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LOCALLY FINITE AFFINE COMPLETE VARIETIES

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Abstract

The main results of the paper are the following: 1. Every locally finite affine complete variety admits a near unanimity term; 2. A locally finite congruence distributive variety is affine complete if and only if all its algebras with no proper subalgebras are affine complete and the variety is generated by one of such algebras. The first of these results sharpens a result of McKenzie asserting that all locally finite affine complete varieties are congruence distributive. The second one generalizes the result by Kaarli and Pixley that characterizes arithmetical affine complete varieties.

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1. Introduction

For an algebra $\mathbf{A} = \langle A, F\rangle$ a function $f : A^m \to A$ is said to be *compatible* if it is compatible with all congruence relations of $\mathbf{A}$. All fundamental operations of $\mathbf{A}$ are compatible by the definition of congruence. Obviously all constant functions and all projections (the functions of the form $(x_1, \dots, x_m) \longrightarrow x_i$) are compatible as well. The functions which can be represented as a composition of fundamental operations, constant functions and projections are called *polynomial functions* of the algebra $\mathbf{A}$. The functions which are compositions of fundamental operations and projections are said to be *term functions* of the algebra $\mathbf{A}$. Clearly all polynomial functions are compatible. An algebra is called *hemiprimal* (respectively *affine complete*) if the term functions (respectively the polynomial functions) are its only compatible functions. The notion of compatibility applies also to the partial functions. An algebra is called *locally affine complete* if all of its partial compatible functions with finite domain are

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polynomial. Note that a partial function is said to be polynomial if it is a restriction of a polynomial function.

An algebra is called *arithmetical* if it is both congruence permutable and congruence distributive (CD). A variety is called CD (respectively *arithmetical, affine complete, locally affine complete*) if all its members are so .

The first examples of affine complete varieties were found by Grätzer [3] and Hu [5]. Grätzer proved that the variety of Boolean algebras is affine complete and Hu generalized this result showing that every variety generated by finitely many independent primal algebras is affine complete. The systematic study of affine complete varieties was initiated in [8]. It was motivated by the fact that the known examples of affine complete varieties enjoyed a nice structure, similar to that of the variety of Boolean algebras. Another point was that it was known already that a variety is locally affine complete if and only if it is arithmetical [11]. Though the notions affine complete and locally affine complete are incomparable, it still gave some hope that there is some similarity between affine complete varieties and arithmetical varieties. Later several aspects of affine complete varieties were studied in [6,7,9]. An excellent survey also providing more background is [12].

It was proved in [8] that all affine complete varieties are residually finite. However, so far this result has remained as the only general result holding without restrictions in any affine complete variety. Several basic properties of those varieties were obtained in [8] under the assumption of the congruence distributivity. In particular, we proved the two results listed in the next theorem.

THEOREM 1.1. *Let V be a* CD *affine complete variety. Then*

(1) *all subdirectly irreducible members of V have no proper subalgebras;*
(2) *if V is of finite type then it is generated by a finite algebra with no proper subalgebras and hence is locally finite.*

The reasons why this assumption was imposed were purely practical: otherwise we were unable to handle the compatible functions at the necessary level of generality. However, even before the paper appeared, it turned out that the requirement about congruence distributivity was natural. McKenzie proved that every locally finite affine complete variety is CD. He has never published his proof which uses methods of tame congruence theory [4]. Recently Kearnes turned our attention to the fact that actually the McKenzie's proof works in a considerably more general situation. Namely, if we call an algebra *hereditarily affine complete* if all its quotient algebras are affine complete then the following is true.

THEOREM 1.2 (McKenzie). *Every finite hereditarily affine complete algebra is* CD.

Since a variety is CD if its free algebra in three generators is CD, Theorem 1.2

implies the original result of McKenzie.

THEOREM 1.3. *Every locally finite affine complete variety is* CD.

It is easily seen that a locally finite variety with all subdirectly irreducible members having no proper subalgebras is actually generated by a single finite algebra with no proper subalgebras. Thus Theorem 1.3 together with Theorem 1.1 implies that every locally finite affine complete variety is generated by a finite algebra with no proper subalgebras.

An m-ary function f (with $m \geq 3$) on a set A is called a *near unanimity function* if $f(a_1, \ldots, a_m) = a$ whenever $|\{i \mid a_i = a\}| \geq m - 1$. A ternary near unanimity function is a *majority function*. A term t of type $\mathscr{F}$ is said to be a *near unanimity term* for a variety V of type $\mathscr{F}$ if it induces near unanimity functions on all members of V. It is well known that a variety admitting a near unanimity term is CD [10]. Even more is true: every algebra admitting a compatible near unanimity function is CD.

All affine complete varieties known so far admit a near unanimity term. Therefore it was natural to ask whether this is a general feature or not. Note that the existence of a near unanimity term is an important property of a variety and it yields not only congruence distributivity. It is known that a clone on a finite set is finitely generated provided it contains a near unanimity function. This implies that every locally finite variety with a near unanimity term is equivalent to a variety of finite type. Another area where near unanimity functions play an essential role is general duality theory. For example, every finite algebra with near unanimity term admits a natural duality [2].

In this paper we prove that every finite hereditarily affine complete algebra admits a near unanimity polynomial. This result implies that all locally finite affine complete varieties admit a near unanimity term. Consequently, they are term equivalent to the varieties of finite type. Moreover, every finite member of any affine complete variety admits a natural duality.

In view of Theorem 1.1 it is natural to state:

PROBLEM. *Given a finite affine complete algebra with no proper subalgebras and belonging to a* CD *variety, when does it generate an affine complete variety?*

A nice solution was obtained in [8] for the arithmetical case: an arithmetical variety generated by a finite algebra with no proper subalgebras is affine complete. Note that in this case we do not require the affine completeness of the generating algebra since all finite algebras of arithmetical varieties are affine complete [11]. Clearly this result does not extend to CD varieties. Indeed, as the example of bounded distributive lattices shows, a finite algebra with no proper subalgebras of a CD variety need not be affine complete.

On the other hand, Example 2.2 in [8] shows that non-arithmetical locally finite affine complete varieties do exist. The idea which was used to construct this example was extended in [7] and it will also be used in the present work (Theorem 4.1). We would even say that the present work grew out of the careful analysis of the aforementioned example, originally constructed by Pixley. Trying to extend the proof to more complicated cases, we came to the following problem: given a system of partial functions, one for every quotient algebra of a finite hereditarily affine complete algebra, when is it possible that all these functions are induced by a single global compatible function? A solution of this problem is presented in the next section of this paper. The main results of the paper (Theorems 3.2 and 4.1) are consequences of results of Section 2.

Thus, trying to answer the above question, we cannot avoid the requirement of affine completeness of the given algebra. However, this is still not sufficient, since in [7] we constructed an example of a finite algebra **A** which is hemiprimal and admits a majority term but does not generate an affine complete variety. More precisely, **A** has a quotient algebra which is not affine complete. So the best we may hope is that every finite hereditarily affine complete algebra with no proper subalgebras generates an affine complete variety. The results of the present paper show that even this conjecture is false. We are able to prove the affine completeness of the variety only if we add the requirement that all members of the variety which have no proper subalgebras are affine complete.

2. Compatible function systems

In this section the tools necessary for obtaining the main results are produced. Throughout the section, A is a set and L is a complete sublattice of the lattice of equivalence relations of A. It is assumed that L contains Δ and ∇, the smallest and the largest equivalence relation of A, respectively. We are going to consider systems of partial functions on quotient sets of A. Let us agree that, if not stated otherwise, a partial function f on A is m-ary, meaning that $f : S \to A$ where $S \subseteq A^m$. Here the subset S is a *domain* of f and denoted by $\operatorname{Dom} f$. The empty functions, that is, the partial functions with their domain empty, are not excluded.

