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Actions of discrete amenable groups into the normalizers of full groups of ergodic transformations

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Dedicated to Professor Yasuyuki Kawahigashi on the occasion of his 60th birthday

Abstract. We apply the Evans–Kishimoto intertwining argument to the classification of actions of discrete amenable groups into the normalizer of a full group of an ergodic transformation. Our proof does not depend on the types of ergodic transformations.

Key words: ergodic transformations, operator algebras, full groups, discrete amenable group, actions
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1. *Introduction*

The purpose of this article is the study actions of discrete amenable groups into the normalizer of a full group of an ergodic transformation on the Lebesgue space. The study of such objects has been motivated by the theory of operator algebras, in particular, the classification of group actions on von Neumann algebras.

The study of automorphism groups of operator algebras is one of the central subjects in the theory of operator algebras, and the classification of automorphisms and group actions has been developed since Connes' seminal works [**5**, **6**]. In particular, classification of actions of discrete amenable groups on injective factors has been completed by many hands [**13**–**16**, **18**, **20**, **21**]. These works heavily depend on the type of factor involved. However, we present a unified approach in [**17**] based on the Evans–Kishimoto method [**9**], and gave a proof that does not depend on the type of the factor in question.

There are corresponding results in ergodic theory. The first result is due to Connes and Krieger [**7**]. They developed the technique of applying ultraproducts to measure spaces and their transformations, and classified transformations (that is, actions of $\mathbb{Z}$) in the normalizer of a full group of type II. The result of Connes and Krieger has been generalized in [**2**] in the case of type II transformation and general discrete amenable groups, in [**1**] in the case of type III$_\lambda$ transformations ($\lambda \neq 0$) and general discrete amenable groups, and finally in

[3] in the case of type III_0 transformation and general discrete amenable groups. These results mentioned above depend on the type of the transformation, and it is natural to expect that our unified approach [17] is valid for the classification of actions of discrete amenable groups into the normalizers of full groups on Lebesgue spaces. In fact, the answer is affirmative and this is the main result of this article.

This classification result is very similar to that of the classification of actions of discrete amenable groups on injective factors. We will explain it in detail. Let $(X, \mathcal{B}, \mu)$ be a Lebesgue space, T an ergodic transformation, and $N[T]$ the normalizer of a full group $[T]$. Let $\alpha : G \to N[T]$ be a homomorphism, which we call an action of G into $N[T]$. The invariant for α is a pair $(N_\alpha, \mathrm{mod}(\alpha))$, where $N_\alpha = \{g \in G \mid \alpha_g \in N[T]\}$ is a normal subgroup of G and $\mathrm{mod}(\alpha)$ is the fundamental homomorphism [11]. (See Theorem 2.4 below for the precise statement of the classification theorem.) However, the invariant of a discrete group action α on a factor $\mathcal{M}$ is the triplet $(N_\alpha, \mathrm{mod}(\alpha), \chi(\alpha))$, where $N_\alpha = \{g \in G \mid \alpha_g \in \mathrm{Cnt}_r(\mathcal{M})\}$ is a normal subgroup of G, $\mathrm{mod}(\alpha)$ is the Connes–Takesaki module, and $\chi(\alpha)$ is the characteristic invariant. (See [15, 17] for details on this notation.) Thus, one can observe the similarity of both classification theorem.

It is also interesting to observe the difference between both classification theorems. Namely, the characteristic invariant $\chi(\alpha)$ does not appear in the ergodic theoretical setting. We consider the case of operator algebras first and explain how the characteristic invariant appears. Let α be an action of a discrete group G on a factor $\mathcal{M}$, and assume $N_\alpha = \{g \in G \mid \alpha_g \in \mathrm{Int}(\mathcal{M})\}$ for simplicity. By the definition of N_α, we can choose $u_n \in U(\mathcal{M})$ with $\alpha_n = \mathrm{Ad}\, u_n$, $n \in N_\alpha$. However, there is no canonical choice of the unitary u_n. Hence, we do not have $u_m u_n = u_{mn}$ and $\alpha_g(u_n) = u_{gng^{-1}}$ in general. A characteristic invariant $\chi(\alpha)$ appears as a cohomological obstruction of these relations.

Next, we consider the ergodic theoretical case. Let $\mathcal{R}_T$ be a Krieger factor associated with T. Then there are canonical homomorphisms $R \in N[T] \to \theta_R \in \mathrm{Aut}(\mathcal{R}_T)$, and $S \in [T] \to U_S \in U(\mathcal{R}_T)$ with $\mathrm{Ad}\, U_S = \theta_S$ and $\theta_R(U_S) = U_{RSR^{-1}}$, $S \in [T]$, $R \in N[T]$. If we lift an action $\alpha : G \to N[T]$ to that on $\mathcal{R}_T$, then the invariant of the lifted action is given by $(N_\alpha, \mathrm{mod}(\alpha), 1)$, due to the above relation between θ_R and U_S. Therefore, the characteristic invariant is trivial in this case.

As we stated at the beginning of this section, our method for the proof of the classification theorem is the application of the Evans–Kishimoto type intertwining argument. To apply it, we need the characterization of full groups and their closures given in [7, 10]. In the study of group actions on operator algebras, two classes of automorphisms play important roles, that is, centrally trivial automorphisms and approximately inner automorphisms. In our case, full groups and their closures correspond to centrally trivial automorphism groups and approximately inner automorphism groups, respectively. Another important tool is the Rohlin type theorem. Combining these results, we first show the cohomology vanishing theorem by the Shapiro type argument in homology theory. Then we obtain the classification theorem by applying the Evans–Kishimoto type intertwining argument.

We expect that this work will shed new light on the relation between ergodic theory and the theory of operator algebras. For example, our result is used in [4] to classify regular subalgebras of type III injective factors.

This paper is organized as follows. In §2, we collect basic facts which will be used in this paper, and state the main results. In §3, we recall the ultraproduct construction of Connes and Krieger, and Ocneanu's Rohlin type theorem. In §4, we show the second cohomology vanishing theorem. In §5, we apply the Evans–Kishimoto type intertwining argument [9] and classify actions of discrete amenable groups into the normalizer of a full group.

2. *Preliminaries*

2.1. *Full groups of ergodic transformations and their normalizers.* In this subsection, we collect known facts on full groups of ergodic transformations and their normalizers which will be used in this article.

Let $(X, \mathcal{B}, \mu)$ be a non-atomic Lebesgue space with $\mu(X) = 1$. (Throughout this article, we treat only non-atomic Lebesgue spaces.) We denote by $\mathrm{Aut}(X, \mu)$ the set of all non-singular transformations. Fix an ergodic transformation $T \in \mathrm{Aut}(X, \mu)$. Let $[T]_*$ be the set of all non-singular bijection $R : A \to B$ for some $A, B \in \mathcal{B}$ such that $Rx \in \{T^n x\}_{n\in\mathbb{Z}}$, $x \in A$. Define the full group of T by $[T] := [T]_* \cap \mathrm{Aut}(X, \mu)$, that is,

$$[T] = \{R \in [T]_* \mid \text{the domain and the range of } R \text{ are both } X\}.$$

We say $E, F \in \mathcal{B}$ are T-equivalent if there exists $R \in [T]_*$ whose domain is E and range is F. A set $E \in \mathcal{B}$ is said to be T-infinite if there exists $F \subset E$ such that $\mu(E\backslash F) > 0$ and F is T-equivalent to E. A set $E \in \mathcal{B}$ is said to be T-finite if it is not T-infinite.

When T is of type II, there exists a unique T-invariant measure m on X ($m(X) < \infty$ when T is of type II_1 and $m(X) = \infty$ when T is of type II_∞). In this case, the following two statements hold: (1) $E \in \mathcal{B}$ is T-finite if and only if $m(E) < \infty$; (2) $E, F \in \mathcal{B}$ are T-equivalent if and only if $m(E) = m(F)$. When T is of type II_1, we always assume μ is the unique T-invariant probability measure.

When T is of type III, then any $E \in \mathcal{B}$ with $\mu(E) > 0$ is T-infinite, and if $E, F \in \mathcal{B}$ satisfy $\mu(E), \mu(F) > 0$, then E and F are T-equivalent. (For instance, see [12, Lemma 8].)

Let $N[T] \subset \mathrm{Aut}(X, \mu)$ be the normalizer of $[T]$. In the following, we use the notation $\hat{\alpha}(t) = \alpha t \alpha^{-1}$ for $t \in [T]$ and $\alpha \in N[T]$.

For $\alpha \in \mathrm{Aut}(X, \mu)$ and $\xi \in L^1(X, \mu)$, define $\alpha_\mu(\xi) \in L^1(X, \mu)$ by

$$\alpha_\mu(\xi)(x) := \xi(\alpha^{-1}x)\frac{d(\mu\circ\alpha^{-1})}{d\mu}(x), \quad \xi \in L^1(X, \mu).$$

Then α_μ is an isometry of $L^1(X, \mu)$, and $(\alpha\beta)_\mu = \alpha_\mu\beta_\mu$ holds for $\alpha, \beta \in \mathrm{Aut}(X, \mu)$.

Let $M(X, \mu)$ (respectively $M_1(X, \mu)$) be the set of complex-valued measures (respectively probability measures) which are absolutely continuous with respect to μ. For $\nu \in M(X, \mu)$, let $\|\nu\| = |\nu|(X)$, where $|\nu|$ is the total variation of ν. Then $M(X, \mu)$ is a Banach space with respect to the norm $\|\nu\|$. For $\xi \in L^1(X, \mu)$, let $\nu_\xi(f) = \int_X \xi(x) f(x) d\mu(x)$. Note that $L^1(X, \mu)$ and $M(X, \mu)$ are isomorphic as Banach spaces by $\xi \mapsto \nu_\xi$. Via this identification, $\alpha_\mu(\xi)$ corresponds to $\alpha(\nu_\xi) = \nu_\xi \circ \alpha^{-1}$. In what follows, we freely use this identification, and we simply denote $\alpha_\mu(\xi)$ by $\alpha(\xi)$ for $\xi \in L^1(X, \mu)$. Thus, $\xi(A)$, $A \in \mathcal{B}$, means $\nu_\xi(A)$.

Recall the topology of $N[T]$ introduced in [12]. For $\alpha, \beta \in \mathrm{Aut}(X, \mu)$, $\{\alpha \neq \beta\}$ denotes the set $\{x \in X \mid \alpha x \neq \beta x\}$. We say a sequence $\{\alpha_n\}_n \subset N[T]$ converges to $\beta \in N[T]$

weakly if $\lim_{n\to\infty} \|\alpha_n(\xi) - \beta(\xi)\| = 0$ for all $\xi \in M(X, \mu)$. Define a metric d_μ by $d_\mu(\alpha, \beta) := \mu(\{\alpha \neq \beta\})$. We say $\{\alpha_n\}_n \subset N[T]$ converges to $\beta \in N[T]$ uniformly if $\lim_{n\to\infty} d_\mu(\alpha_n, \beta) = 0$. This definition does not depend on the choice of equivalence classes of $\mu \in M_1(X, \mu)$. It is shown in [11] that $[T]$ is a Polish group under d_μ.

