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Consecutive Large Gaps in Sequences Defined by Multiplicative Constraints

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Abstract. In this paper we obtain quantitative results on the occurrence of consecutive large gaps between B-free numbers, and consecutive large gaps between nonzero Fourier coefficients of a class of newforms without complex multiplication.

1 Introduction

A recent result of Panaitopol [12] states that if $f_1 < f_2 < \cdots$ is the sequence of square-free numbers, then

$$\limsup_{n\to\infty} \min(f_{n+1} - f_n, f_n - f_{n-1}) = \infty.$$

In the present paper we make further progress in this line of investigation, from several perspectives. We are interested in obtaining a similar result for more general sequences of integers defined by multiplicative constraints. At the same time, we localize the problem in short intervals, and provide quantitative lower bounds for the sizes of gaps as well as for the number of occurrences of pairs of consecutive large gaps. A natural generalization of square-free numbers is supplied by the concept of B-free numbers, which was introduced by Erdős [6]. Let B be a sequence of positive integers $1 < b_1 < b_2 < \cdots$ satisfying

$$(*) \qquad \sum_j \frac{1}{b_j} < \infty, \quad \gcd(b_k, b_j) = 1, k \neq j.$$

Then a number n is called B-free if it is not divisible by any b_k in B. Note that by taking B to be the sequence of squares of prime numbers, the set of B-free numbers coincides with the set of square-free numbers. Note that B-free numbers have positive density, more precisely

$$\lim_{N\to\infty} \frac{1}{N} \#\{1 \le n \le N : \text{n is B-free}\} = \prod_{k=1}^{\infty} \Big(1 - \frac{1}{b_k}\Big) > 0.$$

In spite of this fact, one would expect to find reasonably large gaps, or even pairs of consecutive large gaps, between B-free numbers. Our first result, which establishes the existence of many pairs of consecutive large gaps between B-free numbers in short intervals, is as follows.

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Theorem 1 *Let η, σ be positive real numbers such that $20\sigma > 9 + 3606\eta$. Let B be a sequence of positive integers $1 < b_1 < b_2 < \cdots$ satisfying $\sum_j \frac{1}{b_j} < \infty$ and $\gcd(b_k, b_j) = 1$ for $k \neq j$, and denote by $f_1 < f_2 < \cdots$ the sequence of B-free numbers. For any positive integer N, let $\Phi(N)$ be the largest positive integer for which*

$$\prod_{j=1}^{3\Phi(N)} b_j \leq N^{\eta}.$$

Then for any N large enough in terms of B, η and σ, there exist

$$\gg_{\sigma,\eta,B} \frac{(2\Phi(N))!\, N^{\sigma}}{\prod_{\Phi(N)+1 \leq s \leq 3\Phi(N)} b_s}$$

many values of f_n in $[N, N + N^{\sigma}]$ for which $\min(f_{n+1} - f_n, f_n - f_{n-1}) > \Phi(N)$.

Note that the quality of the above result depends on how sparse the sequence B is. In the case of square-free numbers, that is, in the case when B consists of the squares of all prime numbers, $\Phi(N)$ is asymptotic to $\eta \log N/6 \log\log N$ as $N \to \infty$.

Erdős [6] constructed a set of integers B satisfying $(*)$ such that the gap between two consecutive B-free numbers is unusually large infinitely often. Unaware of this result of Erdős, the authors rediscovered and published it [3] (the only difference between the two published results being that we have a better constant in the exponent). Here we complement this result by constructing such a set B for which we have many simultaneous large gaps between consecutive B-free numbers infinitely often.