The elements of A^m are denoted by lower-case boldface characters. If no other specification occurs, then $\mathbf{a} = (a_1, \dots, a_m)$ where $a_1, \dots, a_m \in A$. If $\rho \in L$, then $\mathbf{a}/\rho = (a_1/\rho, \dots, a_m/\rho) \in (A/\rho)^m$. We write $(\mathbf{a}, \mathbf{b}) \in \rho$ if $\mathbf{a}/\rho = \mathbf{b}/\rho$. The smallest $\rho \in L$ with the property $(\mathbf{a}, \mathbf{b}) \in \rho$ is denoted by $\theta(\mathbf{a}, \mathbf{b})$. A partial function f on A is said to be *compatible* if $(f(\mathbf{a}), f(\mathbf{b})) \in \theta(\mathbf{a}, \mathbf{b})$ for all $\mathbf{a}, \mathbf{b} \in \operatorname{Dom} f$. Clearly the compatibility of f is equivalent to the requirement that f preserves all equivalences $\rho \in L$, that is, $\mathbf{a}, \mathbf{b} \in \operatorname{Dom} f$ and $(\mathbf{a}, \mathbf{b}) \in \rho$ implies $(f(\mathbf{a}), f(\mathbf{b})) \in \rho$. A partial

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function f on the quotient set A/ρ, $\rho \in L$, is said to be compatible if it preserves all equivalences of the form σ/ρ where $\sigma \in L$, $\rho \leq \sigma$. Obviously, if $\rho, \sigma \in L$, $\rho \leq \sigma$, then a compatible partial function on A/ρ induces a compatible partial function on A/σ.

Finally, in what follows we shall identify $\mathbf{a}/\rho_1 \wedge \cdots \wedge \rho_n$ and $(\mathbf{a}/\rho_1, \ldots, \mathbf{a}/\rho_n)$ via the canonical embedding. Hence, $(\mathbf{a}_1/\rho_1, \ldots, \mathbf{a}_n/\rho_n) \in A/\rho_1 \wedge \cdots \wedge \rho_n$ means that there exists $\mathbf{a} \in A^m$ such that $(\mathbf{a}, \mathbf{a}_i) \in \rho_i$, $i = 1, \ldots, n$.

Let now f_ρ be a partial function on A/ρ for any $\rho \in L$. Our aim is to prove that under certain conditions the function system $f = (f_\rho)_{\rho \in L}$ is induced by a single compatible function on A. Definitely these conditions must include the compatibility of all f_ρ but somewhat more is needed.

DEFINITION 2.1. For any $\rho \in L$, let f_ρ be a partial function on A/ρ. Then $f = (f_\rho)_{\rho \in L}$ is said to be a *compatible function system* (CFS) if it satisfies the following two conditions:

($\mathscr{A}$) for every $\rho_1, \rho_2 \in L$ and $\mathbf{a}^i/\rho_i \in \operatorname{Dom} f_{\rho_i}$,

$$f_{\rho_i}(\mathbf{a}^i/\rho_i) = b^i/\rho_i \quad (i = 1, 2) \quad \text{implies} \quad (b^1, b^2) \in \rho_1 \vee \rho_2 \vee \theta(\mathbf{a}^1, \mathbf{a}^2);$$

($\mathscr{B}$) if $f(\mathbf{a}^i/\rho_i) = b^i/\rho_i$, $i = 1, \ldots, n$, and there exist $\mathbf{a} \in A^m$, $\sigma_i \in L$, such that $\rho_i \leq \sigma_i$ and $(\mathbf{a}, \mathbf{a}^i) \in \sigma_i$, $i = 1, \ldots, n$, then $(b^1/\sigma_1, \ldots, b^n/\sigma_n) \in A/\sigma_1 \wedge \cdots \wedge \sigma_n$.

REMARK. We do not require in condition ($\mathscr{B}$) that $\sigma_i \neq \sigma_j$ if $i \neq j$. However, it is easy to see that if ($\mathscr{B}$) is satisfied in this special case then it is also satisfied in general. Indeed, if for example $\sigma_1 = \sigma = \sigma_2$ then $(\mathbf{a}^1, \mathbf{a}^2) \in \sigma$ and by the compatibility of f also $(b^1, b^2) \in \sigma$. Obviously then $(b^2/\sigma_2, \ldots, b^n/\sigma_n) \in A/\sigma_2 \wedge \cdots \wedge \sigma_n$ implies $(b^1/\sigma_1, \ldots, b^n/\sigma_n) \in A/\sigma_1 \wedge \cdots \wedge \sigma_n$.

Let f and g be CFS's on A. We say that g *extends* f if every g_ρ extends f_ρ, $\rho \in L$. In what follows we usually omit the subscript ρ if this does not cause ambiguities. Thus, we write $f(\mathbf{a}/\rho)$ instead of $f_\rho(\mathbf{a}/\rho)$ and $\mathbf{a}/\rho \in \operatorname{Dom} f$ instead of $\mathbf{a}/\rho \in \operatorname{Dom} f_\rho$. Also, if we write $f(\mathbf{a}/\rho) = b/\rho$ then it will always be assumed that $\mathbf{a}/\rho \in \operatorname{Dom} f$.

The next lemma lists the basic properties of CFS's.

LEMMA 2.2. *Let f be a* CFS *on A. Then*

(i) *all f_ρ, $\rho \in L$, are compatible;*

(ii) *if $\rho \leq \sigma$, $\mathbf{a}/\rho, \mathbf{a}/\sigma \in \operatorname{Dom} f$ and $f(\mathbf{a}/\rho) = b/\rho$ then $f(\mathbf{a}/\sigma) = b/\sigma$;*

(iii) *if $\mathbf{a}/\rho_1, \ldots, \mathbf{a}/\rho_n$, $\mathbf{a}/\rho_1 \wedge \cdots \wedge \rho_n \in \operatorname{Dom} f$ then*

$$(f(\mathbf{a}/\rho_1), \ldots, f(\mathbf{a}/\rho_n)) = f(\mathbf{a}/\rho_1 \wedge \cdots \wedge \rho_n).$$

PROOF. The first two properties are direct consequences from Definition 2.1, ($\mathscr{A}$). The third property follows from the second one: if $f(\mathbf{a}/\rho_1 \wedge \cdots \wedge \rho_n) = b/\rho_1 \wedge \cdots \wedge \rho_n$ then by (ii), $(f(\mathbf{a}/\rho_1), \ldots, f(\mathbf{a}/\rho_n)) = (b/\rho_1, \ldots, b/\rho_n) = b/\rho_1 \wedge \cdots \wedge \rho_n$.

A CFS f is said to be *global* if all f_ρ are global functions, that is, Dom $f_\rho = (A/\rho)^m$ for every $\rho \in L$. It follows from Lemma 2.2 that a global CFS is completely determined by f_Δ.

The second claim of Lemma 2.2 asserts that if $\mathbf{a}/\rho \in \operatorname{Dom} f$ and we wish to extend f so that $\mathbf{a}/\sigma$ is in the domain of the extension, then there is at most one way to do so. Similarly, the third claim asserts that if $\mathbf{a}/\rho_1, \ldots, \mathbf{a}/\rho_n \in \operatorname{Dom} f$ and we wish to extend f so that $\mathbf{a}/\rho_1 \wedge \ldots \wedge \rho_n$ is in the domain of the extension, then again there is at most one way to do this. Hence, when trying to prove that a given CFS has a global extension, it is essential first to prove that it has an extension which is closed in the sense of the next definition.

DEFINITION 2.3. A compatible function system f is said to be *closed* if the following two conditions are satisfied:

($\mathscr{C}$) $\mathbf{a}/\rho \in \operatorname{Dom} f,\ \rho \leq \sigma$ implies $\mathbf{a}/\sigma \in \operatorname{Dom} f$;
($\mathscr{D}$) $\mathbf{a}/\rho,\ \mathbf{a}/\sigma \in \operatorname{Dom} f$ implies $\mathbf{a}/\rho \wedge \sigma \in \operatorname{Dom} f$.

