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A SEMI-ANALYTICAL PRICING FORMULA FOR EUROPEAN OPTIONS UNDER THE ROUGH HESTON-CIR MODEL

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Abstract

We combine the rough Heston model and the CIR (Cox–Ingersoll–Ross) interest rate together to form a rough Heston-CIR model, so that both the rough behaviour of the volatility and the stochastic nature of the interest rate can be captured. Despite the convoluted structure and non-Markovian property of this model, it still admits a semi-analytical pricing formula for European options, the implementation of which involves solving a fractional Riccati equation. The rough Heston-CIR model is more general, taking both the rough Heston model and the Heston-CIR model as special cases. The influence of rough volatility and stochastic interest rate is shown to be significant through numerical experiments.

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1. Introduction

Despite the great success of the celebrated Black–Scholes model [6], some of the assumptions made in this particular model in order to achieve analytical simplicity and tractability are at odds with real market data. A typical example is that the implied volatility extracted from market option prices tends to exhibit a "smile" curve, a phenomenon that contradicts the constant-volatility assumption [11]. Thus, different modifications to the Black–Scholes model have been proposed, among which stochastic volatility models have received a lot of attention.

The concept of stochastic volatility models was first considered by Johnson [25] and, since then, option pricing under various stochastic volatility models has been

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widely discussed [26, 29, 32]. At the very early stage, all results were derived using numerical methods because of the difficulty involved in applying analytical approaches after the incorporation of another stochastic source. However, this posed an obstacle for their applications in practice due to the time intensiveness of model calibration, a process that any mathematical model needs to go through before they can be used in real markets, and the slow speed of numerical methods would make this almost impossible. This major flaw of numerical methods prompted the effort in searching for analytical solutions. With the volatility process being assumed to follow another geometric Brownian motion, a series solution was provided by Hull and White [24]. Although their results are very appealing, the zero-correlation assumption between the underlying price and volatility is certainly not appropriate, because this violates the "leverage effects" shown by empirical evidence [4]. A similar situation was faced by the Stein–Stein model [30], in which the adopted volatility process is unable to prevent the volatility taking negative values, although it also admits an analytical solution.

A breakthrough was made by Heston [23] by modelling the volatility with the CIR (Cox–Ingersoll–Ross) process. This model is satisfactory to a great extent, because it not only satisfies a wide range of basic properties, such as the nonnegative and mean-reverting properties, but also admits a closed-form pricing formula for European options, facilitating its applications in practice. Therefore, derivative pricing under the Heston model has been extensively studied [15, 21, 31]. Although the Heston model is still widely used in today's finance industry, there has been empirical evidence showing some of its flaws [5]. As a result, different variations of the Heston model started to appear, including the time-dependent Heston models [16] and regime-switching Heston models [12, 20]. Recently, Gatheral et al. [17] empirically investigated financial time series of realized volatility and they found that the volatility is actually "rough" in the sense that the Hurst parameter H [3] is far less than 0.5, according to real market data. Based on this, a rough Heston model was proposed [13, 14], which admits a semi-analytical pricing formula for European options. Another popular approach in modifying stochastic volatility models is to make the interest rate another random variable, as improved model performance has been shown after incorporating stochastic interest rate into option pricing models [2, 28]. Examples in this category include the Stein–Stein–Hull–White hybrid model [19] and the Heston-CIR hybrid model [18, 22].

In this paper, we combine the rough Heston model and the CIR stochastic interest rate together to formulate a rough Heston-CIR model, in order to capture the effect of rough volatility and stochastic interest rate. Although option pricing under this hybrid model is challenging due to the convoluted model structure, we still manage to present a semi-analytical pricing formula, after the successful derivation of the characteristic function based on the techniques of numeraire change. The characteristic function is written in an affine form, similar to that under the Heston-CIR hybrid model, but it is more computationally intensive, since it involves a fractional Riccati equation that needs to be numerically solved, as a result of incorporating the rough behaviour of the volatility. Finally, numerical experiments are carried out to show various properties of the newly derived formula.

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The rest of the paper is organized as follows. In Section 2, the rough Heston-CIR model and its relationship with some well-known models in the literature are illustrated. In Section 3, a general pricing formula is firstly obtained based on numeraire change, and the involved unknown characteristic function is analytically derived. In Section 4, numerical experiments and discussions are presented, followed by some concluding remarks in the last section.