Theorem 2 *Let $0 < \lambda < \frac{1}{8}$ be a real number. There exists a set of positive integers B satisfying $(*)$ such that, denoting the sequence of B-free numbers by $f_1 < f_2 < \cdots$, there are infinitely many n for which*

$$\min\Big\{ f_{n+j} - f_{n+j-1} : 1 \leq j \leq \big[e^{\sqrt{\lambda \log n \log\log n}}\big] \Big\} > e^{\sqrt{\lambda \log n \log\log n}}.$$

Besides the intrinsic interest about their distribution, the concept of B-free numbers proved to be useful in various contexts, such as in questions concerned with nonvanishing problems for Fourier coefficients of cusp forms of integer weight for congruence subgroups of the full modular group $SL_2(\mathbb{Z})$. To be more specific, let

$$f(z) = \sum_{n=1}^{\infty} a_f(n) q^n \in S_k(\Gamma_0(N), \chi), \quad q = e^{2\pi i z}, \operatorname{Im} z > 0$$

be a nonzero cusp form of integer weight $k \geq 2$ without complex multiplication. Serre introduced the gap function $i_f(n) = \min\{j \geq 0 : a_f(n + j) \neq 0\}$ to study the frequencies of vanishing of Fourier coefficients $a_f(n)$. In [1, 2, 4], various results were obtained on the size of $i_f(n)$ and on the nonvanishing of $a_f(n)$ in the case where $f(z)$ is a newform without complex multiplication. New results on this problem of understanding the behavior of $i_f(n)$ were obtained more recently by Kowalski, Robert

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and Wu [9]. One of our theorems [1] states that $i_f(n) \ll_{f,\Psi} \Psi(n)$ for almost all n, where $\Psi(n)$ is essentially any function monotonically tending to infinity. It was also proved [2] that given $\epsilon > 0$, there exists $M = M(\epsilon, f)$ such that

$$\#\{n \le x : i_f(n) \le M\} \ge (1-\epsilon)x.$$

The proof of Theorem 3 below shows in particular that working in this generality, the above estimates on $i_f(n)$ are the best possible. We obtain a stronger result by constructing many pairs of consecutive large gaps in the Fourier expansion of $f(z)$, when n is confined to a short interval.

Theorem 3 *Let $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k(\Gamma_0(N))$ be a newform of integer weight $k \ge 2$ without complex multiplication having integer Fourier coefficients. Let σ, η be positive real numbers such that $20\sigma > 9 + 3606\eta$. Assume that the set $\{p \text{ prime} : a_f(p) = 0\}$ is infinite and denote these primes by $p_1 < p_2 < \cdots$. For any M, let $\Phi(M)$ be the largest integer for which*

$$\prod_{j=\pi(\Phi(M))+1}^{\pi(\Phi(M))+2\Phi(M)} p_j^2 \le M^{\eta}.$$

Then for M large enough in terms of f, σ, η there are

$$\gg_{f,\sigma,\eta} \frac{(2\Phi(M))!\, M^{\sigma}}{\prod_{j=\pi(\Phi(M))+1}^{\pi(\Phi(M))+2\Phi(M)} p_j^2}$$

many integers $x \in [M, M + M^{\sigma}]$ such that

$$i_f(x - \Phi(M)) = \Phi(M) \quad \textit{and} \quad i_f(x+1) \ge \Phi(M).$$

The proof of Theorem 3 in the case $k > 2$ is the same as in the case $k = 2$, so we chose to state the theorem for a general k. We remark though that in his work on Frobenius distributions and Galois representations, Kumar Murty provides a generalization of the conjecture of Lang and Trotter, which implies that for $k > 2$ the set $\{p \text{ prime} : a_f(p) = 0\}$ from the statement of Theorem 3 is finite (see [11, Conjectures 2.17, 3.1]). Thus for $k > 2$, Theorem 3 would only apply to counterexamples, if any, to Murty's conjecture. The situation is entirely different for $k = 2$. Theorem 3 does apply in particular to the case when $f(z)$ is the newform associated to an elliptic curve over $\mathbb{Q}$ without complex multiplication, since Elkies proved that there are infinitely many supersingular primes for every such elliptic curve. In this case, the size $\Phi(M)$ of the gaps provided in Theorem 3 is dictated by the counting function $\pi_0(x)$ of supersingular primes $\le x$ of a given elliptic curve E. Assuming the generalized Riemann hypothesis, Ram Murty [10] proved that $\pi_0(x) > \sqrt{\log\log x}$. Later, he and Fouvry [7] succeeded in proving a strong unconditional result. Namely, they showed that for any E and for any $\delta > 0$, there exists $x_0(E, \delta)$ such that the inequality