Now we gift $N[T]$ with a topology as follows. We say a sequence $\{\alpha_n\}_n \subset N[T]$ converges to β in $N[T]$ if $\{\alpha_n\}_n$ converges to β weakly, and $\widehat{\alpha_n}(t)$ converges to $\hat{\beta}(t)$ uniformly for all $t \in [T]$. (In fact, we only have to require convergence for $t \in \{T^n\}_{n\in\mathbb{Z}}$.) This is the right topology for $N[T]$. In fact, this topology coincides with the u-topology for a Krieger factor $\mathcal{R}_T$ constructed from (X, μ, T). So we also call this topology the u-topology. It is shown that $N[T]$ is a Polish group in the u-topology [12]. Indeed, let $\{\xi_k\}_{k=1}^{\infty} \subset L^1(X, \mu)$ be a countable dense subset, and define a metric d on $N[T]$ by

$$d(\alpha, \beta) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|\alpha(\xi_k) - \beta(\xi_k)\|}{1 + \|\alpha(\xi_k) - \beta(\xi_k)\|} + \sum_{k\in\mathbb{Z}} \frac{1}{2^{|k|}} \frac{d_\mu(\hat{\alpha}(T^k), \hat{\beta}(T^k))}{1 + d_\mu(\hat{\alpha}(T^k), \hat{\beta}(T^k))}.$$

Then this d makes $N[T]$ a Polish group, and the topology defined by d is nothing but the u-topology on $N[T]$.

We collect elementary results which will be frequently used in what follows. Since proof is easy, we leave it to the readers.

LEMMA 2.1. *The following statements hold.*

(1) $d_\mu(\theta\alpha, \theta\beta) = d_\mu(\alpha, \beta)$, $d_\mu(\alpha\theta, \beta\theta) = d_{\theta(\mu)}(\alpha, \beta)$, $\alpha, \beta, \theta \in N[T]$. *In particular, we have* $d_\mu(\alpha, \mathrm{id}) = d_\mu(\mathrm{id}, \alpha^{-1}) = d_\mu(\alpha^{-1}, \mathrm{id})$, *and* $d_\mu(\hat{\alpha}(t), \hat{\alpha}(t')) = d_{\alpha^{-1}(\mu)}(t, t')$, $\alpha \in N[T]$, $t, t' \in [T]$.

(2) $d_{\nu_1}(\alpha, \beta) \leq \|\nu_1 - \nu_2\| + d_{\nu_2}(\alpha, \beta)$, $\nu_1, \nu_2 \in M_1(X, \mu)$, $\alpha, \beta \in N[T]$.

(3) *Let* $\nu \in M_1(X, \mu)$, $A, B, C, D \in \mathcal{B}$. *Then we have*

$$\nu((A \cup B)\triangle(C \cup D)) \leq \nu(A\triangle C) + \nu(B\triangle D),$$
$$\nu((A \cap B)\triangle(C \cap D)) \leq \nu(A\triangle C) + \nu(B\triangle D).$$

Recall the definition of the fundamental homomorphism [11]. Let $\tilde{X} := X \times \mathbb{R}$ and μ_L be the Lebesgue measure on $\mathbb{R}$. For $R \in \mathrm{Aut}(X, \mu)$ and $t \in \mathbb{R}$, define $\tilde{R}, F_t \in \mathrm{Aut}(\tilde{X}, \mu \times \mu_L)$ by

$$\tilde{R}(x, u) = \bigg(Rx, u - \log \frac{d(\mu \circ R)}{d\mu}(x)\bigg), \quad F_t(x, u) = (x, u + t).$$

Let (Y, ν_Y) be the quotient space by $\tilde{T}$. Namely, let $\zeta(\tilde{T})$ be a measurable partition of $\tilde{X}$ which generates the σ-algebra consisting of all $\tilde{T}$-invariant set. Then $Y = \tilde{X}/\zeta(\tilde{T})$ and ν_Y is a probability measure which is equivalent to $\mu \otimes \mu_L \circ \pi^{-1}$, where $\pi : \tilde{X} \to Y$ is a quotient map. The Lebesgue space (Y, ν_Y) can be also obtained by $L^\infty(Y, \nu_Y) = L^\infty(\tilde{X}, \mu \times \mu_L)^{\tilde{T}}$.

Since $\tilde{T}$ and F_t commute, we get the ergodic flow (Y, ν_Y, F_t), which is called the associated flow of (X, T). Let

$$\mathrm{Aut}_F(Y, \nu_Y) := \{P \in \mathrm{Aut}(Y, \nu_Y) \mid PF_t = F_tP, t \in \mathbb{R}\}.$$

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When R is in $N[T]$, $\tilde{R}$ induces $\mathrm{mod}(R) \in \mathrm{Aut}_F(Y, \nu)$, which is called the fundamental homomorphism. If we lift R to an automorphism of a Krieger factor $\mathcal{R}_T$, $\mathrm{mod}(R)$ is nothing but a Connes–Takesaki module for R [8].

In this article, we do not use the above definition of $\mathrm{mod}(R)$ explicitly, and what we need is the fact $\mathrm{Ker(mod)} = \overline{[T]}$ (closure is taken in the u-topology) and the surjectivity of mod [10, 11].

2.2. *Main results*

Definition 2.2. Let G be a countable discrete group.

(1) A map (or 1-cochain) $v : G \to [T]$ is said to be normalized if $v(e) = \mathrm{id}$. We denote the set of all normalized maps from G into $[T]$ by $C^1(G, [T])$.

(2) A cocycle crossed action of G into $N[T]$ is a pair of maps $\alpha : G \to N[T]$ and $c : G \times G \to [T]$ such that $\alpha_g\alpha_h = c(g, h)\alpha_{gh}$, $\alpha_e = \mathrm{id}$, $c(e, h) = c(g, e) = \mathrm{id}$. When $c(g, h) = \mathrm{id}$ for all $g, h \in G$, we say α is an action of G into $N[T]$.

(3) Let (α, c) be a cocycle crossed action of G into $N[T]$, and $v \in C^1(G, [T])$. A perturbed crossed action $({}_v\alpha, {}_vc)$ of (α, c) by v is defined by

$$ {}_v\alpha_g := v(g)\alpha_g, \quad {}_vc(g, h) = v(g)\hat{\alpha_g}(v(h))c(g, h)v(gh)^{-1}. $$

(4) Let α be an action of G into $N[T]$. We say a map $v \in C^1(G, [T])$ is a 1-cocycle for α if v satisfies the 1-cocycle identity $v(g)\widehat{\alpha_g}(v(h)) = v(gh)$. It is equivalent to that ${}_v\alpha$ is an action.

(5) Let α and β be actions of G into $N[T]$. We say they are cocycle conjugate if there exist $\theta \in N[T]$ and 1-cocycle $v(\cdot)$ such that ${}_v\alpha_g = \theta\beta_g\theta^{-1}$ for all $g \in G$. If θ is chosen in $\overline{[T]}$, then we say they are strongly cocycle conjugate.

Remark

(1) Let (α, c) be a cocycle crossed action of G. (Notion of a p-action is used in [3].) By $(\alpha_g\alpha_h)\alpha_k = \alpha_g(\alpha_h\alpha_k)$, we can deduce the 2-cocycle identity $c(g, h)c(gh, k) = \widehat{\alpha_g}(c(h, k))c(g, hk)$.

(2) In many works, cocycle conjugacy is said to be outer conjugacy. In fact, we must distinguish these two notions for group actions on operator algebras, However, in ergodic theory, we do not have to distinguish them. (We have the canonical homomorphism $u \in [T]$ into the normalizer of a Krieger factor arising from (X, μ, T).)

At first, we show the following theorem.

THEOREM 2.3. *Let (α, c) be a cocycle crossed action of a discrete amenable group into $N[T]$ with $\alpha_g \notin [T]$, $g \neq e$. Then $c(g, h)$ is a coboundary, that is, there exists $v \in C^1(G, [T])$ such that ${}_vc(g, h) = \mathrm{id}$, equivalently ${}_v\alpha$ is a genuine action of G. If $c(g, h)$ is close to* id*, then we can choose v so that it is also close to* id.

See below for a more precise statement.

Let $N_\alpha := \{g \in G \mid \alpha_g \in [T]\}$, which is a normal subgroup of G. Our main result in this article is the following.

THEOREM 2.4. *Let (X, μ) be a Lebesgue space with $\mu(X) = 1$, T an ergodic transformation on (X, μ). Let G be a countable discrete amenable group, and α, β actions of G into $N[T]$. Then α and β are strongly cocycle conjugate if and only if $N_\alpha = N_\beta$ and* $\mathrm{mod}(\alpha) = \mathrm{mod}(\beta)$.

If α and β are strongly cocycle conjugate, then it is obvious that $N_\alpha = N_\beta$ and $\mathrm{mod}(\alpha_g) = \mathrm{mod}(\beta_g)$. (Amenability of G is unnecessary for this implication.) Thus, the problem is to prove the converse implication, and a proof will be presented in subsequent sections. Here we only state the following corollary, which can be easily verified by Theorem 2.4.

COROLLARY 2.5. *Let α and β be actions of G into $N[T]$. Then α and β are cocycle conjugate if and only if $N_\alpha = N_\beta$ and $\mathrm{mod}(\alpha_g) = \theta\, \mathrm{mod}(\beta_g)\theta^{-1}$ for some $\theta \in \mathrm{Aut}_F(Y, \nu_Y)$.*

Proof. Since the 'only if' part is clear, we only have to prove the 'if' part. Suppose $N_\alpha = N_\beta$ and $\mathrm{mod}(\alpha_g) = \theta\, \mathrm{mod}(\beta_g)\theta^{-1}$ for some $\theta \in \mathrm{Aut}_F(Y, \nu_Y)$. By the surjectivity of mod [10], we can take $\sigma \in N[T]$ with $\mathrm{mod}(\sigma) = \theta$. Then $\mathrm{mod}(\alpha_g) = \mathrm{mod}(\sigma\beta_g\sigma^{-1})$ holds, and hence α_g and $\sigma\beta_g\sigma^{-1}$ are strongly cocycle conjugate by Theorem 2.4. □

3. *Ultraproduct of a Lebesgue space and Rohlin type theorem*

We recall the ultraproduct spaces in [7].

Let $\omega \in \beta\mathbb{N}$ be a free ultrafilter on $\mathbb{N}$. For sequences $(A_n)_n, (B_n)_n \subset \mathcal{B}$, define an equivalence relation $(A_n)_n \sim (B_n)_n$ by $\lim_{n\to\omega} \mu(A_n \triangle B_n) = 0$. Let $\mathcal{B}^\omega := \{(A_n)_n \subset \mathcal{B}\}/\!\sim$. This definition depends only on the equivalence class of μ, and $\mathcal{B}^\omega$ is a boolean algebra.

Any $\alpha \in N[T]$ induces a transformation α^ω on $\mathcal{B}^\omega$ by $\alpha^\omega((A_n)_n) := (\alpha(A_n))_n$. Let

$$\mathcal{B}_\omega := \{\hat{A} \in \mathcal{B}^\omega : t^\omega \hat{A} = \hat{A}, t \in [T]\}.$$

We denote by α_ω the restriction of α^ω on $\mathcal{B}_\omega$.

Let $\hat{A} = (A_n) \in \mathcal{B}_\omega$. Then $\lim_{n\to\omega} \chi_{A_n}$ exists in weak-$*$ topology on $L^\infty(X, \nu)$. By the ergodicity of T, this limit is in $\mathbb{C}$, and does not depend on the choice of representative $\hat{A} = (A_n)$. Thus, we can define $\tau : \mathcal{B}_\omega \to \mathbb{C}$ by $\tau(A) := \lim_{n\to\omega} \chi_{A_n}$. We can see $\tau \circ \alpha_\omega = \tau$ for $\alpha \in N[T]$. By [7, Lemma 2.4], for $\alpha \in N[T]$, $\alpha_\omega = \mathrm{id}$ if and only if $\alpha \in [T]$. In fact, we have a stronger result. For $R \in N[T]$, if there exists $\hat{A} \in \mathcal{B}_\omega$ such that $R_\omega \hat{B} = \hat{B}$ for any $\hat{B} \subset \hat{A}$, $\hat{B} \in \mathcal{B}_\omega$, then $R_\omega = \mathrm{id}$, and hence $R \in [T]$ [7, Lemma 2.3]. This means that R_ω is a free transformation if $R_\omega \neq \mathrm{id}$.