2. The rough Heston-CIR model

One of the most popular stochastic volatility models that is still widely adopted in today's financial market is the celebrated Heston model, whose dynamics under a risk-neutral measure $\mathbb{Q}$ can be specified as

$$\begin{aligned} \frac{dS_t}{S_t} &= r\,dt + \sqrt{v_t}\,dW_{1,t}, \\ dv_t &= k(\theta - v_t)\,dt + \sigma\sqrt{v_t}\Big(\rho\,dW_{1,t} + \sqrt{1-\rho^2}\,dW_{2,t}\Big), \end{aligned}$$

where S and v represent, respectively, the underlying asset price and volatility. Here k is the mean-reverting speed of the volatility, θ denotes the corresponding mean-reverting level, σ is the so-called volatility of volatility and r is the constant interest rate. Also, $W_{1,t}$ and $W_{2,t}$ are two standard Brownian motions being independent of each other and ρ is the correlation between the underlying asset price and volatility. Despite its great success, it has also been reported that the Heston model may still fail to capture the main characteristics displayed by real market data and various modifications have thus been proposed. Introducing variable coefficients into the volatility process is one common choice, leading to the time-dependent Heston model [16] and the regime-switching Heston model [12, 20]. Another strand is to incorporate stochastic interest rate into the Heston model, among which the Heston-CIR model [22] is one of the most popular formations and it can be represented by

$$\begin{aligned} \frac{dS_t}{S_t} &= r_t\,dt + \sqrt{v_t}\,dW_{1,t}, \\ dv_t &= k(\theta - v_t)\,dt + \sigma\sqrt{v_t}\Big(\rho\,dW_{1,t} + \sqrt{1-\rho^2}\,dW_{2,t}\Big), \\ dr_t &= \xi(\beta - r_t)\,dt + \eta\sqrt{r_t}\,dW_{3,t}, \end{aligned} \tag{2.1}$$

with $W_{3,t}$ representing another Brownian motion being independent of both $W_{1,t}$ and $W_{2,t}$. Here ξ, β and η are the mean-reverting speed, mean-reverting level and volatility of the interest rate, respectively.

Recently, the use of the CIR-type volatility processes in the Heston as well as its related models has been challenged by Gatheral et al. [17], as it is empirically shown that the Hurst parameter H is far less than 0.5, implying that financial time series of realized volatility are "rough". Therefore, the rough Heston model was introduced [13, 14], by extending the CIR volatility process to a rough process

$$\begin{aligned}\frac{dS_t}{S_t} &= r\,dt + \sqrt{v_t}\,dW_{1,t},\\ v_t &= v_0 + \frac{k}{\gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}(\theta - v_u)\,du\\ &\quad + \frac{\sigma}{\gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\sqrt{v_u}\Big(\rho\,dW_{1,u} + \sqrt{1-\rho^2}\,dW_{2,u}\Big).\end{aligned}$$

Here $\alpha = H + 0.5$ and $\gamma(\cdot)$ denotes the gamma function. Clearly, the rough volatility process here is actually driven by a fractional Brownian motion with a Hurst parameter H.

Although the rough Heston model remarkably admits a semi-analytical solution that can be written in a similar form as the Heston pricing formula, the assumption of constant interest rate is sometimes not appropriate, as a lot of empirical evidence has demonstrated that stochastic interest rate can lead to significantly improved model performance [2, 28]. Therefore, we further introduce the CIR stochastic interest rate into the rough Heston model to formulate the rough Heston-CIR model, so that our model follows the dynamics of

$$\begin{aligned}\frac{dS_t}{S_t} &= r_t\,dt + \sqrt{v_t}\,dW_{1,t},\\ v_t &= v_0 + \frac{k}{\gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}(\theta - v_u)\,du\\ &\quad + \frac{\sigma}{\gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\sqrt{v_u}\Big(\rho\,dW_{1,u} + \sqrt{1-\rho^2}\,dW_{2,u}\Big),\\ dr_t &= \xi(\beta - r_t)\,dt + \eta\sqrt{r_t}\,dW_{3,t}.\end{aligned} \tag{2.2}$$

Note that although a nonzero correlation between the underlying price and interest rate is often assumed [18], we still assume that $W_{3,t}$ is independent of both $W_{1,t}$ and $W_{2,t}$. This is because adding the correlation between the underlying price and interest rate would make it impossible to obtain a semi-closed-form pricing formula, the availability of which is very useful in practice, especially in the recent trend of algorithmic trading. Of course, it is interesting to investigate how to efficiently price options when a full correlation structure is imposed and this will be left for future research.

Having presented the specific model that will be used to evaluate European options in this paper, the details on the derivation of a semi-analytical pricing formula will be provided in the next section.

3. A semi-analytical pricing formula

In this section, a brief discussion is presented over the general approach that can be used in pricing European options when stochastic interest rate is involved, based on which a semi-analytical solution to the characteristic function is derived so that a semi-analytical pricing formula is finally obtained.