It is useful to notice that the conditions ($\mathscr{A}$), ($\mathscr{C}$) and ($\mathscr{D}$) actually imply ($\mathscr{B}$). Indeed, let f be a function system on A which satisfies the aforementioned three conditions and let $\mathbf{a}$, $\mathbf{a}^i$, b^i, ρ_i and σ_i be as in condition ($\mathscr{B}$). Then, because of ($\mathscr{C}$) and ($\mathscr{D}$), $\mathbf{a}_1/\sigma_1, \ldots, \mathbf{a}_n/\sigma_n$ and $\mathbf{a}/\sigma_1 \wedge \cdots \wedge \sigma_n$ are in Dom f. Hence, by Lemma 2.2,

$$\begin{aligned}(b^1/\sigma_1, \ldots, b^n/\sigma_n) &= (f(\mathbf{a}_1/\sigma_1), \ldots, f(\mathbf{a}_n/\sigma_n)) \\ &= (f(\mathbf{a}/\sigma_1), \ldots, f(\mathbf{a}/\sigma_n)) \\ &= f(\mathbf{a}/\sigma_1 \wedge \cdots \wedge \sigma_n) \in A/\sigma_1 \wedge \cdots \wedge \sigma_n.\end{aligned}$$

Now we prove that a CFS has a closed extension provided the lattice L is distributive.

LEMMA 2.4. *For every* CFS f *there exists a* CFS *which extends* f *and satisfies* ($\mathscr{C}$).

PROOF. Let f be a CFS on a set A and define $g = (g_\rho)_{\rho \in L}$ as follows:

$$g(\mathbf{a}/\rho) = b/\rho \quad \Longleftrightarrow \quad \begin{cases} \text{there exist} \quad \tau \in L \quad \text{and} \quad \mathbf{c}/\tau \in \operatorname{Dom} f \\ \text{such that} \quad \tau \leq \rho,\ (\mathbf{a}, \mathbf{c}) \in \rho \quad \text{and} \quad f(\mathbf{c}/\tau) = b/\tau. \end{cases}$$

It follows directly from condition ($\mathscr{A}$) for f that all g_ρ are well-defined. Obviously g extends f and satisfies ($\mathscr{C}$).

Let $g(\mathbf{a}^i/\rho_i) = b^i/\rho_i$, $i = 1, 2$. Then there exist $\tau_i \in L$ and $\mathbf{c}^i/\tau_i \in \operatorname{Dom} f$ such that $\tau_i \le \rho_i$, $(\mathbf{a}^i, \mathbf{c}^i) \in \rho_i$ and $f(\mathbf{c}^i/\tau_i) = b^i/\tau_i$, $i = 1, 2$. Obviously $\theta(\mathbf{c}^1, \mathbf{c}^2) \le \rho_1 \vee \rho_2 \vee \theta(\mathbf{a}^1, \mathbf{a}^2)$ and the compatibility of f implies $(b^1, b^2) \in \tau_1 \vee \tau_2 \vee \theta(\mathbf{c}^1, \mathbf{c}^2) \le \rho_1 \vee \rho_2 \vee \theta(\mathbf{a}^1, \mathbf{a}^2)$. Hence g satisfies $(\mathscr{A})$.

It remains to prove that g satisfies $(\mathscr{B})$. Suppose that $g(\mathbf{a}^i/\rho_i) = b^i/\rho_i$, $\rho_i \le \sigma_i$ and $(\mathbf{a}, \mathbf{a}^i) \in \sigma_i$, $i = 1, \dots, n$. Then there exist $\tau_i \in L$ and $\mathbf{c}^i/\tau_i \in \operatorname{Dom} f$ such that $\tau_i \le \rho_i$, $(\mathbf{a}^i, \mathbf{c}^i) \in \rho_i$ and $f(\mathbf{c}^i/\tau_i) = b^i/\tau_i$, $i = 1, \dots, n$. Now obviously $(\mathbf{a}, \mathbf{c}^i) \in \sigma_i$, $i = 1, \dots, n$, and since f satisfies $(\mathscr{B})$, $(b^1/\sigma_1, \dots, b^n/\sigma_n) \in A/\sigma_1 \wedge \cdots \wedge \sigma_n$. Hence, g satisfies $(\mathscr{B})$, too.

LEMMA 2.5. *If the lattice L is distributive then for every* CFS *f on A there exists a* CFS *which extends f and satisfies $(\mathscr{D})$.*

PROOF. Let f be a CFS on A and define $g = (g_\rho)_{\rho \in L}$ as follows:

$$g(\mathbf{a}/\rho) = b/\rho \iff \begin{cases} \text{there exist } \rho_1, \dots, \rho_n \in L \text{ with } \rho_1 \wedge \cdots \wedge \rho_n = \rho \\ \text{and } f(\mathbf{a}/\rho_i) = b/\rho_i, \; i = 1, \dots, n. \end{cases}$$

First check that all g_ρ are well-defined. Suppose that there exist $\rho_1, \dots, \rho_k$, $\sigma_1, \dots, \sigma_l \in L$ and $b, c \in A$ such that $\rho_1 \wedge \cdots \wedge \rho_k = \rho = \sigma_1 \wedge \cdots \wedge \sigma_l$ and

$$f(\mathbf{a}/\rho_i) = b/\rho_i, \; f(\mathbf{a}/\sigma_j) = c/\sigma_j, \quad i = 1, \dots, k, \; j = 1, \dots, l.$$

Since $\rho_1 \wedge \cdots \wedge \rho_k \wedge \sigma_1 \wedge \cdots \wedge \sigma_l = \rho$ and f satisfies $(\mathscr{B})$, $(b/\rho_1, \dots, b/\rho_k, c/\sigma_1, \dots, c/\sigma_l) \in A/\rho$; let it be equal to d/ρ. Then $(b, d) \in \rho_i$ and $(d, c) \in \sigma_j$, $i = 1, \dots, k$, $j = 1, \dots, l$. Hence both (b, d) and (d, c) are in ρ, implying $b/\rho = c/\rho$.

The next step is to show that g is a CFS. Let $g(\mathbf{a}^1/\rho) = b^1/\rho$ and $g(\mathbf{a}^2/\sigma) = b^2/\sigma$. Then there exist $\rho_1, \dots, \rho_k, \sigma_1, \dots, \sigma_l \in L$ such that $\rho = \rho_1 \wedge \cdots \wedge \rho_k, \sigma = \sigma_1 \wedge \cdots \wedge \sigma_l$ and $f(\mathbf{a}^1/\rho_i) = b^1/\rho_i$, $f(\mathbf{a}^2/\sigma_j) = b^2/\sigma_j$, $i = 1, \dots, k$, $j = 1, \dots, l$. Since f satisfies $(\mathscr{A})$, $(b^1, b^2) \in \rho_i \vee \sigma_j \vee \theta(\mathbf{a}^1, \mathbf{a}^2)$ for all i and j. Applying the distributivity of L, we get $(b^1, b^2) \in \rho \vee \sigma \vee \theta(\mathbf{a}^1, \mathbf{a}^2)$, which means that g satisfies $(\mathscr{A})$ as well

Now we prove that g satisfies $(\mathscr{B})$. Suppose that $g(\mathbf{a}^i/\rho_i) = b^i/\rho_i$, $\rho_i \le \sigma_i$ and $(\mathbf{a}, \mathbf{a}^i) \in \sigma_i$, $i = 1, \dots, n$. Then by definition of g, there exist $\rho_{ij} \in L$ such that $\rho_i = \bigwedge_j \rho_{ij}$ and $f(\mathbf{a}^i/\rho_{ij}) = b^i/\rho_{ij}$, $i = 1, \dots, n$, $j = 1, \dots, k$. (Since repeating ρ_{ij} are not excluded, we may assume that every ρ_i is a meet of the same number k equivalences ρ_{ij}.) Obviously $(\mathbf{a}, \mathbf{a}^i) \in \sigma_i \vee \rho_{ij}$ for every i and j. Since f satisfies $(\mathscr{B})$, we have

$$(b^1/\sigma_1 \vee \rho_{11}, \dots, b^1/\sigma_1 \vee \rho_{1k}, \dots, b^n/\sigma_n \vee \rho_{n1}, \dots, b^n/\sigma_n \vee \rho_{nk}) \in A/\bigwedge_{i,j}(\sigma_i \vee \rho_{ij}).$$

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This means that there is $d \in A$ such that $(d, b^i) \in \sigma_i \vee \rho_{ij}$ for all $i = 1, \ldots, n$, $j = 1, \ldots, k$. Due to the distributivity of L then also $(d, b^i) \in \sigma_i$, $i = 1, \ldots, n$, implying $(b^1/\sigma_1, \ldots, b^n/\sigma_n) \in A/\sigma_1 \wedge \cdots \wedge \sigma_n$.