$$\pi_0(x) > \frac{\log_3 x}{(\log_4 x)^{1+\delta}}$$

holds for $x > x_0(E, \delta)$. Here $\log_k$ stands for the k-fold iterated logarithm function. In the case when the elliptic curve does not have complex multiplication, the precise asymptotics of $\pi_0(x)$ are predicted by the Lang–Trotter conjecture, claiming that

$$\pi_0(x) \sim C_E \frac{\sqrt{x}}{\log x}, \quad \text{as } x \to \infty,$$

where $C_E > 0$ is a constant depending only on E. David and Pappalardi [5] investigated the general problem of the asymptotics of $\#\{p \leq x : a_E(p) = r\}$ in the case $a_E(p) = p + 1 - \#(E/\mathbb{F}_p)$, where $E/\mathbb{F}_p$ is the reduction of E modulo a prime p, and $r \geq 0$ is a given integer. In particular they showed that $\pi_0(x)$ is close to $C_E \frac{\sqrt{x}}{\log x}$ for almost all elliptic curves $Y^2 = X^3 + aX + b$ with a, b integers varying in a large rectangle whose sides grow with x. The Lang–Trotter conjecture implies that the size $\Phi(M)$ of the pairs of consecutive gaps provided by Theorem 3 in the case when $f(z) = f_E(z)$ is the weight 2 newform associated with an elliptic curve E over $\mathbb{Q}$ without complex multiplication satisfies an asymptotic formula of the form

$$\Phi(M) \sim \frac{\eta \log M}{C'_E \sqrt{\log \log M}}, \quad \text{as } M \to \infty,$$

for some constant $C'_E > 0$ depending on C_E.

2 Proof of Theorem 1

Let η, σ, B and Φ be as in the statement of the theorem. We fix a positive integer N and consider the system of congruences

$$\begin{aligned}
x &\equiv 1 \pmod{b_{j_1}},\\
x &\equiv 2 \pmod{b_{j_2}},\\
&\dots\\
x &\equiv \Phi(N) \pmod{b_{j_{\Phi(N)}}},\\
x &\equiv -1 \pmod{b_{j_{\Phi(N)+1}}},\\
x &\equiv -2 \pmod{b_{j_{\Phi(N)+2}}},\\
&\dots\\
x &\equiv -\Phi(N) \pmod{b_{j_{2\Phi(N)}}},
\end{aligned}$$

where we choose in order the moduli $b_{j_1}, b_{j_{\Phi(N)+1}}, b_{j_2}, b_{j_{\Phi(N)+2}}, \dots, b_{j_{\Phi(N)}}, b_{j_{2\Phi(N)}}$ to be

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distinct, and satisfying the following coprimality conditions:

$$(b_{j_2}, 2) = 1 = (b_{j_{\Phi(N)+2}}, 2),$$
$$(b_{j_3}, 3) = 1 = (b_{j_{\Phi(N)+3}}, 3),$$
$$\dots$$
$$(b_{j_{\Phi(N)}}, \Phi(N)) = 1 = (b_{j_{2\Phi(N)}}, \Phi(N)).$$