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3.1. General pricing approach In a risk-neutral pricing world, if we use K to denote the strike price, the price of a European option at the current time $t = 0$ can be calculated as

$$U(S_0, v_0, r_0) = E^Q[e^{-\int_0^T r(s)ds} \max(S_T - K, 0)|S_0, v_0, r_0]. \tag{3.1}$$

As the discounting factor involves the integral of the stochastic interest rate, we introduce a T-forward measure $\mathbb{Q}^T$ [8] to simplify the target expectation. In particular, equation (3.1) can be directly reformulated as

$$U(S_0, v_0, r_0) = P(r_0, 0, T)E^{Q^T}[\max(S_T - K, 0)|S_0, v_0, r_0], \tag{3.2}$$

where $P(r, t, T)$ is defined as

$$P(r, t, T) = E^Q[e^{-\int_t^T r(s)ds}|r_t = r],$$

denoting the price of a T-maturity zero-coupon bond at time t under $\mathbb{Q}$. This implies that we have divided our task of deriving the pricing formula into two steps with the first step figuring out the expression of $P(r, t, T)$, which admits an analytical solution (see its derivation by He and Zhu [22] and also in the Appendix)

$$P(r, t, T) = e^{A_1(t,T) - A_2(t,T)r}, \tag{3.3}$$

where

$$A_1(t, T) = -\xi\beta\Big[\frac{4}{(m-\xi)(m+\xi)} \ln\Big\{\frac{2m + (m+\xi)(e^{m(T-t)} - 1)}{2m}\Big\} + \frac{2}{\xi - m}(T - t)\Big],$$
$$A_2(t, T) = \frac{2(e^{\sqrt{m(T-t)}} - 1)}{2m + (\xi + m)(e^{m(T-t)} - 1)},$$

with $m = \sqrt{\xi^2 + 2\eta^2}$.

Clearly, the task left now is to compute the expectation shown in equation (3.2), which is under the T-forward measure. Therefore, a prior step that needs to be taken is to derive the model dynamics under $\mathbb{Q}^T$, which can be achieved with the techniques illustrated by Brigo and Mercurio [8] using numeraire change. In particular, the numeraires under $\mathbb{Q}$ and $\mathbb{Q}^T$ are, respectively, $N_{1,t} = e^{\int_0^t r(s)\,ds}$ and $N_{2,t} = P(r, t, T)$, implying that

$$dN_{1,t} = N_{1,t} r_t\, dt,$$
$$dN_{2,t} = N_{2,t}\Big[\Big\{\frac{dA_1}{dt} - r_t\frac{dA_2}{dt} - \alpha(\beta - r_t)A_2\Big\}dt - \eta\sqrt{r_t}A_2\, dW_{3,t}\Big].$$

From the dynamics of $N_{1,t}$ and $N_{2,t}$, it is obvious that the volatility terms of $N_{1,t}$ and $N_{2,t}$ can be written as $\sigma^{N_{1,t}} = (0, 0, 0)^T$ and $\sigma^{N_{2,t}} = (0, 0, -\eta\sqrt{r_t}N_{2,t}A_2)^T$, respectively, leading to

$$\begin{bmatrix} dW_{1,t}^{Q^T} \\ dW_{2,t}^{Q^T} \\ dW_{3,t}^{Q^T} \end{bmatrix} = \begin{bmatrix} dW_{1,t} \\ dW_{2,t} \\ dW_{3,t} \end{bmatrix} + C^T \times \Big(\frac{\sigma^{N_{1,t}}}{N_{1,t}} - \frac{\sigma^{N_{2,t}}}{N_{2,t}}\Big)dt,$$

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using the results of Brigo and Mercurio [8]. Here C is defined as

$$C = \begin{bmatrix} 1 & 0 & 0 \\ \rho & \sqrt{1-\rho^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence, one can deduce that

$$\begin{bmatrix} dW_{1,t}^{Q^T} \\ dW_{2,t}^{Q^T} \\ dW_{3,t}^{Q^T} \end{bmatrix} = \begin{bmatrix} dW_{1,t} \\ dW_{2,t} \\ dW_{3,t} + \eta\sqrt{r_t}A_2\,dt \end{bmatrix},$$

which leads to the model dynamics under $\mathbb{Q}^T$

$$\begin{aligned} \frac{dS_t}{S_t} &= r_t\,dt + \sqrt{v_t}\,dW_{1,t}^{Q^T}, \\ v_t &= v_0 + \frac{k}{\gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}(\theta - v_u)\,du \\ &\quad + \frac{\sigma}{\gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\sqrt{v_u}\Big(\rho\,dW_{1,u}^{Q^T} + \sqrt{1-\rho^2}\,dW_{2,u}^{Q^T}\Big), \\ dr_t &= [\xi\beta - (\xi + \eta^2 A_2)r_t]\,dt + \eta\sqrt{r_t}\,dW_{3,t}^{Q^T}. \end{aligned} \tag{3.4}$$

Once we have obtained the model dynamics under the T-forward measure, we can now formally compute the unknown expectation in equation (3.2), which will be discussed in the next subsection.