It remains to prove that g satisfies ($\mathscr{D}$). Suppose that $g(\mathbf{a}/\rho) = b/\rho$ and $g(\mathbf{a}/\sigma) = c/\sigma$. Then, by definition of g, there exist $\rho_1, \ldots, \rho_k, \sigma_1, \ldots, \sigma_l \in L$ such that $\rho = \rho_1 \wedge \cdots \wedge \rho_k$, $\sigma = \sigma_1 \wedge \cdots \wedge \sigma_l$ and $f(\mathbf{a}/\rho_i) = b/\rho_i$, $f(\mathbf{a}/\sigma_j) = c/\sigma_j$, $i = 1, \ldots, k$, $j = 1, \ldots, l$. Since f satisfies ($\mathscr{B}$),

$$\begin{aligned}(b/\rho_1, \ldots, b/\rho_k, c/\sigma_1, \ldots, c/\sigma_l)\\ = (f(\mathbf{a}/\rho_1), \ldots, f(\mathbf{a}/\rho_k), f(\mathbf{a}/\sigma_1), \ldots, f(\mathbf{a}/\sigma_l)) \in A/\rho \wedge \sigma.\end{aligned}$$

Hence there exists $d \in A$ such that $f(\mathbf{a}/\rho_i) = d/\rho_i$, $f(\mathbf{a}/\sigma_j) = d/\sigma_j$, $i = 1, \ldots, k$, $j = 1, \ldots, l$. Since $\rho \wedge \sigma = \rho_1 \wedge \cdots \wedge \rho_k \wedge \sigma_1 \wedge \cdots \wedge \sigma_l$, this implies that $\mathbf{a}/\rho \wedge \sigma \in \operatorname{Dom} g$.

PROPOSITION 2.6. *If the lattice L is distributive then every* CFS *on A has a closed extension.*

PROOF. By Zorn's lemma every CFS has a maximal extension. Due to Lemmas 2.4 and 2.5, the latter must be closed.

3. Near unanimity terms

Now we apply the results of the preceeding section to show that locally finite affine complete varieties admit a near unanimity term. The next theorem is fundamental in this respect but it has another application as well.

THEOREM 3.1. *Let* $\mathbf{A}$ *be a finite hereditarily affine complete algebra, $L = \operatorname{Con} \mathbf{A}$ and let $f = (f_\rho)_{\rho \in L}$ be a* CFS *on A. Then f has a global extension and therefore is induced by some polynomial function of* $\mathbf{A}$.

PROOF. Our proof uses induction on the length of L. If L is trivial then the assertion is trivial. By Theorem 1.2, L is distributive. Hence in view of Proposition 2.6 there exists a closed CFS g on A which extends f. Since A is finite we may assume that g is a maximal closed CFS which extends f. If $\operatorname{Dom} g_\Delta = A$ then we are done. Otherwise there must exist an atom α of L such that $\operatorname{Dom} g_\beta \neq (A/\beta)^m$ where β is the pseudocomplement of α. By the induction hypothesis there is a polynomial function q of $\mathbf{A}$ which induces all g_ρ with $\alpha \leq \rho$.

Now define $h = (h_\rho)_{\rho\in L}$ as follows. First pick an element $\mathbf{a}^0 \in A^m$ such that $\mathbf{a}^0/\beta \notin \operatorname{Dom} g$ and put then $h_\rho = g_\rho$ if $\rho \neq \beta$, and

$$h(\mathbf{a}/\beta) = \begin{cases} g(\mathbf{a}/\beta) & \text{if } \mathbf{a}/\beta \in \operatorname{Dom} g \\ q(\mathbf{a}/\beta) & \text{if } \mathbf{a}/\beta = \mathbf{a}^0/\beta. \end{cases}$$

Thus, $\operatorname{Dom} h_\rho = \operatorname{Dom} g_\rho$ if $\rho \neq \beta$ and $\operatorname{Dom} h_\beta = \operatorname{Dom} g_\beta \cup \{\mathbf{a}^0/\beta\}$. We are going to prove that h is a CFS. Then by Proposition 2.6 it admits a closed extension which contradicts the maximality of g. As a first step, we show that h satisfies $(\mathscr{A})$.

Let $h(\mathbf{a}^i/\rho_i) = b^i/\rho_i$, $i = 1, 2$. Obviously the only non-trivial case is $\mathbf{a}^1/\rho_1 \notin \operatorname{Dom} g$ and $\mathbf{a}^2/\rho_2 \in \operatorname{Dom} g$. Then $\rho_1 = \beta$, $\mathbf{a}^1/\beta = \mathbf{a}^0/\beta$ and $b^1/\beta = q(\mathbf{a}^0)/\beta$. If $\alpha \leq \rho_2$ then $b^2/\rho_2 = q(\mathbf{a}^2)/\rho_2$ and since q is a polynomial,

$$(b^1, b^2) \in \rho_1 \vee \rho_2 \vee \theta(\mathbf{a}^1, \mathbf{a}^2). \tag{1}$$

If $\alpha \not\leq \rho_2$ then $\rho_2 \leq \beta$ and the closedness of g yields $\mathbf{a}^2/\beta \in \operatorname{Dom} g$. Since $\mathbf{a}^1/\beta \notin \operatorname{Dom} g$, we have $(\mathbf{a}^1, \mathbf{a}^2) \notin \beta$. The other consequence of the closedness of g is that $\mathbf{a}^2/\rho_2 \vee \alpha \in \operatorname{Dom} g$. Hence the condition $(\mathscr{A})$ for g implies

$$b^2/\rho_2 \vee \alpha = g(\mathbf{a}^2/\rho_2 \vee \alpha) = q(\mathbf{a}^2)/\rho_2 \vee \alpha$$

and then, by the compatibility of q,

$$(b^1, b^2) \in \rho_1 \vee \rho_2 \vee \alpha \vee \theta(\mathbf{a}^1, \mathbf{a}^2). \tag{2}$$

However, $\theta(\mathbf{a}^1, \mathbf{a}^2) \not\leq \beta$ and therefore $\alpha \leq \theta(\mathbf{a}^1, \mathbf{a}^2)$. Hence, (2) implies (1) and we are done.