Note that one can always find elements b_j of B satisfying the above conditions since the elements of B are pairwise relatively prime. Moreover it is easy to see that such admissible choices for $b_{j_1}, \dots, b_{j_{2\Phi(N)}}$ can be found among the first $3\Phi(N)$ elements of B. Indeed, any prime number $p \le \Phi(N)$ can divide at most one element of B. Therefore there are less than $\Phi(N)$ many elements of B which are divisible by at least one prime number $p \le \Phi(N)$. So at least $2\Phi(N)$ numbers among the first $3\Phi(N)$ elements of B are relatively prime with each of the numbers $2, 3, \dots, \Phi(N)$, and hence the desired moduli $b_{j_1}, \dots, b_{j_{2\Phi(N)}}$ can be chosen from this set of numbers. As a consequence we have

$$\prod_{s=1}^{2\Phi(N)} b_{j_s} \le \prod_{j=1}^{3\Phi(N)} b_j.$$

Next, by the Chinese Remainder Theorem all the solutions x of the above system of congruences form an arithmetic progression $\{b + m\prod_{1\le s\le 2\Phi(N)} b_{j_s} : m \in \mathbb{Z}\}$ for some integer number b. Note that b is relatively prime to the product $\prod_{1\le s\le 2\Phi(N)} b_{j_s}$ since otherwise there will be a prime p such that $p \mid b$ and $p \mid b_{j_s}$ for some $1 \le s \le 2\Phi(N)$. Moreover we either have $b \equiv x \equiv s \pmod{b_{j_s}}$ for some $1 \le s \le \Phi(N)$, or we have $b \equiv x \equiv \Phi(N) - s \pmod{b_{j_s}}$ for some $\Phi(N) + 1 \le s \le 2\Phi(N)$. Both cases give us a contradiction, since $(b_{j_s}, s) = 1$ for $1 \le s \le \Phi(N)$ and since $(b_{j_s}, \Phi(N) - s) = 1$ for $\Phi(N) + 1 \le s \le 2\Phi(N)$. We now employ the following proposition which can be derived from the proof of [3, Theorem 1].

Proposition *Let η, σ be positive real numbers satisfying $20\sigma > 9 + 3606\eta$ and let B be a sequence of pairwise relatively prime positive integers with sum of inverses finite. Then there exists $N_{B,\sigma,\eta}$ such that for any $N \ge N_{B,\sigma,\eta}$ and any relatively prime integers a, b with $1 \le a \le N^\eta$, there are $\gg_{\sigma,\eta,B} \frac{N^\sigma}{a}$ many B-free integers n, with $N \le n \le N + N^\sigma$, for which $n \equiv b \pmod{a}$.*

Note that this result is applicable in our case with $a = \prod_{1\le s\le 2\Phi(N)} b_{j_s}$ and η, σ, B, b as before, since a, b are relatively prime and

$$a = \prod_{1\le s\le 2\Phi(N)} b_{j_s} \le \prod_{j=1}^{3\Phi(N)} b_j \le N^\eta.$$

Using the proposition, it follows that the number of B-free numbers from the interval $[N, N + N^\sigma]$ which belong to the arithmetic progression

$$\{b + m \prod_{1\le s\le 2\Phi(N)} b_{j_s} : m \in \mathbb{Z}\}$$

is

$$\gg_{\sigma,\eta,B} \frac{N^\sigma}{\prod_{1\le s\le 2\Phi(N)} b_{j_s}}.$$

Finally, we take any permutation of the moduli b_{j_s} with $1 \le s \le 2\Phi(N)$ and then apply the Chinese Remainder Theorem as above. In this way we produce $(2\Phi(N))!$ many distinct arithmetic progressions having the same modulus $\prod_{1\le s\le 2\Phi(N)} b_{j_s}$. Hence these arithmetic progressions are disjoint, and their union contains

$$\gg_{\sigma,\eta,B} \frac{(2\Phi(N))!\, N^\sigma}{\prod_{1\le s\le 2\Phi(N)} b_{j_s}} \ge \frac{(2\Phi(N))!\, N^\sigma}{\prod_{\Phi(N)+1\le s\le 3\Phi(N)} b_s}$$

many B-free numbers from the interval $[N, N+N^\sigma]$. This completes the proof of the theorem.