3.2. A semi-analytical pricing formula With the transformation of $y_t = \ln(S_t)$, $t \in [0, T]$, equation (3.2) can be further written as

$$U(y_0, v_0, r_0) = P(r_0, 0, T)[P_1 - KP_2], \tag{3.5}$$

where

$$P_1 = \int_{\ln K}^{\infty} e^y p(y)\,dy \quad \text{and} \quad P_2 = \int_{\ln K}^{\infty} p(y)\,dy,$$

with $p(y)$ being the conditional density function of y_T, given the current information at $t = 0$. It is clear that P_2 is actually the probability of y_T being greater than $\ln K$ and thus

$$P_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\Big[\frac{e^{-j\phi\ln K} f(\phi; T, y_0, v_0, r_0)}{j\phi}\Big]\,d\phi,$$

with j as the imaginary unit and $f(\phi; T, y_0, v_0, r_0)$ representing the characteristic function of the underlying log-price y_T conditional upon all the information at the current time. Although P_1 is not directly a probability, it can be rearranged as

$$P_1 = f(-j; T, y_0, v_0, r_0)\int_{\ln K}^{\infty} \frac{e^y p(y)}{f(-j; T, y_0, v_0, r_0)}\,dy,$$

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where $f(-j; T, y_0, v_0, r_0)$ is obtained by setting $\phi = -j$ in the characteristic function $f(\phi; T, y_0, v_0, r_0)$. In this way, it is clear that the involved integral represents a probability that a certain variable with the density function of $e^y p(y)/f(-j; T, y_0, v_0, r_0)$ is greater than $\ln K$, since it is not difficult to show that

$$\int_{-\infty}^{\infty} e^y p(y)\, dy = f(-j; T, y_0, v_0, r_0).$$

Therefore,

$$P_1 = f(-j; T, y_0, v_0, r_0)\Big[\frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\Big[\frac{e^{-j\phi\ln K} f(\phi - j; T, y_0, v_0, r_0)}{j\phi f(-j; T, y_0, v_0, r_0)}\Big]\, d\phi\Big],$$

using the shift theorem of the Fourier transform [7]. Clearly, the only unknown term involved in the pricing formula (3.5) is the characteristic function, the working out of which would yield the desired result. Although this task is still complicated due to the involvement of both stochastic interest rate and rough volatility, which is neither Markovian nor a semi-martingale, we still manage to obtain a semi-analytical formula, which is presented in the following theorem.

THEOREM 3.1. *If the underlying price, volatility and interest rate follow* (3.4)*, then the characteristic function of the underlying log-price is*

$$f(\phi; T, y_0, v_0, r_0) = e^{C(\phi;T)+D(\phi;T)r_0+I^{1-\alpha}h(\phi;T)v_0+j\phi y_0}, \tag{3.6}$$

where

$$\begin{aligned}
D &= \frac{(m-\xi)(d_1e^{q_1T} + d_2e^{q_2T} + d_3e^{q_3T} + d_4e^{q_4T})}{g_1e^{q_1T} + g_2e^{q_2T} + g_3e^{q_3T} + g_4e^{q_4T}},\\
C &= k\theta\int_0^T h(\phi; s)ds + \xi\beta\Big[p_1T + \frac{2}{\eta^2}\ln\Big\{\frac{p_2}{m}\frac{(\xi-m)e^{q_3T} - (\xi+m)e^{q_4T}}{(\xi-p_2)e^{3q_3T} - (\xi+p_2)e^{q_4T}}\Big\}\Big],\\
d_1 &= m - p_2 - j\phi(m+\xi), \quad d_2 = m + p_2 - j\phi(m-\xi),\\
d_3 &= -m + p_2 + j\phi(m-\xi), \quad d_4 = -m - p_2 + j\phi(m+\xi),\\
g_1 &= -\eta^2(p_2+\xi), \quad g_2 = -(\eta^2+\xi^2-m\xi)(p_2+\xi),\\
g_3 &= -(\eta^2+\xi^2-m\xi)(p_2-\xi), \quad g_4 = -\eta^2(p_2-\xi),\\
q_1 &= \frac{p_2+3m}{2}, \quad q_2 = \frac{p_2+m}{2}, \quad q_3 = \frac{-p_2+m}{2}, \quad q_4 = \frac{-p_2+3m}{2},\\
p_1 &= \frac{2[j\phi(\xi-m) + m - p_2]}{(\xi-p_2)(\xi-m)} + \frac{2(m-p_2)}{\xi^2-m^2} + \frac{6(m-p_2)(j\phi-1)}{\xi^2-p_2^2},\\
p_2 &= \sqrt{m^2 - 2j\phi\eta^2},
\end{aligned} \tag{3.7}$$

with $h(\phi; t)$ satisfying the fractional Riccati equation

$$D^\alpha h(\phi; t) = \tfrac{1}{2}\sigma^2h^2(\phi; t) + (j\phi\rho\sigma - k)h(\phi; t) - \tfrac{1}{2}(j\phi + \phi^2),\ I^{1-\alpha}h(\phi; 0) = 0. \tag{3.8}$$

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Here the fractional integral and derivative of order $\alpha \in (0, 1]$, I^α and D^α are, respectively, defined as

$$I^\alpha h(\phi; t) = \frac{1}{\gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(\phi; s)\, ds,$$
$$D^\alpha h(\phi; t) = \frac{1}{\gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} h(\phi; s)\, ds.$$