Finally, we have to prove that h satisfies $(\mathscr{B})$. Suppose that $h(\mathbf{a}^i/\rho_i) = b^i/\rho_i$, $\rho_i \leq \sigma_i$, and $(\mathbf{a}, \mathbf{a}^i) \in \sigma_i$, $i = 1, \ldots, n$. Since g satisfies $(\mathscr{B})$, we may assume without loss of generality that there is at least one i such that $\rho_i = \beta$ and $\mathbf{a}^i = \mathbf{a}^0$. The case with all $\mathbf{a}^i/\rho_i$ equal to $\mathbf{a}^0/\beta$ is trivial: then

$$(b^1/\sigma_1, \ldots, b^n/\sigma_n) = q(\mathbf{a}^0)/\sigma_1 \wedge \cdots \wedge \sigma_n \in A/\sigma_1 \wedge \cdots \wedge \sigma_n.$$

We first handle the case with $n = 2$, $\mathbf{a}^1/\rho_1 = \mathbf{a}^0/\beta$ and $\mathbf{a}^2/\rho_2 \in \operatorname{Dom} g$, and then show that the general case reduces to this one.

If $\alpha \leq \sigma_2$ then $b^1/\rho_1 = q(\mathbf{a}^1/\rho_1)$ implies $b^1/\sigma_1 = q(\mathbf{a}^1/\sigma_1)$ and due to the closedness of g,

$$b^2/\sigma_2 = g(\mathbf{a}^2/\sigma_2) = q(\mathbf{a}^2/\sigma_2) = q(\mathbf{a}^2)/\sigma_2 = q(\mathbf{a})/\sigma_2.$$

Hence $(b^1/\sigma_1, b^2/\sigma_2) = q(\mathbf{a})/\sigma_1 \wedge \sigma_2 \in A/\sigma_1 \wedge \sigma_2$.

If $\alpha \not\leq \sigma_2$ then $\sigma_2 \leq \beta \leq \sigma_1$. Hence $(b^1/\sigma_1, b^2/\sigma_2) \in A/\sigma_1 \wedge \sigma_2$ is equivalent to $(b^1, b^2) \in \sigma_1$. Since h is a CFS, we have

$$(b^1, b^2) \in \rho_1 \vee \rho_2 \vee \theta(\mathbf{a}^1, \mathbf{a}^2) \leq \sigma_1 \vee \theta(\mathbf{a}^1, \mathbf{a}^2).$$

However, $(\mathbf{a}, \mathbf{a}^1) \in \sigma_1$ and $(\mathbf{a}, \mathbf{a}^2) \in \sigma_2$ imply $(\mathbf{a}^1, \mathbf{a}^2) \in \sigma_1$ and we are done.

Now consider the general case. Assume that $\mathbf{a}^i/\rho_i = \mathbf{a}^0/\beta$ if $i = 1, \ldots, k$, $\mathbf{a}^i/\rho_i \in \operatorname{Dom} g$ if $i = k+1, \ldots, n$ and denote $\tau_1 = \sigma_1 \wedge \cdots \wedge \sigma_k$, $\tau_2 = \sigma_{k+1} \wedge \cdots \wedge \sigma_n$. Since g satisfies $(\mathscr{B})$,

$$(b^{k+1}/\sigma_{k+1}, \ldots, b^n/\sigma_n) = g(\mathbf{a}/\tau_2) = b/\tau_2 \in A/\tau_2.$$

We apply what was proved above for $n = 2$ to the situation with

$$\mathbf{a}^0, \mathbf{a}, \mathbf{a}, b^1, b, \beta, \tau_2, \tau_1, \tau_2 \quad \text{in place of} \quad \mathbf{a}^1, \mathbf{a}^2, \mathbf{a}, b^1, b^2, \rho_1, \rho_2, \sigma_1, \sigma_2,$$

respectively. This is possible, since $(\mathbf{a}^i, \mathbf{a}) \in \sigma_i$ for every $i = 1, \ldots, k$ implies $(\mathbf{a}^0, \mathbf{a}) \in \tau_1$.

Hence there exists $d \in A$ such that $(b^1/\tau_1, b/\tau_2) = d/\tau_1 \wedge \tau_2$. Then obviously $(d, b^1) \in \tau_1 \leq \sigma_i$, $i = 1, \ldots, k$, and in view of $(b, b^j) \in \sigma_j$, $(d, b) \in \tau_2$ implies $(d, b^j) \in \sigma_j$ for $j = k+1, \ldots, n$. Thus $(b^1/\sigma_1, \ldots, b^n/\sigma_n) = d/\sigma_1 \wedge \cdots \wedge \sigma_n$ and we are done.

THEOREM 3.2. *Every finite hereditarily affine complete algebra admits a near unanimity polynomial.*

PROOF. If $|L| \leq 2$ then there is nothing to prove. Let $|L| = m \geq 3$ and define $f = (f_\rho)_{\rho \in L}$ as follows: $f_\rho = \emptyset$ if $\rho \neq \Delta$ and f_Δ is the partial m-ary near unanimity function on $A = A/\Delta$ with domain consisting of all vectors of the form $(a, \ldots, a, b, a, \ldots, a)$ where $a, b \in A$ and b may have an arbitrary position. Obviously f satisfies $(\mathscr{A})$.

To check condition $(\mathscr{B})$ suppose that $f(\mathbf{a}^1) = b^1, \ldots, f(\mathbf{a}^n) = b^n$ and $\sigma_1, \ldots, \sigma_n \in L$ are distinct equivalences such that for some $\mathbf{a} \in A^m$, $(\mathbf{a}, \mathbf{a}^i) \in \sigma_i$, $i = 1, \ldots, n$. If some σ_i is equal to Δ then it follows from $(\mathscr{A})$ that $(b^i, b^j) \in \sigma_j$ for every $j = 1, \ldots, n$ implying $(b^1/\sigma_1, \ldots, b^n/\sigma_n) = b^i \in A$. Thus we may assume that Δ does not occur among σ_i, $i = 1, \ldots, n$, implying $n < m$. Hence by the definition of a near unanimity function, there must exist $j \in \{1, \ldots, m\}$ such that $b^i = a^i_j$ for all $i = 1, \ldots, n$. Then $(b^1/\sigma_1, \ldots, b^n/\sigma_n) = (a^1_j/\sigma_1, \ldots, a^n_j/\sigma_n) = (a_j/\sigma_1, \ldots, a_j/\sigma_n) = a_j/\sigma_1 \wedge \cdots \wedge \sigma_n \in A/\sigma_1 \wedge \cdots \wedge \sigma_n$.

Thus f is a CFS and by Theorem 3.1 it must be induced by some polynomial function of $\mathbf{A}$. This completes the proof.

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Hemiprimal algebras are exactly affine complete algebras all of whose constant functions are term functions. It is easy to see that a hemiprimal algebra is hereditarily affine complete if and only if it is hereditarily hemiprimal, that is, all of its quotient algebras are hemiprimal. Therefore we have the following corollary from Theorem 3.2.

COROLLARY 3.3. *Every finite hereditarily hemiprimal algebra admits a near unanimity term and hence generates a* CD *variety.*

We are especially interested in the algebras **A** having no proper subalgebras. In that case hemiprimality is equivalent to the existence of at least one constant term function. Of course, the hemiprimality is very restrictive condition and it would be pleasant to weaken it in Corollary 3.3. Note that all hemiprimal algebras are *rigid*, that is, they have only the identity automorphism. In general, given a unary term function t of **A**, the set $t(A)$ is a union of Γ-orbits where $\Gamma = \operatorname{Aut}\mathbf{A}$. The extremal case is if $t(A)$ is a Γ-orbit. In [9] we called a finite algebra **A** *weakly diagonal* if it has no proper subalgebras and admits a unary term with this extremal property. The following theorem generalizes Corollary 3.3. The proof follows the idea which was first used in [6].

THEOREM 3.4. *Every weakly diagonal hereditarily affine complete algebra which has no proper subalgebras admits a near unanimity term and hence generates a* CD *variety.*

PROOF. Let **A** be a finite weakly diagonal, hereditarily affine complete algebra which has no proper subalgebras. By Theorem 3.2, **A** admits a near unanimity polynomial and since **A** has no proper subalgebras, we may assume that this polynomial has the form $t(x_1, \dots, x_m, a)$ where $m \geq 3$, t is an $(m+1)$-ary term and $a \in A$. Since **A** is weakly diagonal, there is a unary term u such that $u(A)$ is an orbit under the action of $\Gamma = \operatorname{Aut}\mathbf{A}$. Because **A** has no proper subalgebras, we may assume that $u(A) = \Gamma a$.

