\, |_{\mathcal{H}} = K(\sigma, \tau) \quad \text{for all } \sigma, \tau \in *_{i=1}^n G_i^+.$$

Therefore if h_σ are vectors in $\mathcal{H}$ with only finitely many non-zero, then

$$\sum_{\sigma,\tau\in\Sigma} \big(K(\sigma,\tau) h_\tau, h_\sigma\big) = \sum (V_\tau h_\tau, V_\sigma h_\sigma) = \Big\| \sum V_\tau h_\tau \Big\|^2 \geq 0.$$

Conversely suppose that K is positive semidefinite. Use K to define a semidefinite form on $\mathcal{H}^\Sigma$, the space of all finitely supported functions from Σ into $\mathcal{H}$, by

$$\langle h, k\rangle_K = \sum_{\sigma,\tau\in\Sigma} \big(K(\sigma,\tau) h(\tau), k(\sigma)\big).$$

Then let $\mathcal{H}_K$ denote the Hilbert space completion of $\mathcal{H}^\Sigma / \mathcal{N}$, where $\mathcal{N}$ is the subspace of null vectors in this seminorm. Notice that $\mathcal{H}$ imbeds isometrically as those functions supported on the identity element because $K(0,0) = I$.

Define a contractive representation T of $*_{i=1}^n G_i^+$ on $\mathcal{H}_K$ by the formula

$$(T_\rho h)(\sigma) = \begin{cases} h(\tau) & \text{if } \sigma = \rho\tau \in \Sigma \\ 0 & \text{otherwise.} \end{cases}$$

This defines a partial isometry because of the Toeplitz condition. It is evident that this map is multiplicative, and thus determines a bounded representation. Let T_i denote the restriction of T to G_i^+. The range of $T_i(G_i^+ \setminus \{0\})$ is contained in the closed span of functions supported on

$$\big\{\sigma \in \Sigma : \sigma = g_i \sigma' \text{ for } g_i \in G_i^+ \setminus \{0\}\big\}.$$

The definition of K guarantees that these subspaces of $\mathcal{H}_K$ are pairwise orthogonal. Hence T satisfies condition (†).

https://doi.org/10.4153/CJM-1998-015-5 Published online by Cambridge University Press

Therefore by Theorem 4.1, this representation dilates to an isometric representation V on a Hilbert space $\mathcal{K}$ of $*_{i=1}^n G_i^+$. By Corollary 2.5, there is a $*$-isomorphism of $\mathcal{O}(*_{i=1}^n G_i^+)$ onto $\mathrm{C}^*\big(\{V(\sigma) : \sigma \in *_{i=1}^n G_i^+\}\big)$. Clearly the compression to $\mathcal{H}$, which is a subspace of $\mathcal{H}_K$, is completely positive. Let the composition of these maps be denoted by Φ. We compute for $\sigma \in \Sigma$ and vectors $h, k \in \mathcal{H}$ (and we identify h with the functions $h(\tau) = \delta_{0\tau} h$ in $\mathcal{H}_K$, where $\delta_{0\tau}$ is the Kronecker delta function)

$$\Big(\Phi\big(\lambda(\sigma)\big)h, k\Big) = \big(P_H V(\sigma)h, k\big) = \big\langle T(\sigma)h, k\big\rangle_K = (A_\sigma h, k).$$