3 Proof of Theorem 2

We follow the constructions from [3, 6], with an improvement in the inductive step, which will allow us to obtain long tuples of consecutive large gaps between B-free numbers. Let $\lambda < \frac{1}{8}$ be a real number. We proceed inductively. Fix a k and assume that n_k and $b_1 < b_2 < \cdots < b_{s_k} < n_k$ are already constructed. Denote as usual by $\psi(x, y)$ the number of positive integers $n \le x$ with all prime factors $\le y$. As a consequence of a theorem of Hildebrand and Tenenbaum on the number of smooth numbers for $u = \frac{\log x}{\log y}$,

$$\psi(x, y) = x\rho(u)\Big(1 + O\Big(\frac{\log(u+1)}{\log y}\Big)\Big)$$

uniformly for $y \ge 2$ and $1 \le u \le \exp((\log y)^{\frac{3}{5}-\epsilon})$, where $\log \rho(u) = -u\log u + o(u\log u)$. Then for any fixed $\eta > 0$, $\rho(u) \le e^{-(1-\eta)u\log u}$. Fix a small $\delta > 0$, and for each k set $u = u_k$ and $n = n_k^{u_k}$ so that $(1-\eta)u_k \log u_k = (1+\delta)\log n_k$. Here $u_k = O\big(\frac{\log n_k}{\log\log n_k}\big)$, and $\psi(2n, n_k) \ll n\rho(u_k) \ll n/n_k^{1+\delta}$. Therefore there are integers x, y with $n \le x < y \le 2n$ and $y - x \ge n_k$ such that any integer $m \in [x, y]$ has a prime factor $> n_k$. We set $n_{k+1} = x$. Then the largest prime factor q_m of any $m \in [n_{k+1}, n_{k+1} + n_k]$ satisfies $q_m > n_k$. Here the q_m's are necessarily distinct, since no q_m can divide two different numbers in this interval. Denote now by l_k the integer part of $\sqrt{n_k}$ and choose an integer r_k which is relatively prime with l_k. Let $\mathcal{S}_k$ denote the set of integers inside the interval $[n_{k+1}, n_{k+1} + n_k]$ which are congruent to r_k modulo l_k. Let $\mathcal{T}_k$ be the set of integers in the interval $[n_{k+1}, n_{k+1} + n_k]$ which do not belong to $\mathcal{S}_k$. We list the primes q_m, with $m \in \mathcal{T}_k$ in increasing order, and redenote them by $b_{s_k+1} < b_{s_k+2} < \cdots < b_{s_{k+1}}$ with $s_{k+1} = s_k + \#(\mathcal{T}_k)$. By repeating this construction we obtain a sequence of positive integers $n_1 < n_2 < \cdots < n_k < \cdots$ and a sequence of prime numbers $b_1 < b_2 < \cdots < b_{s_k} < \cdots$. Next, the sum $S = \sum_{s=1}^{\infty} \frac{1}{b_s}$ is convergent. Indeed, $S = \sum_{k=0}^{\infty} S_{k+1}$, with $S_{k+1} = \sum_{s_k+1\le s\le s_{k+1}} \frac{1}{b_s}$, and one has

$$S_{k+1} \le \sum_{\substack{n_k < p \le 2n_k \log n_k \\ p \text{ prime}}} \frac{1}{p} = O\Big(\frac{\log\log n_k}{\log n_k}\Big).$$