Now it is easy to prove that $t(x_1, \dots, x_m, u(x_1))$ is a near unanimity term. Indeed, take arbitrary $x, y \in A$ and let $u(x) = \gamma a$, $u(y) = \delta a$ where $\gamma, \delta \in \Gamma$. Then

$$\begin{aligned} t(x, \dots, x, y, x, \dots, x, u(x)) &= t(x, \dots, x, y, x, \dots, x, \gamma a) \\ &= \gamma(t(\gamma^{-1}x, \dots, \gamma^{-1}x, \gamma^{-1}y, \gamma^{-1}x, \dots, \gamma^{-1}x, a)) = \gamma(\gamma^{-1}x) = x \end{aligned}$$

and

$$\begin{aligned} t(y, x, \dots, x, u(y)) &= t(y, x, \dots, x, \delta a) \\ &= \delta(t(\delta^{-1}y, \delta^{-1}x, \dots, \delta^{-1}x, a)) = \delta(\delta^{-1}x) = x. \end{aligned}$$

The following result sharpens Theorem 1.3.

THEOREM 3.5. *Every locally finite affine complete variety admits a near unanimity term.*

PROOF. Let V be a locally finite affine complete variety and let $\mathbf{A}_i$, $i \in I$, be all subdirectly irreducibles of V. By [8, Corollary 3.2], none of the $\mathbf{A}_i$ have proper subalgebras. Now, if $\mathbf{A}$ is a subalgebra of $\prod_{i \in I} \mathbf{A}_i$ generated by any single element, $\mathbf{A}$ obviously generates V. On the other hand, $\mathbf{A}$ is finite because V is locally finite. Therefore $\mathbf{A}$ can be selected so that it has no proper subalgebras.

Let $\mathbf{F}_1$ be the free algebra in one generator of V. Recall that the elements of $\mathbf{F}_1$ are the unary term functions on $\mathbf{A}$. By Theorem 3.2, $\mathbf{F}_1$ admits a near unanimity polynomial. This means that there exist an $(m + p)$-ary term t and unary terms $w_1, \dots, w_p$ such that

$$t(u(x), \dots, u(x), v(x), u(x), \dots, u(x), w_1(x), \dots, w_p(x)) = u(x) \tag{3}$$

for every two unary terms u and v and for every position of v. Note that (3) is the equality of functions. Thus it holds when replacing the variable x by arbitrary $a \in A$.

We claim that $t(x_1, \dots, x_m, w_1(x_1), \dots, w_p(x_1))$ is then a near unanimity term for V. We have to prove the following two equalities for arbitrary $a, b \in A$:

$$t(a, \dots, a, b, a, \dots, a, w_1(a), \dots, w_p(a)) = a; \tag{4}$$

$$t(b, a, \dots, a, w_1(b), \dots, w_p(b)) = a. \tag{5}$$

(Note that in (4), b is not in the first position.) Since $\mathbf{A}$ has no proper subalgebras, we can find unary terms u and v such that $u(a) = a$ and $v(a) = b$. Hence in view of (3) we have

$$\begin{aligned} &t(a, \dots, a, b, a, \dots, a, w_1(a), \dots, w_p(a)) \\ &\quad = t(u(a), \dots, u(a), v(a), u(a), \dots, u(a), w_1(a), \dots, w_p(a)) \\ &\quad = u(a) = a \end{aligned}$$

proving (4). Likewise, choosing the terms u and v so that $u(b) = b$ and $v(b) = a$, we get the equality (5).

This completes the proof.

4. On finite algebras generating affine complete varieties

We return to the problem we raised in the introduction: given a finite algebra $\mathbf{A}$ which has no proper subalgebras and generates a CD variety V, when is V affine complete? Obviously **A** has to be hereditarily affine complete but as we shall see soon, it is still not enough. First we prove that every algebra $\mathbf{B} \in V$ which contains **A** as a subalgebra, is affine complete.

THEOREM 4.1. *Let V be a* CD *variety generated by a finite hereditarily affine complete algebra* **A** *which has no proper subalgebras. Then all algebras $\mathbf{B} \in V$ which have a subalgebra isomorphic to* **A** *are affine complete.*

PROOF. By Jónsson's lemma, $\operatorname{Var}\mathbf{A}$ is contained in $\operatorname{SPH}\mathbf{A}$. Thus, let $\mathbf{B} \in \operatorname{SPH}\mathbf{A}$ and suppose that **A** is contained in **B** as a subalgebra. Since **A** has no proper subalgebras, **B** is subdirect in $\prod_{i\in I}\mathbf{A}_i$ where $\mathbf{A}_i$ are quotient algebras of **A**; let

$$\mathbf{A}_i = \mathbf{A}/\tau_i, \quad \tau_i \in L = \operatorname{Con}\mathbf{A}, \quad i \in I.$$

We denote the members of $\prod_{i\in I}\mathbf{A}_i$ as families $(a_i/\tau_i)_{i\in I}$ where $a_i \in A$. It is easily seen that the embedding $\mathbf{B} \leq \prod_{i\in I}\mathbf{A}_i$ can be chosen so that every $a \in A$ considered as an element of B is represented as $(a/\tau_i)_{i\in I}$.

Consider an arbitrary compatible function f on **B**. Because of the compatibility, the function f factors as $f = (f_i)_{i\in I}$ where f_i is an m-ary compatible function on $\mathbf{A}_i$, $i \in I$.

Now introduce some notation based on an enumeration of $f(A^m)$:

$$\begin{aligned} f(A^m) &= \{s_1, \dots, s_p\}, & & \\ s_j &= (s_j^i/\tau_i)_{i\in I}, & & s_j^i \in A,\ j = 1, \dots, p, \\ \mathbf{s}^i &= (s_1^i, \dots, s_p^i), & & i \in I. \end{aligned}$$

For every $\rho \in L = \operatorname{Con}\mathbf{A}$ define an $(m+p)$-ary partial function g_ρ on $\mathbf{A}/\rho$ as follows:

$$\begin{aligned} \operatorname{Dom} g_\rho &= (A/\rho)^m \times \{\mathbf{s}^i/\rho \mid i \in I_\rho\}, \\ g_\rho(\mathbf{a}/\rho, \mathbf{s}^i/\rho) &= f_i(\mathbf{a}/\rho) \quad \text{where} \quad I_\rho = \{i \in I \mid \tau_i = \rho\}. \end{aligned}$$

First check that all g_ρ are well-defined. Indeed, suppose that $i \neq j$, $\tau_i = \rho = \tau_j$ and $\mathbf{s}^i/\rho = \mathbf{s}^j/\rho$ and let $f(\mathbf{a}) = f(a_1, \dots, a_m) = s_u$. Then $f_i(\mathbf{a}/\rho) = s_u^i/\rho = s_u^j/\rho = f_j(\mathbf{a}/\rho)$.