The main tool of this article is the following Rohlin type theorem, essentially due to Ocneanu [18]. (The following formulation is presented in [17].)

THEOREM 3.1. *Let (α, c) be a cocycle crossed action of a discrete amenable group G into $N[T]$ such that $\alpha_{g,\omega} \neq \mathrm{id}$ for all $g \neq e$. Let $K \Subset G$, $\varepsilon > 0$, and S be a (K, ε)-invariant set. (The notation $K \Subset G$ means that K is a finite subset of G.) Then there exists a partition of unity $\{\hat{E}_s\}_{s\in S} \subset \mathcal{B}_\omega$ such that:*

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(1) $\sum_{s\in S_g} \tau(\alpha_{g,\omega}\hat{E}_s \triangle \hat{E}_{gs}) < 5\varepsilon^{\frac{1}{2}}, \quad g \in K$;

(2) $\sum_{s\in S\setminus S_{g^{-1}}} \tau(\hat{E}_s) < 3\varepsilon^{\frac{1}{2}}$,

where $S_g := S \cap g^{-1}S$.

Note that we have $gs \in S_{g^{-1}} = S \cap gS$ for $s \in S_g$.

The proof of [18] is based on the following two facts, that is, the freeness of actions on central sequence algebras, and the ultraproduct technique. In our case, freeness holds as we remarked before Theorem 3.1. Hence, the proof of [17, 18] can be applied in our case by a suitable modification.

Remark. The formulation of our Rohlin type theorem is different from that of the Rohlin theorem by Ornstein and Weiss [19]. The main reason is the use of the ultraproduct technique. Using the ultraproduct technique, Ocneanu showed a very strong result in [18, Lemma 6.3]. Namely, under the same assumption in Theorem 3.1, for any $e \notin A \Subset G$ and $\delta > 0$, he showed the existence of the partition of unity $\{E_i\}_{i=0}^N \subset \mathcal{B}_\omega$ such that $\tau(E_0) < \delta$ and $E_i \cap \alpha_{g,\omega}E_i = \emptyset$ for any $g \in A$, and $i = 1, 2, \ldots, N$. (Thus, $\{\alpha_{g,\omega}E_i\}_{g\in A}$ are disjoint for any $i = 1, \ldots, N$.) This lemma is an important step in constructing a Rohlin tower. (Note that $\tau \circ \alpha_\omega = \tau$ for $\alpha \in N[T]$.) Combining with Zorn's lemma, we can construct a single Rohlin tower as in Theorem 3.1.

In what follows, we say α is an ultrafree action of G if $\alpha_{g,\omega} \neq \mathrm{id}$ for any $g \in G$, $g \neq e$, to distinguish from the usual freeness of actions on Lebesgue spaces.

LEMMA 3.2. *Let A, B be finite sets, $\{E_a\}_{a\in A} \subset \mathcal{B}_\omega$ a partition of X, and $\{P_{a,b}\}_{a\in A, b\in B} \subset [T]$. Choose representatives $E_a = (E_a^n)_n$ such that $E_a^n \cap E_{a'}^n = \emptyset$ for $a \neq a'$, $\bigsqcup_{a\in A} E_a^n = X$. Then for any $\varepsilon > 0$, $\Phi \Subset M_1(X, \mu)$, there exists $N \in \omega$, $\{Z_a^n\}_{a\in A} \subset \mathcal{B}$, $R_b^n \in [T]$, $n \in N$, $b \in B$, such that:*

(1) $\nu(P_{a,b}^{-1}E_a^n \triangle E_a^n) < \varepsilon, n \in N, \nu \in \Phi$;

(2) $Z_a^n \subset E_a^n$, $P_{a,b}Z_a^n \subset E_a^n$, $n \in N$;

(3) $\nu(E_a^n \setminus Z_a^n) < \varepsilon$, $\nu(E_a^n \setminus P_{a,b}Z_a^n) < \varepsilon, n \in N, \nu \in \Phi$;

(4) $R_b^n x = P_{a,b}x, n \in N, x \in Z_a^n$.

Proof. Since $P_{a,b}E_a = E_a$ by [7, Lemma 2.4], there exists $N \in \omega$ such that

$$P_{a,b}(\nu)\bigg(\bigg(E_a^n \cup \bigcup_{b\in B} P_{a,b}^{-1}E_a^n\bigg)\bigg\backslash\bigg(E_a^n \cap \bigcap_{b\in B} P_{a,b}^{-1}E_a^n\bigg)\bigg) < \frac{\varepsilon}{2}$$

for $n \in N, a \in A, b \in B, \nu \in \Phi$.

Let $Y_a^n := E_a \cap \bigcap_{b\in B} P_{a,b}^{-1}E_a^n$. Clearly we have $Y_a^n, P_{a,b}Y_a^n \subset E_a^n$. Moreover,

$$\nu(P_{a,b}^{-1}E_a^n \triangle E_a^n) < \frac{\varepsilon}{2}, \quad \nu(E_a^n \setminus Y_a^n) < \frac{\varepsilon}{2}, \quad \nu(E_a^n \setminus P_{a,b}Y_a^n) = P_{a,b}(\nu)(P_{a,b}^{-1}E_a^n \setminus Y_a^n) < \frac{\varepsilon}{2}$$

hold for $n \in N$, $\nu \in \Phi$. Let $Y^n := \bigsqcup_{a\in A} Y_a^n$. Thus, we can define $R_{0,b}^n \in [T]_*$ with $\mathrm{Dom}(R_{0,b}^n) = Y^n$ by $R_{0,b}^n x = P_{a,b}x$, $x \in Y_a^n$. If $X\setminus Y^n$ and $X\setminus R_{0,b}^n Y^n$ are T-equivalent, then we can extend $R_{0,b}^n$ to an element $R_b^n \in [T]$.

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At first, let us assume that Y^n is T-finite. (Thus, so is $R_{0,b}Y^n$.) Such a case can happen if T is of type II. Then $X\backslash Y^n$ and $X\backslash R_{0,b}^n Y^n$ are T-equivalent. Hence, we can extend $R_{0,b}$ to $R_b \in [T]$. Set $Z_a^n := Y_a^n$. Then all the statements in the lemma are satisfied.

Next, let us assume that Y^n is T-infinite. (Hence, so is $R_{0,b}^n Y^n$.) Take $W_k \subset Y^n$, $k \in \mathbb{N}$, such that $W_k \subset W_{k+1}$, $\bigcup_k W_k = Y^n$, and $Y^n \backslash W_k$ are T-infinite for all k. Set $Z_{a,k}^n := Y_a^n \cap W_k$. Of course, we have $Z_{a,k}^n \subset Z_{a,k+1}^n$, $\bigcup_k Z_{a,k}^n = Y_a^n$, $\bigsqcup_{a\in A} Z_{a,k}^n = Y^n \cap W_k = W_k$, and $Z_{a,k}^n$, $P_{a,b}Z_{a,k}^n \subset E_a^n$. Thus, $\{Z_{a,k}^n\}_{a\in A}$ satisfies condition (2).

Take sufficiently large k such that

$$\nu(Y_a^n \backslash Z_{a,k}^n) < \frac{\varepsilon}{2}, \quad \nu(P_{a,b}Y_a^n \backslash P_{a,b}Z_{a,k}^n) < \frac{\varepsilon}{2}$$

for $a \in A$, $b \in B$, $\nu \in \Phi$. Then it is clear that $\{Z_{a,k}^n\}$ satisfies condition (3). By the choice of $\{W_k\}$, $X\backslash \bigsqcup_{a\in A} Z_{a,k}^n \supset Y^n \backslash W_k$ and $X\backslash R_{0,b}^n \bigsqcup_{a\in A} Z_{a,k}^n \supset R_{0,b}^n(Y^n\backslash W_k)$. It follows that $X\backslash \bigsqcup_{a\in A} Z_{a,k}^n$ and $X\backslash R_{0,b}^n \bigsqcup_{a\in A} Z_{a,k}^n$ are both T-infinite and hence are equivalent. Thus, $Z_a^n := Z_{a,k}^n$ satisfies all statements in the lemma. □

Now we can combine Theorem 3.1 and Lemma 3.2 as follows.

PROPOSITION 3.3. *Let G be a discrete amenable group, and (α, c) an ultrafree cocycle crossed action of G into $N[T]$. Let $K \Subset G$ and $\varepsilon > 0$ be given, and S a (K, ε)-invariant set. Let B, C be finite sets, $\{P_{s,b}\}_{s\in S, b\in B} \subset [T]$, $\{\nu_s^c\}_{s\in S, c\in C} \Subset M_1(X, \mu)$. Then for any $\delta > 0$, there exists a partition $\{E_s\}_{s\in S} \subset \mathcal{B}$ of X, $E_s \supset Z_s$, and $R_b \in [T]$, $b \in B$, such that:*

(1) $\sum_{s\in S_g} \nu_s^c(\alpha_g E_s \triangle E_{gs}) < 5\varepsilon^{1/2}$, $g \in K$, $c \in C$;
(2) $\sum_{s\in S\backslash S_{g^{-1}}} \nu_s^c(E_s) < 3\varepsilon^{1/2}$, $g \in K$, $c \in C$;
(3) $\nu_s^c(P_{s,b}^{-1}E_s \triangle E_s) < \delta$, $s \in S$, $b \in B$, $c \in C$;
(4) $P_{s,b}Z_s \subset E_s$, $s \in S$, $b \in B$;
(5) $\nu_s^c(E_s\backslash Z_s) < \delta$, $\nu_s^c(E_s\backslash P_{s,b}Z_s) < \delta$, $s \in S$, $b \in B$, $c \in C$;
(6) $R_b x = P_{s,b}x$, $s \in S$, $b \in B$, $x \in Z_s$,

where $S_g := S \cap g^{-1}S$.

Proof. Let $\{\hat{E}_s\}_{s\in S} \subset \mathcal{B}_\omega$ be a Rohlin partition as in Theorem 3.1. Since $\tau(\hat{A}) = \lim_{n\to\omega} \chi_{A_n}$ for $\hat{A} = (A_n)_n \in \mathcal{B}_\omega$, $\tau(\hat{A}) = \lim_{n\to\omega} \nu(A^n)$ for any $\nu \in M_1(X, \mu)$. Choose a representative $\hat{E}_s = (E_s^n)_n$ such that $E_s^n \cap E_{s'}^n = \emptyset$, $\bigsqcup_{s\in S} E_s^n = X$. By Theorem 3.1:

(1) $\lim_{n\to\omega} \sum_{s\in S_g} \nu_s^c(\alpha_g E_s^n \triangle E_{gs}^n) < 5\varepsilon^{1/2}$, $g \in K$;
(2) $\lim_{n\to\omega} \sum_{s\in S\backslash S_{g^{-1}}} \nu_s^c(E_s^n) < 3\varepsilon^{1/2}$, $g \in K$

hold for any $\{\nu_s^c\}_{s\in S, c\in C} \subset M_1(X, \mu)$. Thus, there exists $N_1 \in \omega$ such that

$$\sum_{s\in S_g} \nu_s^c(\alpha_g E_s^n \triangle E_{gs}^n) < 5\varepsilon^{1/2}, \quad g \in K, c \in C,$$

$$\sum_{s\in S\backslash S_{g^{-1}}} \nu_s^c(E_s^n) < 3\varepsilon^{1/2}, \quad g \in K, c \in C$$

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for all $n \in N_1$. By Lemma 3.2, there exists $N_2 \in \omega$, $Z_s^n \subset E_s^n$, and $R_b^n \in [T]$, $(n \in N_2)$, such that

$$\begin{aligned}
&\nu_s^c(P_{s,b}^{-1}E_s^n \triangle E_s^n) < \delta, \quad s \in S, b \in B,\\
&P_{s,b}Z_s^n \subset E_s^n, \quad s \in S, b \in B,\\
&\nu_s^c(E_s^n \backslash Z_s^n) < \delta, \quad \nu_s^c(E_s^n \backslash P_{s,b}Z_s^n) < \delta, \quad s \in S, b \in B, c \in C,\\
&R_b^n x = P_{s,b}x, \quad s \in S, b \in B, x \in Z_s^n
\end{aligned}$$

for any $n \in N_2$. Fix $n \in N_1 \cap N_2$, and set $E_s := E_s^n$, $Z_s := Z_s^n$, $R_b := R_b^n$. Then these E_s, Z_s, R_b are desired objects. □

4. *Cohomology vanishing*

At first, we show the following second cohomology vanishing result, which is shown in [**3**, Theorem 1.3]. We present the proof for readers' convenience.