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Clearly $u_k \log n_k \le \log n_{k+1} \le \log 2 + u_k \log n_k$, and since $u_k \ll \frac{\log n_k}{\log\log n_k}$, we have

$$\frac{\log\log n_{k+1}}{\log n_{k+1}} \ll \frac{\log\log n_k}{u_k \log n_k}.$$

This proves that $S = \sum_{s=1}^{\infty} \frac{1}{b_s} < \infty$. Returning now to a fixed k, let us remark that the set $\mathcal{S}_k$ is a finite arithmetic progression whose elements are relatively prime with its modulus. Then by the lower bound aspect of the linear sieve (see [8, Theorem 8.4]) it follows that a positive proportion of the elements of $\mathcal{S}_k$ are relatively prime with any of the primes $b_1 < b_2 < \cdots < b_{s_k}$. We also know that none of the primes $b_{s_k+1} < b_{s_k+2} < \cdots < b_{s_{k+1}}$ can divide any of the elements of $\mathcal{S}_k$, since each of the primes $b_{s_k+1} < b_{s_k+2} < \cdots < b_{s_{k+1}}$ coincides with a q_m that divides one of the elements of $\mathcal{T}_k$, and, as we remarked before, no q_m can divide two integers in $[n_{k+1}, n_{k+1} + n_k] = \mathcal{S}_k \cup \mathcal{T}_k$. Also, each b_j with $j > s_{k+1}$ is larger than any element of $\mathcal{S}_k$, thus no such b_j can divide any element of $\mathcal{S}_k$. Consequently, a positive proportion of the elements of $\mathcal{S}_k$ is B-free. We also know that no element of $\mathcal{T}_k$ is B-free, since by our construction for each integer m in $\mathcal{T}_k$ the largest prime factor q_m of m coincides with one of the elements b_j of B. Recall that $\mathcal{S}_k$ consists of about $[\sqrt{n_k}]$ many elements in an arithmetic progression of modulus about the size of $\sqrt{n_k}$. In conclusion, the interval $[n_{k+1}, n_{k+1} + n_k]$ contains $\gg \sqrt{n_k}$ many B-free numbers, and the gap between any two of them is at least as large as the modulus of the arithmetic progression on which $\mathcal{S}_k$ is supported, that is, this gap is $\gg \sqrt{n_k}$. Lastly, since $n_{k+1} \le 2n_k^{u_k}$ and $u_k \log u_k = \frac{(1+\delta)}{(1-\eta)} \log n_k$, we see that

$$u_k \le (1+c)\frac{\log n_k}{\log\log n_k},$$

for some small constant $c > 0$ depending on δ and η. Combining these, we have $\sqrt{n_k} > e^{\sqrt{\lambda \log n_{k+1} \log\log n_{k+1}}}$, for any fixed $\lambda < \frac{1}{8}$ and small enough $c, \delta, \eta > 0$. This completes the proof of the theorem.

4 Proof of Theorem 3

Let $f(z) = \sum_{n=1}^{\infty} a_f(n) q^n \in S_k(\Gamma_0(N))$ be a newform without complex multiplication, and define $B_f = \{p \text{ prime } : a_f(p) = 0\} \cup \{2\} \cup \{q \text{ prime } : q|N\}$. Since B_f is infinite by our assumption, we may write all the primes in B_f in increasing order as $2 \le p_1 < p_2 < \cdots$. Serre proved that for any $\epsilon > 0$,

$$|B_f \cap [2, z]| \ll_{\epsilon, f} \frac{z}{(\log z)^{\frac{3}{2}-\epsilon}}.$$

Consequently, $\sum_{p \in B_f} \frac{1}{p}$ is convergent. Next, let M and $\Phi(M)$ be positive integers, where $\Phi(M)$ will be defined explicitly later as an increasing function of M. Consider the system of congruences

$$x \equiv 1 + p_{j_1} \pmod{p_{j_1}^2},$$
$$x \equiv 2 + p_{j_2} \pmod{p_{j_2}^2},$$
$$\dots$$
$$x \equiv \Phi(M) + p_{j_{\Phi(M)}} \pmod{p_{j_{\Phi(M)}}^2},$$
$$x \equiv -1 + p_{j_{\Phi(M)+1}} \pmod{p_{j_{\Phi(M)+1}}^2},$$
$$x \equiv -2 + p_{j_{\Phi(M)+2}} \pmod{p_{j_{\Phi(M)+2}}^2},$$
$$\dots$$
$$x \equiv -\Phi(M) + p_{j_{2\Phi(M)}} \pmod{p_{j_{2\Phi(M)}}^2},$$