PROOF. With the underlying price process being specified in (3.4), the underlying log-price can be explicitly formulated as

$$y_T = y_0 + \int_0^T r_s - \frac{1}{2} v_s\, ds + \int_0^T \sqrt{v_s}\, dW_{1,s}$$

and thus the target characteristic function $f(\phi; T, y_0, v_0, r_0)$ can be calculated through

$$\begin{aligned} f(\phi; T, y_0, v_0, r_0) &= E^{Q^T}[e^{j\phi y_T} | y_0, v_0, r_0] \\ &= e^{j\phi y_0} E^{Q^T}[e^{j\phi \int_0^T r_s\, ds} | r_0] E^{Q^T}[e^{j\phi\left(-\int_0^T v_s/2\, ds + \int_0^T \sqrt{v_s}\, dW_{1,s}\right)} | y_0, v_0], \end{aligned} \quad (3.9)$$

since r_t is independent of v_t. This means that all we need is to compute the two expectations involved in the above equation.

In particular, if we denote

$$w(\phi; r_t, t) = E^{Q^T}[e^{j\phi \int_t^T r_s\, ds} | r_t],$$

the PDE (partial differential equation) governing $w(\phi; r_t, t)$ can be derived as

$$\frac{\partial w}{\partial t} + [\xi\beta - \{\xi + \eta^2 A_2(t, T)\} r_t] \frac{\partial w}{\partial r_t} + \frac{1}{2}\eta^2 r_t \frac{\partial^2 w}{\partial r_t^2} + j\phi r_t w = 0,$$

with $w(\phi; r_T, T) = 1$, using the Feynman–Kac theorem [27]. By assuming the expression of $w(\phi; r_t, t)$ as

$$w(\phi; r_t, t) = e^{C_1(\phi;\tau) + D(\phi;\tau) r_t}, \qquad \tau = T - t,$$

together with $D(\phi; 0) = C_1(\phi; 0) = 0$, it is not difficult to find that $D(\phi; \tau)$ and $C_1(\phi; \tau)$ should respectively satisfy

$$\frac{dD}{d\tau} = \frac{1}{2}\eta^2 D^2 - [\xi + \eta^2 A_2(t, T)] D + j\phi,$$
$$\frac{dC_1}{d\tau} = \xi\beta D.$$

Although the ODE (ordinary differential equation) for $D(\phi; \tau)$ is a time-dependent Riccati equation, which is usually difficult to solve, we manage to present a completely analytical solution using the software Maple, as shown in equation (3.7). Note that besides Maple, the procedure presented by He and Zhu [22] can also be used to obtain a series solution with its convergence guaranteed. Once $D(\phi; \tau)$ has been worked out,

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the expression of $C_1(\phi;\tau)$ can be obtained by directly integrating its governing ODE, which yields

$$C_1(\phi;\tau) = \xi\beta\Big[p_1\tau + \frac{2}{\eta^2}\ln\Big\{\frac{p_2}{m}\frac{(\xi-m)e^{q_3\tau}-(\xi+m)e^{q_4\tau}}{(\xi-p_2)e^{3q_3\tau}-(\xi+p_2)e^{q_4\tau}}\Big\}\Big].$$

This certainly yields the representation of the first expectation in (3.9),

$$E^{Q^T}[e^{j\phi\int_0^T r_s ds}|r_0] = w(\phi; r_0, 0). \tag{3.10}$$

On the other hand, we can straightforwardly obtain

$$E^{Q^T}[e^{j\phi\{-\int_0^T (v_s/2)\,ds+\int_0^T \sqrt{v_s}\,dW_{1,s}\}}|y_0, v_0] = e^{C_2(\phi;T)+I^{1-\alpha}h(\phi;T)v_0}, \tag{3.11}$$

where $h(\phi;t)$ is the solution (3.8) and $C_2(\phi;T) = k\theta\int_0^T h(\phi;s)\,ds$, using the results of Euch and Rosenbaum [14]. Therefore, combining equations (3.9), (3.10) and (3.11) finally yields the desired formula. This completes the proof. □

Note that the key property that leads to the success in deriving an analytical formula for the characteristic function is the joint affine structure of the processes (y_t, v_t, r_t). Since v_t solves a stochastic convolution equation instead of an SDE (stochastic differential equation), this affine property should be understood in the sense of Abi Jaber et al. [1]. This indicates that it should be possible to construct more general models, such as those with more than one volatility factor and/or additional interest rate factors, and such extension will be left for future research.

The price of a European option under the rough Heston-CIR model (2.2) can be calculated with the semi-analytical formula (3.5), with the involved characteristic function being presented in (3.6). Of course, once a formula has been derived, it is necessary to check its validity, and the properties of the new formula should also be investigated, both of which will be discussed in the next section.