We show now that $g = (g_\rho)_{\rho\in L}$ is a CFS. Let $\mathbf{a}^1 = (a_1^1, \dots, a_m^1)$, $\mathbf{a}^2 = (a_1^2, \dots, a_m^2)$ be arbitrary elements of A^m, $i, j \in I$ and

$$\begin{aligned} g(\mathbf{a}^1/\tau_i, \mathbf{s}^i/\tau_i) &= f_i(\mathbf{a}^1/\tau_i) = b^1/\tau_i, \\ g(\mathbf{a}^2/\tau_j, \mathbf{s}^j/\tau_j) &= f_j(\mathbf{a}^2/\tau_j) = b^2/\tau_j. \end{aligned}$$

It follows from the definition of g that $(\mathbf{a}^1/\tau_j, \mathbf{s}^j/\tau_j) \in \operatorname{Dom} g$. Let $g(\mathbf{a}^1/\tau_j, \mathbf{s}^j/\tau_j) = f_j(\mathbf{a}^1/\tau_j) = b/\tau_j$. By compatibility of f_j we have immediately

$$(b^2, b) \in \tau_j \vee \theta(\mathbf{a}^1, \mathbf{a}^2). \tag{6}$$

On the other hand, if $f(\mathbf{a}^1) = s_u$ then

$$b^1/\tau_i = f_i(\mathbf{a}^1/\tau_i) = s_u^i/\tau_i, \tag{7}$$

$$b/\tau_j = f_j(\mathbf{a}^1/\tau_j) = s_u^j/\tau_j. \tag{8}$$

Obviously $(s_u^i, s_u^j) \in \theta(\mathbf{s}^i, \mathbf{s}^j)$; hence by (7) and (8),

$$(b, b^1) \in \tau_i \vee \tau_j \vee \theta(\mathbf{s}^i, \mathbf{s}^j). \tag{9}$$

Now (6) and (9) imply $(b^1, b^2) \in \tau_i \vee \tau_j \vee \theta(\mathbf{a}^1, \mathbf{a}^2) \vee \theta(\mathbf{s}^i, \mathbf{s}^j)$ meaning that g satisfies $(\mathscr{A})$.

The next step is to prove that g satisfies $(\mathscr{B})$ as well. Let $\rho_l \in L$ and $(\mathbf{a}^l/\rho_l, \mathbf{c}^l/\rho_l) \in \operatorname{Dom} g$, $l = 1, \dots, r$. This means that for every $l \in \{1, \dots, r\}$ there is $i_l \in I$ such that $\rho_l = \tau_{i_l}$ and $\mathbf{c}^l = \mathbf{s}^{i_l}$. Let

$$g(\mathbf{a}^l/\rho_l, \mathbf{c}^l/\rho_l) = f_{i_l}(\mathbf{a}^l/\rho_l) = b^l/\rho_l, \quad l = 1, \dots, r.$$

Suppose that there are $\mathbf{a} = (a_1, \dots, a_m) \in A^m$, $\mathbf{c} = (c_1, \dots, c_p) \in A^p$ and $\sigma_l \in L$ such that

$$\rho_l \le \sigma_l, \qquad (\mathbf{a}^l, \mathbf{a}), (\mathbf{c}^l, \mathbf{c}) \in \sigma_l, \quad l = 1, \dots, r.$$

By the definition of g, all $(\mathbf{a}/\rho_l, \mathbf{c}^l/\rho_l)$ are in $\operatorname{Dom} g$. Let $g(\mathbf{a}/\rho_l, \mathbf{c}^l/\rho_l) = d^l/\rho_l$, $l = 1, \dots, r$. Since f_{i_l} is compatible, $(\mathbf{a}, \mathbf{a}^l) \in \sigma_l$ implies

$$(b^l, d^l) \in \sigma_l, \quad l = 1, \dots, r. \tag{10}$$

On the other hand, if $f(\mathbf{a}) = s_u$ then $d^l/\rho_l = f_{i_l}(\mathbf{a}/\rho_l) = s_u^{i_l}/\rho_l$ and consequently

$$(d_l, s_u^{i_l}) \in \sigma_l, \quad l = 1, \dots, r. \tag{11}$$

Because $\mathbf{c}^l = \mathbf{s}^{i_l}$ and $(\mathbf{c}^l, \mathbf{c}) \in \sigma_l$, we also have

$$(c_u, s_u^{i_l}) \in \sigma_l, \quad l = 1, \dots, r. \tag{12}$$

Now (10), (11) and (12) imply $(c_u, b^l) \in \sigma_l$ for every $l = 1, \dots, r$ and we are done.

It remains to show that f is a polynomial function. By Theorem 3.1, g is induced by some $(m+p)$-ary polynomial q of $\mathbf{A}$. We prove that the function $f = f(x_1, \dots, x_m)$ coincides with the m-ary polynomial function $q(x_1, \dots, x_m, s_1, \dots, s_p)$. To do that,

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it is enough to verify that they induce the same function on each $\mathbf{A}_i$, $i \in I$. Take arbitrary $a_1, \ldots, a_m \in A$ and compute:

$$(q(a_1, \ldots, a_m, s_1, \ldots, s_p))_i = g_{\tau_i}(a_1/\tau_i, \ldots, a_m/\tau_i, s_1^i/\tau_i, \ldots, s_p^i/\tau_i)$$
$$= f_i(a_1/\tau_i, \ldots, a_m/\tau_i) = (f(a_1, \ldots, a_m))_i.$$

The proof is complete.

COROLLARY 4.2. *If* **A** *is a weakly diagonal hereditarily affine complete algebra which has no proper subalgebras then* Var **A** *is affine complete.*

PROOF. By Theorem 3.4, Var **A** is CD. Also it is easy to see (cf. [9, Lemma 2.1]) that if **A** is weakly diagonal then every **B** $\in$ Var **A** contains a subalgebra isomorphic to **A**.

Since every algebra of a locally finite variety contains a subalgebra which has no proper subalgebras, we have the following characterization of locally finite affine complete varieties.

COROLLARY 4.3. *A locally finite variety V is affine complete if and only if the following conditions are satisfied:*

(1) *V is* CD*;*
(2) *V is generated by an algebra which has no proper subalgebras;*
(3) *all algebras in V which have no proper subalgebras are affine complete.*

Recall that it was noticed in [12] that if V satisfies the conditions (1) and (2) then it is affine complete whenever all its finite members are. Thus Corollary 4.3 significantly sharpens this result. Now, if we want to check whether a given locally finite CD variety is affine complete, we need to locate its members which have no proper subalgebras. The next proposition will be useful in this respect.

PROPOSITION 4.4. *Let V be a locally finite variety which has an algebra with no proper subalgebras. Then V contains an algebra* **L** *with the property that* **A** $\in V$ *has no proper subalgebras if and only if it is a homomorphic image of* **L**. *The algebra* **L** *is unique, up to isomorphism, and it can be characterized as a minimal subalgebra of the free algebra* $\mathbf{F}_1$ *in one generator of V.*

PROOF. All algebras of V which have no proper subalgebras are homomorphic images of the finite algebra $\mathbf{F}_1$, so we have only finitely many of them. Clearly any minimal subalgebra in their direct product can be chosen as **L**. The uniqueness of **L** is obvious.

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Our final claim follows by applying some elementary semigroup theory. We can think of elements of $\mathbf{F}_1$ as unary term functions on the generating algebra of V. Thus the set F_1 is a monoid under the composition of functions and every algebra in V can be regarded as an act over that monoid. An algebra $\mathbf{A}$ has no proper subalgebras if and only if it has no proper F_1-subacts. Minimal left ideals of the monoid F_1 are exactly universes of minimal subalgebras of $\mathbf{F}_1$. Now, if I is any minimal left ideal of F_1 and $\mathbf{A}$ is an algebra in V with no proper subalgebras then $Ia = A$ for every $a \in A$. Hence the size of A is not greater than the size of I implying that the subalgebra $\mathbf{I} \leq \mathbf{F}_1$ is isomorphic to $\mathbf{L}$.

5. Examples

Recall that a ternary function p on a set A is a *Pixley function* if

$$p(x, y, y) = p(x, y, x) = p(y, y, x) = x$$

for all $x, y \in A$. An important example of Pixley function is the so-called (*ternary*) *discriminator* d defined by $d(x, x, z) = z$ and $d(x, y, z) = x$ if $x \neq y$. It is well known that every algebra admitting a compatible Pixley function is arithmetical. Moreover, a variety is arithmetical if and only if there exists a Pixley term, that is, a ternary term determining a Pixley function on every member of the variety.