where we choose in order the odd primes $p_{j_1}, p_{j_{\Phi(M)+1}}, p_{j_2}, p_{j_{\Phi(M)+2}}, \dots, p_{j_{\Phi(M)}}, p_{j_{2\Phi(M)}}$ to be distinct, not dividing N, and satisfying the following coprimality conditions:

$$(p_{j_2}, 2) = 1 = (p_{j_{\Phi(M)+2}}, 2),$$
$$(p_{j_3}, 3) = 1 = (p_{j_{\Phi(M)+3}}, 3),$$
$$\dots$$
$$(p_{j_{\Phi(M)}}, \Phi(M)) = 1 = (p_{j_{2\Phi(M)}}, \Phi(M)).$$

Since B_f is infinite, one can always find primes p_{j_s} in B_f as above, where $1 \le s \le 2\Phi(M)$. Moreover, for M large enough, we have

$$\prod_{s=1}^{2\Phi(M)} p_{j_s} \le \prod_{j=\pi(\Phi(M))+1}^{\pi(\Phi(M))+2\Phi(M)} p_j,$$

where $\pi(x)$ denotes the number of primes $\le x$. We define $\Phi(M)$ to be the largest integer satisfying the inequality

$$\prod_{j=\pi(\Phi(M))+1}^{\pi(\Phi(M))+2\Phi(M)} p_j^2 \le M^\eta,$$

where η is as in the statement of the theorem. By the Chinese Remainder Theorem all the solutions x of the above system of congruences constitute an arithmetic progression of the form

$$\{b + m \prod_{s=1}^{2\Phi(M)} p_{j_s}^2 : m \in \mathbb{Z}\}$$

for some integer b. Note that if p_{j_s} divides b for some $1 \le s \le 2\Phi(M)$, then either $s + p_{j_s} \equiv x \equiv b (\mathrm{mod} p_{j_s}^2)$ when $1 \le s \le \Phi(M)$ or $\Phi(M) - s + p_{j_s} \equiv x \equiv b(\mathrm{mod}$

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$p_{j_s}^2$) when $\Phi(M)+1 \leq s \leq 2\Phi(M)$. Both possibilities give us that p_{j_s} divides s, a contradiction. In conclusion b is relatively prime to the modulus $\prod_{s=1}^{2\Phi(M)} p_{j_s}^2$. Using the proposition from Section 2, it follows that for $M \geq M_{B_f,\sigma,\eta}$, $20\sigma > 9 + 3606\eta$ and

$$a = \prod_{s=1}^{2\Phi(M)} p_{j_s}^2 \leq \prod_{j=\pi(\Phi(M))+1}^{\pi(\Phi(M))+2\Phi(M)} p_j^2 \leq M^\eta,$$

the number of B_f-free integers $x \in [M, M+M^\sigma]$ which belong to the arithmetic progression

$$\Big\{ b + m \prod_{s=1}^{2\Phi(M)} p_{j_s}^2 : m \in \mathbb{Z} \Big\} \quad \text{is} \quad \gg_{B_f,\sigma,\eta} \frac{M^\sigma}{\prod_{s=1}^{2\Phi(M)} p_{j_s}^2}.$$

Let us also observe that by taking permutations of the moduli $p_{j_s}^2$ for $1 \leq s \leq 2\Phi(M)$ and applying the Chinese Remainder Theorem, one can produce $(2\Phi(M))!$ many disjoint arithmetic progressions with the same modulus $\prod_{s=1}^{2\Phi(M)} p_{j_s}^2$. Hence in this way one can find

$$\gg_{B_f,\sigma,\eta} \frac{(2\Phi(M))!\, M^\sigma}{\prod_{s=1}^{2\Phi(M)} p_{j_s}^2} \geq \frac{(2\Phi(M))!\, M^\sigma}{\prod_{j=\pi(\Phi(M))+1}^{\pi(\Phi(M))+2\Phi(M)} p_j^2},$$