Thus the Fourier coefficients of Φ are A_σ as desired. ∎

References

1. J. Agler, *The Arveson extension theorem and coanalytic models*. Integral Equations Operator Theory **5**(1982), 608–631.
2. N. I. Akhiezer, *The classical moment problem*. Olivier and Boyd, Edinburgh, Scotland, 1965.
3. W. B. Arveson, *Subalgebras of C*-algebras*. Acta Math. **123**(1969), 141–224.
4. ______, *Continuous analogues of Fock space*. Mem. Amer. Math. Soc. (409) **80**(1989).
5. D. P. Blecher and V. I. Paulsen, *Explicit construction of universal operator algebras and applications to polynomial factorization*. Proc. Amer. Math. Soc. **112**(1991), 839–850.
6. J. W. Bunce, *Models for n-tuples of noncommuting operators*. J. Func. Anal. **57**(1984), 21–30.
7. L. A. Coburn, *The C^*–algebra generated by an isometry, I*. Bull. Amer. Math. Soc. **13**(1967), 722–726.
8. J. Cuntz, *Simple C^*–algebras generated by isometries*. Comm. Math. Phys. **57**(1977), 173–185.
9. ______, *K-theory for certain C^*–algebras*. Ann. Math. **113**(1981), 181–197.
10. K. R. Davidson, *C*-algebras by example*. Fields Inst. Monographs **6**, Amer. Math. Soc., Providence, RI, 1996.
11. K. R. Davidson and D. R. Pitts, *Invariant subspaces and hyperreflexivity for free semigroup algebras*. Proc. London Math. Soc., to appear.
12. ______, *The algebraic structure of non-commutative analytic Toeplitz algebras*. Math. Ann., to appear.
13. H. T. Dinh, *Discrete product systems and their C^*–algebras*. J. Funct. Anal. **102**(1991), 1–34.
14. ______, *On generalized Cuntz C^*–algebras*. J. Operator Theory **30**(1993), 123–135.
15. R. G. Douglas, *On the C*-algebra of a one-parameter semigroup of isometries*. Acta Math. **128**(1972), 143–152.
16. A. Frahzo, *Models for non-commuting operators*. J. Funct. Anal. **48**(1982), 1–11.
17. M. Laca and I. Raeburn, *Semigroup crossed products and the Toeplitz algebras of non-abelian groups*. J. Funct. Anal. **139**(1996), 415–440.
18. M. Laca, *Purely infinite simple Toeplitz algebras*. Preprint, 1997.
19. W. Mlak, *Unitary dilations in case of ordered groups*. Ann. Polon. Math. **17**(1960), 321–328.
20. A. Nica, *C*-algebras generated by isometries and Wiener–Hopf operators*. J. Operator Theory **27**(1992), 17–52.
21. V. I. Paulsen, *Completely bounded maps and dilations*. Pitman Res. Notes Math. Ser. **146**, Longman Sci. Tech., Harlow, 1986.
22. G. Popescu, *Isometric dilations for infinite sequences of noncommuting operators*. Trans. Amer. Math. Soc. **316**(1989), 523–536.
23. ______, *Von Neumann inequality for $(B(H)^n)_1$*. Math. Scand. **68**(1991), 292–304.
24. ______, *Functional calculus for noncommuting operators*. Mich. J. Math. **42**(1995), 345–356.
25. ______, *Non-commutative disc algebras and their representations*. Proc. Amer. Math. Soc. **124**(1996), 2137–2148.
26. ______, *Positive-definite functions on free semigroups*. Can. J. Math. **48**(1996), 887–896.
27. ______, *Noncommutative joint dilations and free product operator algebras*. Pacific J. Math, to appear.
28. ______, *Positive definite kernels on free product semigroups and universal algebras*. Math. Scand., to appear.
29. W. Rudin, *Fourier Analysis on Groups*. Interscience Publishers, New York, 1962.
30. W. F. Stinespring, *Positive functions on C^*-algebras*. Proc. Amer. Math. Soc. **6**(1955), 211–216.

31. B. Sz. Nagy, *Sur les contractions de l'espace de Hilbert.* Acta. Sci. Math. (Szeged) **15**(1953), 87–92.
32. ______, *Transformations de l'espace de Hilbert, fonctions de type positif sur un groupe*. Acta. Sci. Math. (Szeged) **15**(1954), 104–114.
33. B. Sz.Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert space*. Akademiai Kaidó, Budapest, 1970.
34. J. von Neumann, *Eine spektraltheorie für allgemeine operatoren eines unitären raumes*. Math. Nachr. **4**(1951), 49–131.

Pure Mathematics Department
University of Waterloo
Waterloo, Ontario
N2L 3G1
email: krdavidson@math.uwaterloo.ca

Division of Mathematics and Statistics
University of Texas at San Antonio
San Antonio, TX 78249
USA
gpopescu@sphere.math.utsa.edu