THEOREM 4.1. *Let T be a transformation of type II_∞ or type III, and (γ, c) a cocycle crossed action of a discrete group G into $N[T]$. Then $c(g, h)$ is a coboundary, that is, there exists $u \in C^1(G, [T])$ such that ${}_u c(g, h) = \mathrm{id}$.*

Proof. Since T is of type II_∞ or type III, there exists a partition $\{E_h\}_{h\in G}$ of X such that each E_h is T-infinite. Let $\{f_{g,h}\}_{g,h\in G} \subset [T]$ be an array for $\{E_g\}_{g\in G}$, that is, $\{E_g\}_{g\in G}$ is a partition of X and $f_{g,h} \in [T]_*$ is a bijection from E_h onto E_g such that $f_{g,h}f_{h,k} = f_{g,k}$. Take $v_g^0 \in [T]_*$ with $\mathrm{Dom}(v_g^0) = \gamma_g E_e$ and $\mathrm{Ran}(v_g^0) = E_e$. Define $v(g) \in [T]$ by $f_{h,e}v_g^0\gamma_g(f_{e,h})$ on $\gamma_g E_h$. Then we have ${}_v\gamma_g : E_h \to E_h$ and $\widehat{{}_v\gamma_g}(f_{h,k}) = f_{h,k}$ for any $g, h, k \in G$. Replacing (γ, c) with $({}_v\gamma, {}_vc)$, we may assume $\gamma_g E_k = E_k$ and $\widehat{\gamma_g}(f_{h,k}) = f_{h,k}$. Since $\gamma_g\gamma_h = c(g, h)\gamma_{gh}$, we also have $c(g, h)E_k = E_k$ and $\widehat{c(g, h)}(f_{k,l}) = f_{k,l}$.

Next define $u(g) \in [T]$ by $u(g) = c(g, l)^{-1} f_{gl,l}$ on E_l. Note $u(g)$ sends E_l to E_{gl}, and hence so does ${}_u\gamma_g$. Hence, for $x \in E_l$,

$$\begin{aligned}
{}_u\gamma_g {}_u\gamma_h x &= u(g)\gamma_g c(h, l)^{-1} f_{hl,l}\gamma_h x = u(g)\gamma_g c(h, l)^{-1}\gamma_h f_{hl,l}x\\
&= u(g)\widehat{\gamma_g}(c(h, l))^{-1}\gamma_g\gamma_h f_{hl,l}x = u(g)\widehat{\gamma_g}(c(h, l))^{-1}c(g, h)\gamma_{gh} f_{hl,l}x\\
&= c(g, hl)^{-1} f_{ghl,hl}\widehat{\gamma_g}(c(h, l))^{-1}c(g, h)\gamma_{gh} f_{hl,l}x\\
&= c(g, hl)^{-1}\widehat{\gamma_g}(c(h, l))^{-1}c(g, h)\gamma_{gh} f_{ghl,hl} f_{hl,l}x\\
&= c(gh, l)^{-1} f_{ghl,l}\gamma_{gh}x = u(gh)\gamma_{gh}x.
\end{aligned}$$

This implies that ${}_u\gamma$ is an action, and ${}_uc(g, h) = u(g)\widehat{\gamma_g}(u(h))c(g, h)u(gh)^{-1} = \mathrm{id}$ holds. □

In Theorem 4.1, we have no estimation on the choice of $u(g)$, even if $c(g, h)$ is close to id. The rest of this section is devoted to solving this problem. From now on, we always assume that G is a discrete amenable group.

For all $g \in G$ and $S \Subset G$, fix a bijection $l(g) : S \to S$ such that $l(g)s = gs$ if $gs \in S$.

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LEMMA 4.2. *Let (γ, c) be an ultrafree cocycle crossed action of G. For any $\varepsilon > 0$, $K \Subset G$, $\mu \in \Phi \Subset M_1(X, \mu)$, there exists $w \in C^1(G, [T])$ such that*

$$d_\nu({}_w c(g, h), \mathrm{id}) < \varepsilon, \quad g, h \in K, \nu \in \Phi.$$

Moreover, for given $\varepsilon > 0$, $e \in K \Subset G$, there exist $\delta > 0$ and $S \Subset G$, which depend only on K and $\varepsilon > 0$, such that if

$$\|c(g, h)(\xi) - \xi\| < \delta, \quad d_\nu(\widehat{c(g, h)}(t), t) < \delta, \quad g, h \in S, t \in \Lambda, \xi, \nu \in \Phi$$

for some cocycle crossed action (γ, c), $\Lambda \Subset [T]$ and $\Phi \Subset M_1(X, \mu)$, then we can choose $w \in C^1(G, [T])$ so that it further satisfies

$$\|w(g)(\xi) - \xi\| < \varepsilon, \quad d_\nu(\widehat{w(g)}(t), t) < \varepsilon, \quad g \in K, \xi, \nu \in \Phi, t \in \Lambda.$$

Proof. Choose $\varepsilon' > 0$ with $11\sqrt{\varepsilon'} < \varepsilon$, and let $S' \subset G$ be a $(K \cup K^2, \varepsilon')$-invariant set and $S = S' \cup K$. Choose δ such that $5\delta|S| + 11\sqrt{\varepsilon'} < \varepsilon$.

By applying Proposition 3.3, we can take a Rohlin partition $\{E_s\}_{s \in S'} \subset \mathcal{B}$, $Z_s \subset E_s$, $w(g) \in [T]$, $g \in K$, such that:

(1) $E_{l(g)s} \supset c(g, s)^{-1} Z_{l(g)s}$, $g \in K \cup K^2$, $s \in S'$;
(2) $\nu(E_s \backslash Z_s) < \delta$, $\nu(E_{l(g)s} \backslash c(g, s)^{-1} Z_{l(g)s}) < \delta$, $g \in K \cup K^2$, $s \in S'$, $\nu \in \Phi$;
(3) $\nu(c(gh, k)^{-1} c(g, h)^{-1} \widehat{\gamma}_g(c(h, k))(E_{ghk} \backslash Z_{ghk})) < \delta$, $g, h \in K$, $k \in S'gh \cap S'h$, $\nu \in \Phi$;
(4) $\nu(c(gh, k)^{-1} c(g, h)^{-1} \gamma_g(E_{hk} \backslash Z_{hk})) < \delta$, $g, h \in K$, $k \in S'h$, $\nu \in \Phi$;
(5) $\nu(E_{ghk} \triangle c(gh, k)^{-1} c(g, h)^{-1} \widehat{\gamma}_g(c(h, k)) E_{ghk}) < \delta$, $g, h \in K$, $k \in S'h$, $\nu \in \Phi$;
(6) $\nu(E_{ghk} \triangle c(gh, k)^{-1} c(g, h)^{-1} E_{ghk}) < \delta$, $g, h \in K$, $k \in S'gh \cap S'h$, $\nu \in \Phi$;
(7) $\sum_{k \in S'gh \cap S'h} \nu(c(gh, k)^{-1} c(g, h)^{-1} (E_{ghk} \triangle \gamma_g E_{hk})) < 5\sqrt{\varepsilon'}$, $g, h \in K$, $\nu \in \Phi$;
(8) $\sum_{k \in S' \backslash S'(gh)^{-1}} \nu(E_s) < 3\sqrt{\varepsilon'} g \in K \cup K^2$, $\nu \in \Phi$;
(9) $w(g)x = c(g, s)^{-1} x$, $x \in Z_{l(g)s}$, $g \in K$, $s \in S'$.

Here we applied Proposition 3.3 for

$$\begin{aligned} B = \{c(g, s)^{-1} \mid g \in K, s \in S'\} &\cup \{c(gh, k)^{-1} c(g, h)^{-1} \mid g, h \in K, k \in S'\} \\ &\cup \{c(gh, k)^{-1} c(g, h)^{-1} \widehat{\gamma}_g(c(h, k)) \mid g, h \in K, k \in S'\} \end{aligned}$$

and

$$\begin{aligned} C = \Phi &\cup \{\nu(c(gh, k)^{-1} c(g, h)^{-1} \widehat{\gamma}_g(c(h, k)) \cdot) \mid \nu \in \Phi, g, h \in K, k \in S'\} \\ &\cup \{\nu(c(gh, k)^{-1} c(g, h)^{-1} \gamma_g \cdot) \mid \nu \in \Phi, g, h \in K, k \in S'\} \\ &\cup \{\nu(c(gh, k)^{-1} c(g, h)^{-1} \cdot) \mid \nu \in \Phi, g, h \in K, k \in S'\}. \end{aligned}$$

We define $w(g) = \mathrm{id}$ if $g \notin G$.

Let

$$\begin{aligned} W^0_{g,h,k} = c(gh, k)^{-1} Z_{ghk} &\cap c(gh, k)^{-1} c(g, h)^{-1} \gamma_g Z_{hk} \\ &\cap c(gh, k)^{-1} c(g, h)^{-1} \widehat{\gamma}_g(c(h, k)) Z_{ghk} \end{aligned}$$

for $k \in S'_{g,h} \cap S'_h$ and

$$W_{g,h} = \bigcup_{k \in S'_{gh} \cap S'_h} W^0_{g,h,k}.$$

We can verify $w(g)\hat{\gamma}_g(w(h))c(g,h)w(gh)^{-1} = \mathrm{id}$ on $W_{g,h}$ as follows. Take $x \in W^0_{g,h,k}$. Since $x \in c(gh,k)^{-1}Z_{ghk}$, we have $w(gh)^{-1}x = c(gh,k)x$. Thus, we have

$$\gamma_g^{-1}c(g,h)w(gh)^{-1}x = \gamma_g^{-1}c(g,h)c(gh,k)x.$$

Since $x \in c(gh,k)^{-1}c(g,h)^{-1}\gamma_g Z_{hk}$, $\gamma_g^{-1}c(g,h)c(gh,k)x \in Z_{hk}$ holds. Hence, we have

$$w(h)\gamma_g^{-1}c(g,h)c(gh,k)x = c(h,k)^{-1}\gamma_g^{-1}c(g,h)c(gh,k)x.$$

Since $x \in c(gh,k)^{-1}c(g,h)^{-1}\widehat{\gamma_g}(c(h,k))Z_{ghk}$,

$$\widehat{\gamma_g}(w(h))c(g,h)w(gh)^{-1}x = \widehat{\gamma_g}(c(h,k))^{-1}c(g,h)c(gh,k)x \in Z_{ghk}$$

holds, and hence we have

$$w(g)\gamma_g c(h,k)^{-1}\gamma_g^{-1}c(g,h)c(gh,k)x = c(g,hk)^{-1}\gamma_g c(h,k)^{-1}\gamma_g^{-1}c(g,h)c(gh,k)x = x$$

by the 2-cocycle identity. These computations show

$$_w c(g,h) = w(g)\widehat{\gamma_g}(w(h))c(g,h)w(gh)^{-1} = \mathrm{id}$$

on $W_{g,h}$. Thus, we have $\{_w c(g,h) \neq \mathrm{id}\} \subset X \backslash W_{g,h}$.