We remark that our solution procedure here also offers an alternative analytical solution to European option prices under the Heston-CIR model. Specifically, if the underlying asset price, volatility and interest are assumed to follow equation (2.1), the price of a European option is

$$C(y_0, v_0, r_0) = P(r_0, 0, T)[P_1^0 - KP_2^0],$$

where

$$P_1^0 = ch(-j;T,y_0,v_0,r_0)\Big\{\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\Big[\frac{e^{-j\phi\ln K}ch(\phi-j;T,y_0,v_0,r_0)}{j\phi ch(-j;T,y_0,v_0,r_0)}\Big]d\phi\Big\},$$

$$P_2^0 = \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\Big[\frac{e^{-j\phi\ln K}ch(\phi;T,y_0,v_0,r_0)}{j\phi}\Big]d\phi.$$

Here the characteristic function $ch(\phi;T,y_0,v_0,r_0)$ is defined as

$$ch(\phi;T,y_0,v_0,r_0) = e^{E_1(\phi;T)+E_2(\phi;T)r_0+E_3(\phi;T)v_0+j\phi y_0},$$

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with

$$
\begin{aligned}
E_1 &= \frac{k\theta}{\sigma^2}\Big\{[d-(j\phi\rho\sigma-k)]T - 2\ln\Big(\frac{1-ge^{dT}}{1-g}\Big)\Big\} \\
&\quad + \xi\beta\Big\{p_1T + \frac{2}{\eta^2}\ln\Big(\frac{p_2}{m}\frac{(\xi-m)e^{q_3T}-(\xi+m)e^{q_4T}}{(\xi-p_2)e^{3q_3T}-(\xi+p_2)e^{q_4T}}\Big)\Big\}, \\
E_2 &= D = \frac{(m-\xi)(d_1e^{q_1T}+d_2e^{q_2T}+d_3e^{q_3T}+d_4e^{q_4T})}{g_1e^{q_1T}+g_2e^{q_2T}+g_3e^{q_3T}+g_4e^{q_4T}}, \\
E_3 &= \frac{d-(j\phi\rho\sigma-k)}{\sigma^2}\cdot\frac{1-e^{dT}}{1-ge^{dT}}.
\end{aligned}
$$

Compared with the series solution under the same model provided by He and Zhu [22], this alternative solution is much faster in numerical computation; it takes this formula about 0.12 s to calculate one option price, while the series solution in [22] needs about 3.77 and 32.65 s when using 10 and 100 terms, respectively. The CPU time here is obtained using Matlab R2016b on a PC with the following specifications: Intel(R) Core(TM) i5-3470 3.20 GHz CPU and 8.0 GB of RAM. Of course, one should not devalue the approach proposed in [22], since it is possible to extend their solution technique to other time-dependent Riccati equations which could not be analytically solved with numerical methods using software.

4. Numerical experiments and discussions

In this section, the accuracy of the newly derived formula will be verified and the influence of stochastic interest rate and rough volatility will then be shown. Unless otherwise stated in the following, the parameters used in this section are listed as follows. The mean-reverting speed, long-term mean and volatility of the volatility, k, θ and σ, are assumed to be 10, 0.2 and 0.1, respectively, and those of the interest rate satisfy $\xi = 5$, $\beta = 0.01$ and $\eta = 0.02$. The rough parameter α is set to be 0.6, which is consistent with the empirical results in [17]. Other parameters include $S_0 = K = 100$, $v_0 = 0.05$, $r_0 = 0.03$, $\rho = -0.5$ and $T = 1$.

We point out that the realization of (3.5) still depends on how the fractional Riccati equation (3.8) can be numerically solved. We first rewrite equation (3.8) as

$$
h(\phi;t) = \frac{1}{\gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(\phi;h(\phi,s))\,ds, \qquad t\in(0,T],
$$

with

$$
F(\phi;x) = \tfrac{1}{2}\sigma^2x^2 + (j\phi\rho\sigma-k)x - \tfrac{1}{2}(j\phi+\phi^2).
$$

By uniformly discretizing the time interval $[0,T]$ into N sub-intervals with $dt = T/N$ and $t_n = n\,dt$, $n = 0,1,\ldots,N$, the determination of the function $h(\phi;t), t\in[0,T]$ reduces to finding the discrete values of $h(\phi;t_{n+1})$, $n = 0,1,\ldots,N-1$ with the initial value $h(\phi;t_0) = 0$. Thus, this is actually a forward process; for a given $n = 0,1,\ldots,N-1$, $h(\phi;t_{n+1})$ needs to be determined based on all the already derived values $h(\phi;t_j)$,