B_f-free integers $x \in [M, M+M^\sigma]$ satisfying the above congruences. Note that $p_{j_1} || x-1$ (where the notation means that p_{j_1} divides $x-1$ but $p_{j_1}^2$ does not divide $x-1$). Hence using the multiplicativity of the Fourier coefficients of a newform we have

$$a_f(x-1) = a_f(p_{j_1}) a_f\Big(\frac{x-1}{p_{j_1}}\Big) = 0.$$

Similarly, using $p_{j_2} || x-2, \dots, p_{j_{\Phi(M)}} || x-\Phi(M)$ and $p_{j_{\Phi(M)+1}} || x+1, \dots, p_{j_{\Phi(2M)}} || x+\Phi(M)$ we see that $a_f(x \pm s) = 0$ for $1 \leq s \leq \Phi(M)$. Next, we claim that $a_f(x) \neq 0$. To show this, let us write $x = \prod_{j=1}^r q_j^{k_j}$ as the prime factorization of x. Then $a_f(x) = \prod_{j=1}^r a_f(q_j^{k_j})$, and note that since x is B_f-free, q_j is not in B_f for any $1 \leq j \leq r$ and consequently $a_f(q_j) \neq 0$. Let us now see that $a_f(q_j^u) \neq 0$ for any $u \geq 1$. We proceed by induction on u. Let ν_j be the exact exponent of q_j in $a_f(q_j)$. Note that ν_j is well defined since $a_f(q_j) \neq 0$. We may put $\nu_{q_j}(a_f(q_j)) = \nu_j$, where ν_{q_j} denotes the q_j-adic valuation. Let us see by induction that $\nu_{q_j}(a_f(q_j^u)) = u\nu_j$ for any $u \geq 1$. Since $f(z)$ is a newform of weight k and q_j does not divide N, the Fourier coefficients of $f(z)$ satisfy the recursion $a_f(q_j^{u+1}) = a_f(q_j)a_f(q_j^u) - q_j^{k-1}a_f(q_j^{u-1})$ for $u \geq 1$. Taking the q_j-valuation of the terms on the right side of the recursion and using the induction hypothesis we obtain

$$\nu(a_f(q_j)a_f(q_j^u)) = \nu(a_f(q_j)) + \nu(a_f(q_j^u)) = (u+1)\nu_j$$

and

$$\nu(q_j^{k-1}a_f(q_j^{u-1})) = \nu(q_j^{k-1}) + \nu(a_f(q_j^{u-1})) = k-1+(u-1)\nu_j.$$

Note that Deligne's proof of the Weil conjectures implies the precise bound

$$|a_f(q_j)| \leq 2q_j^{\frac{k-1}{2}}.$$

Since q_j is not in B_f, q_j is odd and we have $\nu_j \leq [\frac{k-1}{2}]$. This forces $(u+1)\nu_j < k-1+(u-1)\nu_j$. Hence $\nu(a_f(q_j^{u+1})) = (u+1)\nu_j$ and $a_f(q_j^{u+1}) \neq 0$. This completes the induction step and proves our claim that $a_f(x) \neq 0$. Finally, by the definition of the gap function, $i_f(x - \Phi(M)) = \Phi(M)$ and $i_f(x+1) \geq \Phi(M)$. This completes the proof of the theorem.

In the statement of Theorem 3 we concentrated on large values of $\Phi(M)$, which give us large consecutive gaps in the Fourier expansion of $f(z)$. One can, of course, repeat the proof of Theorem 3 with smaller values of $\Phi(M)$, in particular with $\Phi(M) = K$ a constant independent of M, thereby obtaining a positive proportion, depending on K, of consecutive gaps of size at least K.

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