We will show $\nu(X\backslash W_{g,h}) < \varepsilon$ for $\nu \in \Phi$. By condition (2), we have

$$\nu(E_{ghk}\backslash c(gh,k)^{-1}Z_{ghk}) < \delta, \quad \nu \in \Phi, g, h \in Kk \in S'_{gh}.$$

For $g, h \in K$, $k \in S'_{gh} \cap S'_h$, $\nu \in \Phi$, we have

$$\begin{aligned}
&\nu(E_{ghk} \triangle c(gh,k)^{-1}c(g,h)^{-1}\gamma_g Z_{hk})\\
&\quad \leq \nu(E_{ghk} \triangle c(gh,k)^{-1}c(g,h)^{-1}E_{ghk}) + \nu(c(gh,k)^{-1}c(g,h)^{-1}(E_{ghk} \triangle \gamma_g Z_{hk}))\\
&\quad \leq \delta + \nu(c(gh,k)^{-1}c(g,h)^{-1}(E_{ghk} \triangle \gamma_g Z_{hk})) \quad \text{(by condition (6))}\\
&\quad \leq \delta + \nu(c(gh,k)^{-1}c(g,h)^{-1}(E_{ghk} \triangle \gamma_g E_{hk}))\\
&\qquad + \nu(c(gh,k)^{-1}c(g,h)^{-1}\gamma_g(E_{hk} \triangle Z_{hk}))\\
&\quad \leq 2\delta + \nu(c(gh,k)^{-1}c(g,h)^{-1}(E_{ghk} \triangle \gamma_g E_{hk})) \quad \text{(by condition (4))}
\end{aligned}$$

and

$$\begin{aligned}
&\nu(E_{ghk} \triangle c(gh,k)^{-1}c(g,h)^{-1}\widehat{\gamma_g}(c(h,k))Z_{ghk})\\
&\quad \leq \nu(E_{ghk} \triangle c(gh,k)^{-1}c(g,h)^{-1}\widehat{\gamma_g}(c(h,k))E_{ghk})\\
&\qquad + \nu(c(gh,k)^{-1}c(g,h)^{-1}\widehat{\gamma_g}(c(h,k))(E_{ghk} \triangle Z_{ghk}))\\
&\quad < 2\delta \quad \text{(by conditions (5) and (3))}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\nu(E_{ghk} \triangle W^0_{g,h,k}) &\leq \nu(E_{ghk}\backslash c(gh,k)^{-1}Z_{ghk}) + \nu(E_{ghk} \triangle c(gh,k)^{-1}c(g,h)^{-1}\gamma_g Z_{hk})\\
&\qquad + \nu(E_{ghk} \triangle c(gh,k)^{-1}c(g,h)^{-1}\gamma_g c(h,k)\gamma_g^{-1}Z_{ghk})\\
&< 5\delta + \nu(c(gh,k)^{-1}c(g,h)^{-1}(E_{ghk} \triangle \gamma_g E_{hk}))
\end{aligned}$$

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follows. Then

$$\begin{aligned}\nu\bigg(\bigg(\bigcup_{k\in S'_{gh}\cap S'_h} E_{ghk}\bigg)\triangle W_{g,h}\bigg) &\leq \sum_{k\in S'_{gh}\cap S'_h} \nu(E_{ghk}\triangle W^0_{g,h,k})\\ &< 5\delta|S| + \sum_{k\in S'_{gh}\cap S'_h} \nu(c(gh,k)^{-1}c(g,h)^{-1}(E_{ghk}\triangle\gamma_g E_{hk}))\\ &< 5\delta|S| + 5\sqrt{\varepsilon'} \quad \text{(by condition (7))}\end{aligned}$$

holds.

Finally, we have

$$\begin{aligned}\nu(X\backslash W_{g,h}) &\leq \nu\bigg(X\backslash \bigcup_{k\in S'_{gh}\cap S'_h} E_{ghk}\bigg) + \nu\bigg(\bigg(\bigcup_{k\in S'_{gh}\cap S'_h} E_{ghk}\bigg)\triangle W_{g,h}\bigg)\\ &= \nu\bigg(\bigcup_{k\in S\backslash(S'_{g^{-1}}\cap S'_{(gh)^{-1}})} E_k\bigg) + \nu\bigg(\bigg(\bigcup_{k\in S'_{gh}\cap S'_h} E_{ghk}\bigg)\triangle W_{g,h}\bigg)\\ &< \sum_{k\in S'\backslash S'_{g^{-1}}} \nu(E_k) + \sum_{k\in S'\backslash S'_{(gh)^{-1}}} \nu(E_s) + 5\delta|S| + 5\sqrt{\varepsilon'}\\ &< 5\delta|S| + 11\sqrt{\varepsilon'} \quad \text{(by condition (8))}\\ &< \varepsilon.\end{aligned}$$

(Note $ghk \in S'_{g^{-1}} \cap S'_{(gh)^{-1}}$ for $k \in S'_h \cap S'_{gh}$.) This inequality implies

$$d_\nu({}_w c(g,h), \mathrm{id}) < \varepsilon \quad \text{for } g, h \in K, \nu \in \Phi.$$

Assume

$$\|c(g,h)(\xi) - \xi\| < \delta, \quad d_\nu(\widehat{c(g,h)}(t), (t)) < \delta, \quad g, h \in S, t \in \Lambda, \xi, \nu \in \Phi.$$

We show

$$\|w(g)(\xi) - \xi\| < \varepsilon, \quad d_\nu(\widehat{w(g)}(t), t) < \varepsilon, \quad g \in K, \xi, \nu \in \Phi, t \in \Lambda.$$

Let $Z := \bigsqcup_{s\in S'} Z_s$. By the definition of $w(g)$, $w(g)Z = \bigsqcup_{s\in S'} c(g,s)^{-1}Z_s$ holds. Then we have

$$\begin{aligned}&\|w(g)(\xi) - \xi\|\\ &\quad= \int_X |w(g)(\xi)(x) - \xi(x)|\,d\mu(x)\\ &\quad= \int_{w(g)Z} |w(g)(\xi)(x) - \xi(x)|\,d\mu(x) + \int_{X\backslash w(g)Z} |w(g)(\xi)(x) - \xi(x)|\,d\mu(x)\\ &\quad= \sum_{s\in S'} \int_{c(g,s)^{-1}Z_{l(g)s}} |w(g)^{-1}(\xi)(x) - \xi(x)|\,d\mu(x)\\ &\qquad+ \int_{X\backslash w(g)Z} |w(g)(\xi)(x) - \xi(x)|\,d\mu(x).\end{aligned}$$

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Note

$$w(g)(\xi)(x) = \xi(w(g)^{-1}x)\frac{d(\mu\circ w(g)^{-1})}{d\mu}(x)$$
$$= \xi(c(g,s)x)\frac{d(\mu\circ c(g,s))}{d\mu}(x) = c(g,s)^{-1}(\xi)(x)$$

for $x \in c(g,s)^{-1}Z_{l(g)s}$, when we regard ξ as an element of $L^1(X,\mu)$. Thus, the first term is estimated as follows:

$$\sum_{s\in S'}\int_{c(g,s)^{-1}Z_{l(g)s}} |w(g)^{-1}(\xi)(x) - \xi(x)|\,d\mu(x)$$
$$= \sum_{s\in S'}\int_{c(g,s)^{-1}Z_{l(g)s}} |c(g,s)^{-1}(\xi)(x) - \xi(x)|\,d\mu(x)$$
$$\le \sum_{s\in S'} \|c(g,s)^{-1}(\xi) - \xi\| < \delta|S|.$$

To estimate the second term, one should note

$$\xi(X\backslash Z) = \sum_{s\in S'}\xi(E_s\backslash Z_s) < \delta|S'|, \quad \xi(X\backslash w(g)Z) = \sum_{s\in S'}\xi(E_s\backslash c(g,s)^{-1}Z_s) < \delta|S'|$$

by condition (2). Hence,

$$\int_{X\backslash w(g)Z} |w(g)(\xi)(x) - \xi(x)|\,d\mu(x)$$
$$\le \int_{X\backslash w(g)Z} w(g)(\xi)(x)\,d\mu(x) + \int_{X\backslash w(g)Z}\xi(x)\,d\mu(x)$$
$$= \int_{X\backslash w(g)Z}\xi(w(g)^{-1}x)\frac{d(\mu\circ w(g)^{-1})}{d\mu}d\mu(x) + \xi(X\backslash w(g)Z)$$
$$= \int_{X\backslash Z}\xi(x)\,d\mu(x) + \xi(X\backslash w(g)Z) = \xi(X\backslash Z) + \xi(X\backslash w(g)Z) < 2|S'|\delta,$$

and we obtain $\|w(g)(\xi) - \xi\| < 3\delta|S'| < \varepsilon$.

We next show

$$d_\nu(\widehat{w(g)}(t), t) < \varepsilon, \quad g\in G, \nu\in\Phi, t\in\Lambda.$$

By the assumption

$$\|c(g,s)(\nu) - \nu\| < \delta, \quad d_\nu(\widehat{c(g,s)}(t), t) < \delta, \quad t\in\Lambda, g,s\in S, \nu\in\Phi,$$

$$d_\nu(\widehat{c(g,s)}^{-1}(t), t) = d_{c(g,s)(\nu)}(t, \widehat{c(g,s)}(t))$$
$$\le \|c(g,s)(\nu) - \nu\| + d_\nu(\widehat{c(g,s)}(t), t) < 2\delta$$

holds. We can further assume

$$\nu(E_{l(g)s}\triangle c(g,s)^{-1}t^{-1}E_{l(g)s}) < \delta, \quad \nu(c(g,s)^{-1}t^{-1}(E_{l(g)s}\triangle Z_{l(g)s})) < \delta$$

for $t\in\Lambda$, $s\in S'$, $g\in K$ in the choice of Z_s and E_s.