We shall use the abbreviation 'FACS algebra' introduced in [9] for referring to finite arithmetical affine complete algebras with no proper subalgebras.

In [6] we considered the following algebra $\mathbf{A} = \langle A; F\rangle$: $A = \{a, b, c, d\}$, $F = \{f, g, h\}$, g and h are unary operations defined by

$$g(a) = g(b) = b, \quad g(c) = g(d) = d, \quad h(a) = h(b) = c, \quad h(c) = h(d) = a$$

and f is a 4-ary operation defined by the following conditions:

(1) $f(x, y, z, a)$ and $f(x, y, z, b)$ are Pixley functions in x, y, z;
(2) f is compatible with the equivalence θ determined by the partition $A = \{a, b\} \cup \{c, d\}$ and induces the discriminator on A/θ;
(3) if $w \in \{c, d\}$ or $|\{x, y, z\}| = 3$ then $f(x, y, z, w) \in \{a, d\}$.

It is easy to see that θ is the only non-trivial congruence of $\mathbf{A}$.

This algebra was constructed as a counter-example to Pixley's conjecture that every FACS algebra generates an arithmetical variety. Note that a finite arithmetical algebra is affine complete if and only if it admits a Pixley polynomial. Thus all finite arithmetical affine complete algebras are hereditarily affine complete.

Later Pixley conjectured (private communication) that maybe FACS algebras still generate affine complete or at least CD varieties. Moreover, concerning the above example **A** he thought that even if Var **A** does not happen to be affine complete, maybe we do get an affine complete variety if we add a suitable majority function to the set F. There is only one way of defining a majority function m on A so that the structure of **A** will not change too drastically: we require that m is compatible with θ and $m(x, y, z) \in \{a, d\}$ if $|\{x, y, z\}| = 3$. The resulting algebra $\langle A; f, g, h, m\rangle$ will be denoted by $\mathbf{A}^*$. Clearly **A** and $\mathbf{A}^*$ have the same congruences.

We are going to show that none of the varieties $V = \operatorname{Var}\mathbf{A}$ and $V^* = \operatorname{Var}\mathbf{A}^*$ is affine complete. In fact V is not even CD.

It is clear from the definition that the 2-element algebras $\mathbf{A}/\theta$ and $\mathbf{A}^*/\theta$ have a non-trivial automorphism. Let $B \subseteq A^2$ be a graph subuniverse determined by this automorphism. Clearly

$$B = A^2\backslash\theta = \{(a, d), (b, d), (a, c), (b, c), (c, b), (d, b), (c, a), (d, a)\}.$$

We also consider the set $C = \{(a, d), (b, d), (a, c), (d, b), (c, a), (d, a)\}$ which is B without (b, c) and (c, b).

Obviously, $g(B) \subseteq C$ and $h(B) \subseteq C$. Also it is easy to see that $f(B^4) \subseteq C$. Indeed, if $x_i, y_i \in A$, $i = 1, 2, 3, 4$, and

$$f((x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)) \in \{(b, c), (c, b)\}$$

then $x_4, y_4 \in \{a, b\}$, hence $(x_4, y_4) \notin B$.

Thus the equivalence ρ determined by the partition $B = C \cup \{(b, c), (c, b)\}$ is a congruence of **B** and the quotient algebra $\mathbf{B}/\rho$ is term equivalent to a pointed set. Consequently the variety generated by $\mathbf{B}/\rho$ is not congruence distributive.

In what follows we need to know unary term functions of $\mathbf{A}^*$. By the lemma of Baker and Pixley [1], the term functions of a finite algebra with a majority term are characterized as those which preserve all subuniverses of the direct square of this algebra. Therefore we first find out all subuniverses of $\mathbf{A}^2$. Suppose that a subuniverse S of $\mathbf{A}^2$ contains a diagonal element. Then, since **A** has no proper subalgebras, the whole diagonal is contained in S. Hence, S is a subuniverse of $(\mathbf{A}^+)^2$ where $\mathbf{A}^+$ is the algebra obtained from **A** by adding constants as new fundamental operations. Clearly $\mathbf{A}^+$ is a hemiprimal arithmetical algebra. It is known that the only subuniverses of the square of a finite hemiprimal arithmetical algebra are congruences. Thus in our case $S \in \{\Delta, \theta, \nabla\}$. Using unary operations g and h we see that every subuniverse of $\mathbf{A}^2$ generated by $(x, y) \in \theta$ intersects Δ. Hence, the subuniverses of $\mathbf{A}^2$ which are not congruences must be contained in B.

Since $f(B^4), g(B), h(B) \subseteq C$, the C, $C \cup \{(b, c)\}$, $C \cup \{(c, b)\}$ and B are subuniverses. On the other hand, the scheme

$$(a, d) \xrightarrow{g} (b, d) \xrightarrow{h} (c, a) \xrightarrow{g} (d, b) \xrightarrow{h} (a, c) \xrightarrow{fh} (d, a) \xrightarrow{fh} (a, d)$$

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shows that C is a minimal subuniverse. (Here fh is a composition of f and h defined by $(fh)(x) = f(h(x), h(x), h(x), h(x))$.)

Thus

$$\Delta, \quad \theta, \quad \nabla, \quad C, \quad C \cup \{(b,c)\}, \quad C \cup \{(c,b)\}, \quad B$$

is the list of all subuniverses of $\mathbf{A}^2$. It is easy to check that the majority function m preserves all these subsets of A^2, hence this is also the list of all subuniverses of $(\mathbf{A}^*)^2$.

It is an easy exercise to check that there are exactly 15 unary functions on A which preserve these 7 subsets of A^2. Only one of them is bijective (the identity map), 8 functions have 3 as the size of their range and the remaining 6 functions map A onto 2-element subsets $\{a, c\}$, $\{b, d\}$ or $\{a, d\}$. In view of Proposition 4.4, these 6 functions must form a minimal subalgebra of $\mathbf{F}_1^*$. Indeed, clearly a minimal left ideal of the monoid F_1 must consist of functions of the minimal possible range. Since our variety contains a 6-element algebra $\mathbf{C}^*$ with no proper subalgebras, the free algebra $\mathbf{F}_1^*$ must contain at least a 6-element minimal subalgebra.

Now we show that $\mathbf{C}^*$ is not affine complete. Since V^* is CD, $\mathbf{C}^*$ has exactly 3 non-trivial congruences: two kernels of projections $\mathbf{C}^* \to \mathbf{A}^*$ and their join. Obviously the function α given by the scheme

$$(a,d) \overset{\alpha}{\longleftrightarrow} (d,a), \quad (b,d) \overset{\alpha}{\longleftrightarrow} (c,a), \quad (d,b) \overset{\alpha}{\longleftrightarrow} (a,c)$$

is compatible with all of them. However, as we shall soon see, α is not a polynomial of $\mathbf{C}^*$ since there is a subuniverse of $(\mathbf{C}^*)^2$ which contains the diagonal but is not preserved by α.

We represent the elements of C^2 as quadruples $(x, y, z, w) = ((x, y), (y, w))$ where $x, y, z, w \in A$ and $(x, y), (z, w) \in C$. Consider the subuniverse D of $(\mathbf{C}^*)^2$ generated by (a, c, b, d). Applying all 15 unary term functions to this quadruple, we see that D contains the diagonal. Though not important for our purposes, we mention that $|D| = 15$ and thus $\mathbf{D}^*$ is isomorphic to the free algebra $\mathbf{F}_1^*$. Since there is only one bijective term function, for every $(x, y, z, w) \in D$ either $(x, y, z, w) = (a, c, b, d)$ or $|\{x, y, z, w\}| \leq 3$. Hence, $\alpha(a, c, b, d) = (d, b, c, a) \notin D$ yielding $\alpha(D) \not\subseteq D$.

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