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$j = 0, 1, \ldots, n$. In this case, for each $n = N-1, N-2, \ldots, 0$, we implement the fractional Adams method [9, 10, 14], which is indeed a two-step predictor–corrector process. Based on the predicted value $\hat{h}(\phi; t_{n+1})$ found by using the scheme

$$\hat{h}(\phi; t_{n+1}) = \sum_{j=0}^{n} a_j^{n+1} F(\phi, h(\phi; t_j)),$$
$$a_j^{n+1} = \frac{(dt)^\alpha}{\gamma(\alpha+1)}[(n-j+1)^\alpha - (n-j)^\alpha], \quad 0 \le j \le n,$$

in the first step, the corrected value $h(\phi; t_{n+1})$ can be obtained in the second step through

$$h(\phi; t_{n+1}) = \sum_{j=0}^{n} b_j^{n+1} F(\phi, h(\phi; t_j)) + b_{n+1}^{n+1} F(\phi, \hat{h}(\phi; t_{n+1})),$$

where

$$b_0^{n+1} = \frac{(dt)^\alpha}{\gamma(\alpha+2)}[n^{\alpha+1} - (n-\alpha)(n+1)^\alpha],$$
$$b_j^{n+1} = \frac{(dt)^\alpha}{\gamma(\alpha+2)}[(n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}], \quad 1 \le j \le n,$$
$$b_{n+1}^{n+1} = \frac{(dt)^\alpha}{\gamma(\alpha+2)}.$$

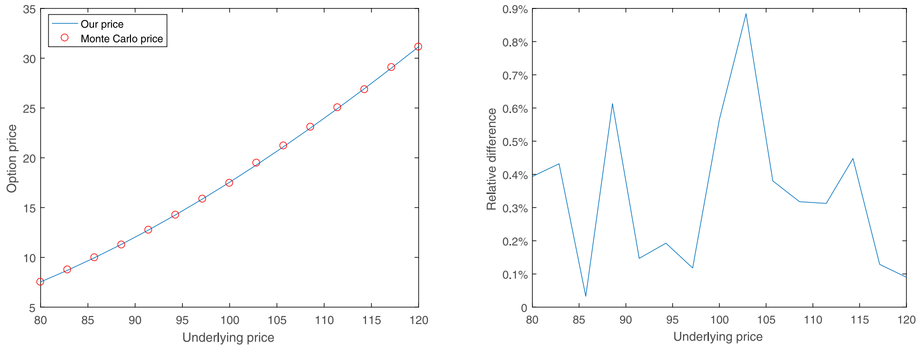
Once we have derived all $h(\phi; t_{n+1})$, $n = 0, 1, \ldots, N-1$, the Riemann integral can be directly computed through the trapezoidal rule, while the fractional integral involved in the pricing formula should be rearranged as

$$I^{1-\alpha} h(\phi; T) = \frac{1}{\gamma(1-\alpha)} \int_0^T h(\phi; t)(T-t)^{-\alpha}\, dt$$
$$= \frac{1}{\gamma(2-\alpha)} \int_0^T (T-t)^{1-\alpha}\, dh(\phi; t),$$

using integration by parts, after which a normal trapezoidal rule can be applied to obtain the value of the fractional integral.

Having been aware of how the newly derived pricing formula can be numerically implemented, the accuracy of the formula needs to be checked first before it can be applied in practice. Depicted in Figure 1 is the numerical comparison of option prices calculated with our formula (our prices) and those from Monte Carlo simulation (Monte Carlo prices). One can observe clearly in Figure 1(a) that Monte Carlo prices are closely located around our prices in a point-wise manner, which demonstrates the accuracy of our formula. Further evidence is also shown in Figure 1(b) by calculating the relative difference between our prices and Monte Carlo prices, the result of which verifies our formula, with the maximum relative difference being less than 0.9%.

With the confidence in our formula, we can now start to investigate its properties, the first of which that needs to be mentioned is that our model is quite general and



(a) Our price vs Monte Carlo price.

(b) The relative difference between our price and Monte Carlo price.

Figure 1. The comparison of option prices calculated with our formula and those obtained through Monte Carlo simulation.

it contains several well-known models as special cases. In particular, what is shown in Figure 2(a) is the degeneration of our model to the rough Heston model, which is achieved through introducing a scale parameter $\epsilon \in [0, 1]$ such that $\xi = 5 * \epsilon$ and $\eta = 0.02 * \epsilon$. Such a degeneration happens when $\epsilon = 0$, corresponding to the case where both ξ and η are zero, which is reasonable since, in this case, the interest rate in our model is no longer stochastic, but a constant. It can also be observed that the option price under our model is a monotonic increasing and decreasing function of the scale parameter when the long-term mean of the interest is greater and smaller than the current interest rate, respectively. This is also expected as the interest rate tends to climb (decline) when the long-term mean of the interest rate is larger (smaller) than its initial value, and a higher mean-reverting speed resulting from a greater scale parameter will further accelerate this process, leading to higher option premiums. On the other hand, as the rough volatility model will go back to the CIR process when α approaches 1, it is natural for us to check whether our pricing formula can also display such a property and thus Figure 2(b) shows the option prices under our model and the Heston-CIR model with respect to α. It is not difficult to find that our price actually increases with α and it approaches the Heston-CIR price when α is close to 1.