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Let $B_{g,s,t} := \{\widehat{c(g,s)^{-1}}(t) = t\}$. Then we have $\nu(X \backslash B_{g,s,t}) < 2\delta$, $\nu \in \Phi$. We can see $\widehat{w(g)}(t) = \widehat{c(g,s)^{-1}}(t)$ on $c(g,s)^{-1}Z_{l(g)s} \cap c(g,s)t^{-1}Z_{l(g)s}$ as above. Thus, $\widehat{w(g)}(t) = t$ holds on

$$\bigcup_{s \in S'} c(g,s)^{-1}Z_{l(g)s} \cap c(g,s)t^{-1}Z_{l(g)s} \cap B_{g,s,t}.$$

We will show

$$\nu\bigg(X \backslash \bigcup_{s \in S'} c(g,s)^{-1}Z_{l(g)s} \cap c(g,s)t^{-1}Z_{l(g)s} \cap B_{g,s,t}\bigg) < \varepsilon.$$

At first, we have

$$\begin{aligned}
&\nu(E_{l(g)s} \triangle (c(g,s)^{-1}Z_{l(g)s} \cap c(g,s)^{-1}t^{-1}Z_{l(g)s}))\\
&\quad \leq \nu(E_{l(g)s} \triangle c(g,s)^{-1}Z_{l(g)s}) + \nu(E_{l(g)s} \triangle c(g,s)^{-1}t^{-1}Z_{l(g)s})\\
&\quad < \delta + \nu(E_{l(g)s} \triangle \cap c(g,s)^{-1}t^{-1}E_{l(g)s}) + \nu(c(g,s)^{-1}t^{-1}(E_{l(g)s} \triangle Z_{l(g)s})) \quad \text{(by (2))}\\
&\quad < 3\delta.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\nu\bigg(X \backslash \bigcup_{s \in S'} c(g,s)^{-1}Z_{l(g)s} \cap c(g,s)^{-1}t^{-1}Z_{l(g)s} \cap B_{g,s,t}\bigg)\\
&\quad \leq \nu\bigg(X \backslash \bigcup_{s \in S'} c(g,s)^{-1}Z_{l(g)s} \cap c(g,s)^{-1}t^{-1}Z_{l(g)s}\bigg) + \nu\bigg(X \backslash \bigcup_{s \in S'} B_{g,s,t}\bigg)\\
&\quad \leq \sum_{s \in S'} \nu(E_{l(g)s} \triangle (c(g,s)^{-1}Z_{l(g)s} \cap c(g,s)^{-1}t^{-1}Z_{l(g)s})) + \sum_{s \in S'} \nu(X \backslash B_{g,s,t})\\
&\quad < 5|S'|\delta < \varepsilon
\end{aligned}$$

holds, and we obtain $d_\nu(\widehat{w(g)}(t), t) < \varepsilon$ for $g \in K$, $\nu \in \Phi$, $t \in \Lambda$. □

LEMMA 4.3. *For any $e \in K \Subset G$ and $\varepsilon > 0$, there exist $S \Subset G$ and $\delta > 0$ satisfying the following property: for any $\mu \in \Phi \Subset M_1(X, \mu)$, an ultrafree cocycle crossed action (γ, c) of G, and $u \in C^1(G, [T])$ with*

$$d_\nu(\widehat{\gamma}_g(u(s))^{-1}u(g)^{-1}u(gs), \mathrm{id}) < \delta, \quad g, s \in S, \nu \in \Phi,$$

there exists $w \in [T]$ such that

$$d_\nu(w^{-1}u(g)\widehat{\gamma}_g(w), \mathrm{id}) < \varepsilon, \quad \nu \in \Phi, g \in K.$$

Proof. Let $K \Subset G$, $\varepsilon > 0$ be given. Take $\varepsilon' > 0$ such that $8\sqrt{\varepsilon'} < \varepsilon$. Let S' be a (K, ε')-invariant set and set $S = S' \cup K$. Choose $\delta > 0$ such that $4|S'|\delta + 8\sqrt{\varepsilon'} < \varepsilon$. Let a cocycle crossed action (γ, c), $\Phi \Subset M_1(X, \mu)$, and $u \in C^1(G, [T])$ satisfying the condition

$$d_\nu(\widehat{\gamma}_g(u(s))^{-1}u(g)^{-1}u(gs), \mathrm{id}) < \delta, \quad g, s \in S, \nu \in \Phi$$

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be given. By Proposition 3.3, choose a partition $\{E_s\}_{s\in S'}$ of X, $E_s \supset Z_s$, and $w \in [T]$ such that:

(1) $u(s)Z_s \subset E_s$;
(2) $\nu(\gamma_g(E_s\backslash Z_s)) < \delta$, $\nu(E_s\backslash u(s)Z_s) < \delta$, $g \in K$, $\nu \in \Phi$;
(3) $\nu(\widehat{\gamma_g}(u(s))^{-1}u(g)^{-1}(E_{gs}\backslash Z_{gs})) < \delta$, $g \in K$, $s \in S'g$, $\nu \in \Phi$,
(4) $\nu(E_{gs}\triangle\widehat{\gamma_g}(u(s))^{-1}g^{-1}E_{gs}) < \delta$, $g \in K$, $s \in S'g$, $\nu \in \Phi$;
(5) $\sum_{s\in S'g}\nu(E_{gs}\triangle\gamma_g E_s) < 5\sqrt{\varepsilon'}$, $g \in K$, $\nu \in \Phi$;
(6) $\sum_{s\in S'\backslash S'_{g^{-1}}}\nu(E_s) < 3\sqrt{\varepsilon'}$, $g \in K$, $\nu \in \Phi$;
(7) $wx = u(s)x$, $x \in Z_s$.

Let

$$W_g := \bigcup_{s\in S'_g}\{\widehat{\gamma_g}(u(s))^{-1}u(g)^{-1}u(gs) = \mathrm{id}\} \cap \gamma_g Z_s \cap \widehat{\gamma_g}(u(s))^{-1}u(g)^{-1}Z_{gs}.$$

We can verify that $w^{-1}u(g)\widehat{\gamma_g}(w) = \mathrm{id}$ on W_g, $g \in K$, as in the proof of Lemma 4.2.

Next we show $\nu(X\backslash W_g) < \varepsilon$. We have

$$\nu(E_{gs}\triangle\gamma_g Z_s) \le \nu(E_{gs}\triangle\gamma_g E_s) + \nu(\gamma_g E_s\backslash\gamma_g Z_s) < \nu(E_{gs}\triangle\gamma_g E_s) + \delta$$

by condition (2), and

$$\begin{aligned}&\nu(E_{gs}\triangle\widehat{\gamma_g}(u(s))^{-1}u(g)^{-1}Z_{gs})\\ &\quad\le \nu(E_{gs}\triangle\widehat{\gamma_g}(u(s))^{-1}u(g)^{-1}E_{gs}) + \nu(\widehat{\gamma_g}(u(s))^{-1}u(g)^{-1}(E_{gs}\backslash Z_{gs})) < 2\delta\end{aligned}$$

by conditions (3) and (4). Hence, we have

$$\nu(E_{gs}\triangle(\gamma_g Z_s \cap \widehat{\gamma_g}(u(s))^{-1}u(g)^{-1}E_{gs})) < 3\delta + \nu(E_{gs}\triangle\gamma_g E_s).$$

Then we have

$$\begin{aligned}&\nu\bigg(\bigcup_{s\in S'_g}E_{gs}\triangle\bigcup_{s\in S'_g}(\gamma_g Z_s\cap\widehat{\gamma_g}(u(s))^{-1}u(g)^{-1}E_{gs})\bigg)\\ &\quad\le\sum_{s\in S'_g}\nu(E_{gs}\triangle(\gamma_g Z_s\cap\widehat{\gamma_g}(u(s))^{-1}u(g)^{-1}E_{gs}))\\ &\quad<\sum_{s\in S'_g}(3\delta+\nu(E_{gs}\triangle\gamma_g E_s)) < 3|S'|\delta + 5\sqrt{\varepsilon'}\end{aligned}$$

by condition (5). Hence, we get

$$\begin{aligned}&\nu\bigg(X\backslash\bigcup_{s\in S'_g}(\gamma_g Z_s\cap\widehat{\gamma_g}(u(s))^{-1}u(g)^{-1}E_{gs})\bigg)\\ &\quad\le\sum_{s\in S'\backslash S'_{g^{-1}}}\nu(E_s)+\nu\bigg(\bigcup_{s\in S'_g}E_{gs}\triangle\bigcup_{s\in S'_g}(\gamma_g Z_s\cap\widehat{\gamma_g}(u(s))^{-1}u(g)^{-1}E_{gs})\bigg)\\ &\quad< 3|S'|\delta + 8\sqrt{\varepsilon'}\end{aligned}$$

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by condition (6). By the assumption

$$d_\nu(\widehat{\gamma_g}(u(s))^{-1}u(g)^{-1}u(gs), \mathrm{id}) < \delta, \quad g, s \in S, \nu \in \Phi,$$

we have $\nu(X \setminus \bigcup_{s \in S'g} \{\widehat{\gamma_g}(u(s))^{-1}u(g)^{-1}u(gs) = \mathrm{id}\}) < |S'|\delta$. Hence,

$$\nu(X \setminus W_g) < 4|S'|\delta + 8\sqrt{\varepsilon'} < \varepsilon$$

holds. □

THEOREM 4.4. *Let (γ, c) be an ultrafree cocycle crossed action of G. Then there exists $u \in C^1(G, [T])$ such that ${}_u c(g, h) = \mathrm{id}$, and hence ${}_u\gamma$ is an action.*

Moreover, for any $e \in K \Subset G$, $\varepsilon > 0$, there exists $S \Subset G$, $\delta > 0$, which depends only on K and ε, or on cocycle crossed action (γ, c), such that if

$$d_\nu(c(g, h), \mathrm{id}) < \delta, \quad g, h \in S, \nu \in \Phi$$

for some $\Phi \Subset M_1(X, \mu)$ with $\mu \in \Phi$, then we can choose $u \in C^1(G, [T])$ so that

$$d_\nu(u(g), \mathrm{id}) < \varepsilon, \quad g \in K, \nu \in \Phi.$$

Proof. At first, we treat a type II_∞ or type III case.

Let $e \in K \Subset G$ and $\varepsilon > 0$ be given, and take $S \Subset G$ and $\delta > 0$ as in Lemma 4.3. Assume $d_\nu(c(g, h), \mathrm{id}) < \delta$ for $g, h \in S$, $\nu \in \Phi \Subset M_1(X, \mu)$. There exists $v \in C^1(G, [T])$ such that ${}_v c(g, h) = \mathrm{id}$ by Theorem 4.1. Hence, $c(g, h) = \widehat{\gamma_g}(v(h))^{-1}v(g)^{-1}v(gh)$ holds and

$$d_\nu(\widehat{\gamma_g}(v(h))^{-1}v(g)^{-1}v(gh), \mathrm{id}) < \delta, \quad g, h \in S, \nu \in \Phi.$$

By Lemma 4.3, there exists $w \in [T]$ such that

$$d_\nu(w^{-1}v(g)\widehat{\gamma_g}(w), \mathrm{id}) < \varepsilon, \quad \nu \in \Phi, g \in K.$$

Define $u(g) := w^{-1}v(g)\widehat{\gamma_g}(w)$. Then we obtain $d_\nu(u(g), \mathrm{id}) < \varepsilon$ for $g \in K$, $\nu \in \Phi$, and

$${}_u c(g, h) = u(g)\widehat{\gamma_g}(u(h))c(g, h)u(gh)^{-1} = w^{-1}v(g)\widehat{\gamma_g}(v(h))c(g, h)v(gh)^{-1}w = \mathrm{id}.$$

Hence, we have proved the theorem for the type II_∞ and type III cases.

Next, we assume T is of type II_1. In this case, we can assume that μ is the unique T-invariant probability measure and choose Φ as $\Phi = \{\mu\}$. Let us take an increasing sequence $\{K_n\}_n \Subset G$ and decreasing sequence $\{\varepsilon_n\}_n$ such that $e \in K_n$, $\bigcup_{n=1}^\infty K_n = G$, and $\sum_n \varepsilon_n < \infty$. Take S_n and δ_n for K_n and $\varepsilon_n > 0$ as in Lemma 4.3. We can choose S_n and δ_n so that $S_n \subset S_{n+1}$, $\delta_n > \delta_{n+1}$.

For given $K \Subset G$ and $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $K \subset K_N$, $\varepsilon > \sum_{k=N}^\infty \varepsilon_k$. By Lemma 4.2, take $S_N \Subset G$ and $\delta_N > 0$ for K_N and $\varepsilon_N > 0$. Again by Lemma 4.2, we can perturb (γ, c) by some $w \in C^1(G, [T])$ so that

$$d_\mu({}_w c(g, h), \mathrm{id}) < \varepsilon_N, \quad g, h \in K_N, \quad d_\mu({}_w c(g, h), \mathrm{id}) < \frac{\delta_N}{2}, \quad g, h \in S_N.$$

Set

$$(\gamma^{(N)}, c_N) := ({}_w\gamma, {}_w c), \quad u_N(g) = 1.$$

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We will inductively construct a family of cocycle crossed actions $(\gamma^{(n)}, c_n)$ and normalized maps $\{u_n\} \subset C^1(G, [T])$, $n \geq N$, such that:

(1.n) $(\gamma^{(n)}, c_n) = ({}_{u_n}\gamma^{(n-1)}, {}_{u_n}c_{n-1})$;

(2.n) $d_\mu(c_n(g, h), \mathrm{id}) < \varepsilon_n$, $g, h \in K_n$;

(3.n) $d_\mu(c_n(g, h), \mathrm{id}) < \delta_n/2$, $g, h \in S_n$;

(4.n) $d_\mu(u_n(g), \mathrm{id}) < \varepsilon_{n-1}$, $g \in K_{n-1}$.

Here we regard $\gamma^{(N-1)} = \gamma^{(N)}$, $c_{N-1}(c, h) = c_N(g, h)$. Clearly we have (1.$N$), (2.$N$), (3.$N$), and (4.$N$).

Assume we have done up to the nth step.

By Lemma 4.2, we choose $\bar{u}_{n+1} \in C^1(G, [T])$ such that:

($a.n+1$) $d_\mu(\bar{u}_{n+1}(g)\widehat{\gamma_g^{(n)}}(\bar{u}_{n+1}(h))c_n(g, h)\bar{u}_{n+1}(gh)^{-1}, \mathrm{id}) < \varepsilon_{n+1}$, $g, h \in K_{n+1}$;

($b.n+1$) $d_\mu(\bar{u}_{n+1}(g)\widehat{\gamma_g^{(n)}}(\bar{u}_{n+1}(h))c_n(g, h)\bar{u}_{n+1}(gh)^{-1}, \mathrm{id}) < \delta_{n+1}/2$, $g, h \in S_{n+1}$.

By condition ($b.n+1$), we have

$$d_\mu\big(\widehat{\gamma_g^{(n)}}(\bar{u}_{n+1}(h))^{-1}\bar{u}_{n+1}(g)^{-1}\bar{u}_{n+1}(gh), c_n(g, h)\big) < \frac{\delta_{n+1}}{2}, \quad g, h \in S_{n+1}.$$

Combining with condition (3.n), we get

$$d_\mu\big(\widehat{\gamma_g^{(n)}}(\bar{u}_{n+1}(h))^{-1}\bar{u}_{n+1}(g)^{-1}\bar{u}_{n+1}(gh), \mathrm{id}\big) < \delta_n, \quad g, h \in S_n.$$

By Lemma 4.3, there exists $w \in [T]$ such that $d_\mu(w^{-1}\bar{u}_{n+1}(g)\widehat{\gamma_g^{(n)}}(w), \mathrm{id}) < \varepsilon_n$ for $g \in K_n$. Here set $u_{n+1}(g) := w^{-1}\bar{u}_{n+1}(g)\widehat{\gamma_g^{(n)}}(w)$. Then we get condition (4.$n+1$). Define a cocycle crossed action $(\gamma^{(n+1)}, c_{n+1})$ as condition (1.$n+1$). Then we get conditions (2.$n+1$) and (3.$n+1$) from conditions ($a.n+1$) and ($b.n+1$), respectively, and the induction is complete.

Let $v_n(g) := u_n(g)u_{n-1}(g)\cdots u_N(g)$. We have $(\gamma^n, c_n) = ({}_{v_n}\gamma^{(N)}, {}_{v_n}c_N)$ by the construction. Fix $L \in \mathbb{N}$ and take any $g \in K_L$. By condition (4.n),

$$d_\mu(v_n(g), v_{n-1}(g)) = d_\mu(u_n(g), \mathrm{id}) < \varepsilon_{n-1}, \quad n \geq L+1$$

holds. So $\{v_n(g)\}_n$ is a Cauchy sequence and hence $v_n(g)$ converges to some $v(g) \in [T]$ uniformly. Note that $v_n(g)^{-1}$ converges to $v(g)^{-1}$ automatically, since μ is the invariant measure for $[T]$. Combining with condition (2.n), we obtain ${}_vc(g, h) = \mathrm{id}$ for all $g, h \in G$.

If $g \in K_N$, then

$$d_\mu(v_n(g), \mathrm{id}) = d_\mu(v_n(g), v_N(g)) \leq \sum_{k=N}^{n-1} d_\mu(v_{k+1}(g), v_k(g)) < \sum_{k=N}^{n-1} \varepsilon_k.$$

Hence, we have $d_\mu(v(g), \mathrm{id}) \leq \sum_{k=N}^{\infty} \varepsilon_k < \varepsilon$. Set $S := S_N \cup K_N$, $\delta := \min\{\delta_N/2, \varepsilon_N\}$. If $d_\mu(g, h) < \delta$ for $g, h \in S$, then we have $d_\mu(v(g), \mathrm{id}) < \varepsilon$ for $g \in K_N$. Note that S and δ depend only on K and ε. □

5. *Classification*

LEMMA 5.1. *Let α and β be actions of G into $N[T]$ with* $\mathrm{mod}(\alpha_g) = \mathrm{mod}(\beta_g)$. *Then for any $\varepsilon > 0$, $K \Subset G$, $\mu \in \Phi \Subset M_1(X, \mu)$, $\Lambda \Subset [T]$, there exists $w \in C^1(G, [T])$ such that:*

(1) $\|{}_w\alpha_g(\xi) - \beta_g(\xi)\| < \varepsilon$, $g \in K$, $\xi \in \Phi$;

(2) $d_\nu(\widehat{{}_w\alpha_g}(t), \widehat{\beta_g}(t)) < \varepsilon$, $g \in K$, $t \in \Lambda$, $\nu \in \Phi$;

(3) *let* $c(g, h) := w(g)\widehat{\alpha_g}(w(h))w(gh)^{-1}$. *Then*

$$\|c(g, h)(\xi) - \xi\| < \varepsilon, \quad d_\nu(\widehat{c(g, h)}(t), t) < \varepsilon, \quad g, h \in K, \xi, \nu \in \Phi, t \in \Lambda.$$

Proof. By enlarging K, we may assume $e \in K = K^{-1} \Subset G$. Let

$$\tilde{\Phi} := \{\beta_{gh}(\xi) \mid g, h \in K, \xi \in \Phi\}, \quad \tilde{\Lambda} := \{\widehat{\beta_{gh}}(t) \mid g, h \in K, t \in \Lambda\}.$$

By the assumption, $\beta_g\alpha_g^{-1} \in \mathrm{Ker(mod)} = \overline{[T]}$. Hence, we can take $w \in C^1(G, [T])$ so that

$$\|{}_w\alpha_{gh}(\xi) - \beta_{gh}(\xi)\| < \frac{\varepsilon}{7}, \quad d_\nu(\widehat{{}_w\alpha_{gh}}(t), \widehat{\beta_{gh}}(t)) < \frac{\varepsilon}{7}$$

for $g, h \in K$, $\nu, \xi \in \bigcup_{g\in K} \beta_g(\tilde{\Phi})$, $t \in \bigcup_{g\in K} \beta_g(\tilde{\Lambda})$. Obviously, we have conditions (1) and (2).

Then for $g, h \in K$, $\eta \in \tilde{\Phi}$, we have

$$\begin{aligned}\|{}_w\alpha_g {}_w\alpha_h(\eta) - \beta_{gh}(\eta)\| &\leq \|{}_w\alpha_g {}_w\alpha_h(\eta) - {}_w\alpha_g\beta_h(\eta)\| + \|{}_w\alpha_g\beta_h(\eta) - \beta_g\beta_h(\eta)\| \\ &\leq \|{}_w\alpha_h(\eta) - \beta_h(\eta)\| + \|{}_w\alpha_g\beta_h(\eta) - \beta_g\beta_h(\eta)\| < \frac{2\varepsilon}{7}.\end{aligned}$$

Thus,

$$\begin{aligned}\|c(g, h)\beta_{gh}(\eta) - \beta_{gh}(\eta)\| &\leq \|c(g, h)\beta_{gh}(\eta) - c(g, h)\alpha_{gh}(\eta)\| \\ &\quad + \|c(g, h)\alpha_{gh}(\eta) - \beta_{gh}(\eta)\| \\ &< \frac{3\varepsilon}{7}\end{aligned}$$

holds for $g, h \in K$, $\eta \in \tilde{\Phi}$. Hence, we get $\|c(g, h)(\xi) - \xi\| < 3\varepsilon/7$ for $g, h \in K$, $\xi \in \Phi$.

For $g \in K$, $t \in \tilde{\Lambda}$, $\nu \in \tilde{\Phi}$, we have

$$\begin{aligned}d_\nu(\widehat{c(g, h)}\widehat{{}_w\alpha_{gh}}(t), \widehat{\beta_{gh}}(t)) &= d_\nu(\widehat{{}_w\alpha_g}\widehat{{}_w\alpha_h}(t), \widehat{\beta_{gh}}(t)) \\ &\leq d_\nu(\widehat{{}_w\alpha_g}\widehat{{}_w\alpha_h}(t), \widehat{{}_w\alpha_g}\widehat{\beta_h}(t)) + d_\nu(\widehat{{}_w\alpha_g}\widehat{\beta_h}(t), \widehat{\beta_{gh}}(t)) \\ &\leq d_{{}_w\alpha_g^{-1}(\nu)}(\widehat{{}_w\alpha_h}(t), \widehat{\beta_h}(t)) + \frac{\varepsilon}{7} \\ &\leq d_{\beta_g^{-1}(\nu)}(\widehat{{}_w\alpha_h}(t), \widehat{\beta_h}(t)) + \|{}_w\alpha_g^{-1}(\nu) - \beta_g^{-1}(\nu)\| + \frac{\varepsilon}{7} \\ &\leq d_{\beta_g^{-1}(\nu)}(\widehat{{}_w\alpha_h}(t), \widehat{\beta_h}(t)) + \frac{2\varepsilon}{7} < \frac{3\varepsilon}{7}.\end{aligned}$$

By noting $\|c(g, h)(\nu) - \nu\| \leq 3\varepsilon/7$ for $g, h \in K$, $\nu \in \Phi$, we have

$$\begin{aligned}d_\nu(\widehat{c(g, h)}\widehat{\beta_{gh}}(t), \widehat{\beta_{gh}}(t)) &\leq d_\nu(\widehat{c(g, h)}\widehat{\beta_{gh}}(t), \widehat{c(g, h)}\widehat{{}_w\alpha_{gh}}(t)) \\ &\quad + d_\nu(\widehat{c(g, h)}\widehat{{}_w\alpha_{gh}}(t), \widehat{\beta_{gh}}(t))\end{aligned}$$

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$$\leq d_{c(g,h)^{-1}(\nu)}(\widehat{\beta_{gh}}(t), \widehat{{}_w\alpha_{gh}}(t)) + \frac{3\varepsilon}{7}$$
$$\leq d_\nu(\widehat{\beta_{gh}}(t), \widehat{{}_w\alpha_{gh}}(t)) + \frac{6\varepsilon}{7} < \varepsilon$$

for $\nu \in \Phi$, $g, h \in K$, $t \in \tilde{\Lambda}$. Thus, $d_\nu(\widehat{c(g,h)}(t), t) < \varepsilon$ holds for $g, h \in K$, $t \in \Lambda$, $\nu \in \Phi$. □