The influence of introducing the stochastic interest rate as well as rough volatility is demonstrated in Figure 3. In specifics, option prices calculated with our model are compared with those under the rough Heston model in Figure 3(a) and one can again observe that our model with a long-term mean of the stochastic interest rate being greater (smaller) than the initial interest rate typically produces higher (lower) option prices compared with the rough Heston model, and a larger mean-reverting speed has a positive impact on this trend. On the other hand, it is interesting to observe in Figure 3(b) that the gap between our price and the Heston-CIR price is relatively large when the time to expiry is small, implying that the introduced rough volatility has a significant impact on the prices of short-tenor options, which are among the

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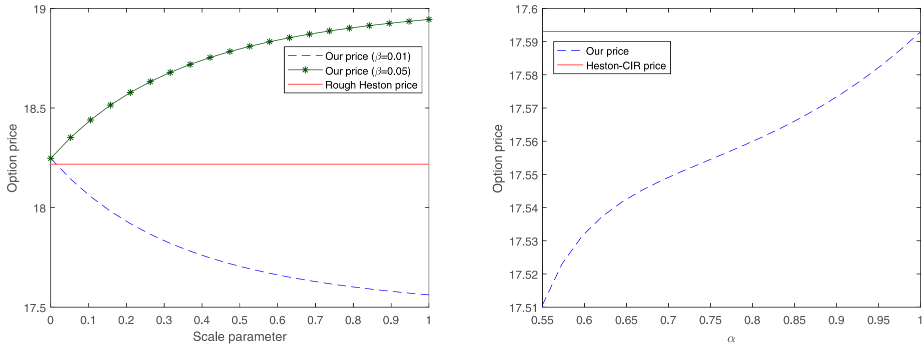
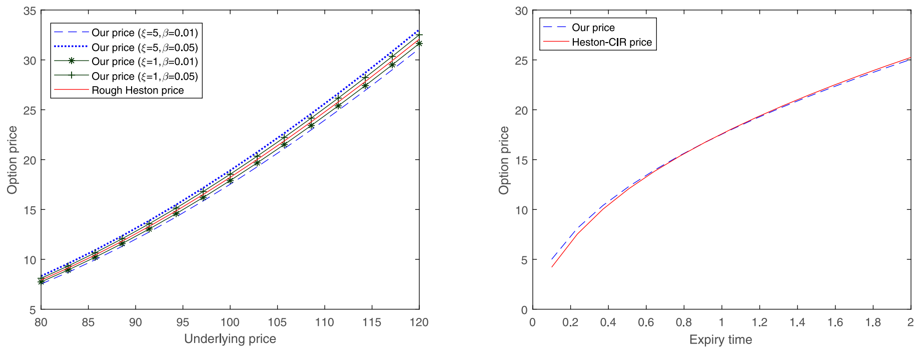


(a) Degeneration to the rough Heston model.

(b) Degeneration to the Heston-CIR model.

FIGURE 2. Special cases of the rough Heston-CIR model.



(a) Our model vs the rough Heston model.

(b) Our model vs the Heston-CIR model.

FIGURE 3. Comparison of our prices with those under the rough Heston model and the Heston-CIR model.

most liquid option contracts in real markets. One possible explanation for this is the mean-reverting feature, with which the rough behaviour of the volatility would become insignificant when the volatility approaches the long-term mean over time.

5. Conclusion

We considered the pricing of European options under the rough Heston-CIR model, which is introduced to capture the main characteristics exhibited by real market data, including the rough behaviour of the volatility and the stochastic nature of the interest rate. The successful derivation of the characteristic function in an affine form gives rise to a semi-analytical pricing formula. Numerical experiments were carried out to compare option prices under our model and those obtained from the rough Heston model as well as the Heston-CIR model, and results demonstrate that the introduction of the rough volatility and stochastic interest rate has a significant impact on option

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prices, implying the potential of our model to be applied in practice.

Appendix

As $P(r, t, T)$ represents the price of a zero-coupon bond at time t with T being the expiry, the Feynman–Kac theorem indicates that it is the solution to the PDE system

$$\begin{cases} \dfrac{\partial P}{\partial t} + \alpha(\beta - r)\dfrac{\partial P}{\partial r} + \dfrac{1}{2}\eta^2 r \dfrac{\partial^2 P}{\partial r^2} - rP = 0, \\ P(r, t, T)|_{t=T} = 1. \end{cases}$$

With $P(r, t, T)$ assumed to be in the form of (3.3), it is straightforward to derive the ODEs governing $A_1(t, T)$ and $A_2(t, T)$,

$$\begin{cases} \dfrac{dA_2}{dt} = \dfrac{1}{2}\eta^2 A_2^2 + \alpha A_2 - 1, \quad A_2(T, T) = 0, \\ \dfrac{dA_1}{dt} = \alpha\beta A_2, \quad A_1(T, T) = 0. \end{cases}$$

The function $A_2(t, T)$ can be derived using the standard technique developed to deal with Riccati equations, based on which $A_1(t, T)$ can then be obtained through integrating $A_2(t, T)$, leading to the desired result.

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