LEMMA 5.2. *Let α and β be actions of G into $N[T]$ with* $\mathrm{mod}(\alpha_g) = \mathrm{mod}(\beta_g)$. *For any $\varepsilon > 0$, $K \Subset G$, $\Lambda \Subset [T]$, $\Phi \Subset M_1(X, \mu)$, there exists $v \in C^1(G, [T])$ such that*

$$\|{}_v\alpha_g(\xi) - \beta_g(\xi)\| < \varepsilon, \quad g \in K, \nu \in \Phi,$$
$$d_\nu(\widehat{{}_v\alpha_g}(t), \widehat{\beta_g}(t)) < \varepsilon, \quad g \in K, t \in \Lambda, \nu \in \Phi,$$
$$d_\nu(v(g)\widehat{\alpha_g}(v(h))v(gh)^{-1}, \mathrm{id}) < \varepsilon, \quad g, h \in K, \nu \in \Phi.$$

Proof. Let $\tilde{\Phi} := \{\beta_g(\xi) \mid g \in K, \xi \in \Phi\}$, $\tilde{\Lambda} := \{\widehat{\beta_g}(t) \mid g \in K, t \in \Lambda\}$. Choose $\delta > 0$ and S for $\varepsilon/3 > 0$ and K as in Lemma 4.2. By Lemma 5.1, there exists $u \in C^1(G, [T])$ such that

$$c(g,h) := u(g)\widehat{\alpha_g}(u(h))u(gh)^{-1},$$
$$\|{}_u\alpha_g(\xi) - \beta_g(\xi)\| < \frac{\varepsilon}{3}, \quad g \in K, \xi \in \Phi,$$
$$\|c(g,h)(\xi) - \xi\| < \delta, \quad g, h \in S, \xi \in \tilde{\Phi},$$
$$d_\nu(\widehat{c(g,h)}(t), t) < \delta, \quad g, h \in S, t \in \tilde{\Lambda}, \nu \in \tilde{\Phi}.$$

By Lemma 4.2, there exists $w \in C^1(G, [T])$ such that

$$d_\nu(w(g)\widehat{{}_u\alpha_g}(w(h))c(g,h)w(gh)^{-1}, \mathrm{id}) < \frac{\varepsilon}{3}, \quad g, h \in K, \nu \in \Phi$$

and

$$\|w(g)(\xi) - \xi\| < \frac{\varepsilon}{3}, \quad d_\nu(\widehat{w(g)}(t), t) < \frac{\varepsilon}{3}, \quad g \in K, \xi, \nu \in \tilde{\Phi}, t \in \tilde{\Lambda}.$$

Let $v(g) := w(g)u(g)$. Then we have

$$d_\nu(v(g)\widehat{\alpha_g}(v(h))v(gh)^{-1}, \mathrm{id}) < \varepsilon, \quad g, h \in K, \nu \in \Phi.$$

We can verify the first inequality as follows. For $g \in K, \xi \in \Phi$,

$$\|{}_v\alpha_g(\xi) - \beta_g(\xi)\| \leq \|w(g)u(g)\alpha_g(\xi) - w(g)\beta_g(\xi)\| + \|w(g)\beta_g(\xi) - \beta_g(\xi)\|$$
$$< \frac{2\varepsilon}{3} < \varepsilon$$

since $\beta_g(\xi) \in \tilde{\Phi}$. Similarly, we have

$$d_\nu(\widehat{{}_v\alpha_g}(t), \widehat{\beta_g}(t)) \leq d_\nu(\widehat{w(g)_u\alpha_g}(t), \widehat{w(g)\beta_g}(t)) + d_\nu(\widehat{w(g)\beta_g}(t), \widehat{\beta_g}(t))$$
$$\leq d_{w(g)(\nu)}(\widehat{{}_u\alpha_g}(t), \widehat{\beta_g}(t)) + \frac{\varepsilon}{3}$$
$$\leq \|w(g)(\nu) - \nu\| + d_\nu(\widehat{{}_u\alpha_g}(t), \widehat{\beta_g}(t)) + \frac{\varepsilon}{3} < \varepsilon$$

for $g \in K, t \in \Lambda, \nu \in \Phi$. □

THEOREM 5.3. *Let α and β be ultrafree actions of G into $N[T]$ with* $\mathrm{mod}(\alpha_g) = \mathrm{mod}(\beta_g)$. *Then there exists a sequence $\{u_n(\cdot)\}$ of 1-cocycles for α_g such that* $\lim_{n\to\infty} {}_{u_n}\alpha_g = \beta_g$ *in the u-topology.*

Proof. By Lemma 5.2, there exists a sequence $\{v_n\} \subset C^1(G, [T])$ of normalized maps such that $\lim_{n\to\infty} {}_{v_n}\alpha_g = \beta_g$ in the u-topology, and $\lim_{n\to\infty} d_\mu(v_n(g)\widehat{\alpha_g}(v_n(h))v_n(gh)^{-1}, \mathrm{id}) = 0$. Let $\alpha^{(n)} = {}_{v_n}\alpha$ and $c_n(g,h) = v_n(g)\widehat{\alpha_g}(v_n(h))v_n(gh)^{-1}$. By Theorem 4.4, there exists a sequence $\{w_n\} \subset C^1(G, [T])$ such that

$$w_n(g)\widehat{\alpha_g^{(n)}}(w_n(h))c_n(g,h)w_n(gh)^{-1} = 1, \quad \lim_{n\to\infty} d_\mu(w_n(g), \mathrm{id}) = 0.$$

Then it turns out that $u_n(g) := w_n(g)v_n(g)$ is a 1-cocycle for α_g, and $\lim_{n\to\infty} {}_{u_n}\alpha_g = \beta_g$ holds in the u-topology. □

LEMMA 5.4. *Let $K \Subset G$ and $\varepsilon > 0$ be given. Then there exist $S \Subset G$ and $\delta > 0$ satisfying the following: for any action γ of G, a 1-cocycle $u(\cdot)$ for γ, $\Phi \Subset M_1(X,\mu)$ with $\mu \in \Phi$ and $\Lambda \Subset [T]$ satisfying*

$$\|u(s)(\xi) - \xi\| < \delta, \quad d_\nu(\widehat{u(s)}(t), t) < \delta, \quad s \in S, \xi, \nu \in \Phi, t \in \Lambda,$$

there exists $w \in [T]$ such that

$$d_\nu(u(g)\widehat{\gamma_g}(w)w^{-1}, 1) < \varepsilon, \|w(\xi) - \xi\| < \varepsilon, d_\nu(w(t), t) < \varepsilon, g \in K, \xi, \nu \in \Phi, t \in \Lambda.$$

Proof. Take $\varepsilon_1 > 0$ with $8\varepsilon_1^{1/2} < \varepsilon$, and let S be a (K, ε_1)-invariant set. Choose $\delta > 0$ with $8\varepsilon_1^{1/2} + 3|S|\delta < \varepsilon$, $4|S|\delta < \varepsilon$.

By Proposition 3.3, take a partition $\{E_s\}_{s\in S}$ of X, $Z_s \subset E_s$, and $w \in [T]$ such that:

(1) $u(s)Z_s \subset E_s$, $s \in S$;
(2) $\nu(E_s \setminus Z_s) < \delta$, $\nu(E_s \setminus u(s)Z_s) < \delta$, $s \in S$, $\nu \in \Phi$;
(3) $\nu(u(gs)\gamma_g(E_s \setminus Z_s)) < \delta$, $g \in K$, $s \in S_g$, $\nu \in \Phi$;
(4) $\nu(u(s)t^{-1}(E_s \setminus Z_s)) < \delta$, $s \in S$, $t \in \Lambda$, $\nu \in \Phi$;
(5) $\nu(u(s)E_s \triangle E_s) < \delta$, $s \in S$, $\nu \in \Phi$;
(6) $\nu(E_{gs} \triangle \widehat{\gamma_g}(u(s))E_{gs}) < \delta$, $s \in S_g$, $\nu \in \Phi$;
(7) $\nu(E_s \triangle u(s)t^{-1}E_s) < \delta$, $s \in S$, $t \in \Lambda$, $\nu \in \Phi$;
(8) $\sum_{s\in S_g} u(gs)^{-1}(\nu)(\gamma_g E_s \triangle E_{gs}) < 5\varepsilon_1^{1/2}$, $g \in K$, $\nu \in \Phi$;
(9) $\sum_{s\in S\setminus S_{g^{-1}}} \nu(E_s) < 3\varepsilon_1^{1/2}$, $g \in K$;
(10) $wx = u(s)x$, $x \in Z_s$.

In the following proof, the letters g, s, and ν denote an elements in K, S, and Φ, respectively. As in the proof of Lemma 4.2, we can see that

$$u(g)\widehat{\gamma_g}(w)w^{-1}x = u(g)\gamma_g u(s)\gamma_g^{-1}u(gs)^{-1}x = x$$

for $x \in u(gs)Z_{gs} \cap u(gs)\gamma_g Z_s$.

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We have

$$\begin{aligned}
&\nu(E_{gs}\triangle(u(gs)Z_{gs}\cap u(gs)\gamma_g Z_s))\\
&\quad\le \nu(E_{gs}\backslash u(gs)Z_{gs})+\nu(E_{gs}\triangle u(gs)\gamma_g Z_s)\\
&\quad< \delta+\nu(E_{gs}\triangle u(gs)\gamma_g E_s)+\nu(u(gs)\gamma_g(E_s)\backslash u(gs)\gamma_g Z_s)\quad\text{(by (2))}\\
&\quad< 2\delta+\nu(E_{gs}\triangle u(gs)E_{gs})+\nu(u(gs)(E_{gs}\triangle\gamma_g E_s))\quad\text{(by (3))}\\
&\quad< 3\delta+u(gs)^{-1}(\nu)(E_{gs}\triangle\gamma_g E_s)\quad\text{(by (5)).}
\end{aligned}$$

Thus,

$$\begin{aligned}
&\nu\bigg(X\backslash\bigcup_{s\in S_g}u(gs)Z_{gs}\cap u(gs)\gamma_g Z_s\bigg)\\
&\quad\le \nu(X\backslash\bigsqcup_{s\in S_g}E_{gs})+\sum_{s\in S_g}\nu(E_{gs}\triangle(u(gs)Z_{gs}\cap u(gs)\gamma_g Z_s))\\
&\quad\le \sum_{s\in S\backslash S_{g-1}}\nu(E_s)+\sum_{s\in S_g}(3\delta+u(gs)^{-1}(\nu)(E_{gs}\triangle\gamma_g E_s))\\
&\quad< 3\varepsilon_1^{1/2}+3|S|\delta+5\varepsilon_1^{1/2}=8\varepsilon_1^{1/2}+3|S|\delta<\varepsilon
\end{aligned}$$

holds. Hence, $\nu(\{u(g)\widehat{\gamma_g}(w)w^{-1}\neq\mathrm{id}\})<\varepsilon$ for $g\in K$ and $\nu\in\Phi$, which implies

$$d_\nu(u(g)\widehat{\gamma_g}(w)w^{-1},\mathrm{id})<\varepsilon,\quad g\in K,\nu\in\Phi.$$

We next show $\|w(\xi)-\xi\|<\varepsilon$ and $d_\nu(\widehat{w}(t),t)<\varepsilon$. Let $Z=\bigsqcup_{s\in S}Z_s$. As in the proof of Lemma 4.2, we can see $w(\xi)(x)=u(s)(\xi)(x)$ on $u(s)Z_s$, and

$$\int_{X\backslash wZ}|w(\xi)(x)-\xi(x)|\,d\mu(x)<2|S|\delta$$

by using conditions (2) and (10). If $u(s)$ satisfies $\|u(s)(\xi)-\xi\|<\delta$ for $s\in S$, then

$$\begin{aligned}
\|w(\xi)-\xi\|&=\sum_{s\in S}\int_{u(s)Z_s}|w(\xi)(x)-\xi(x)|\,d\mu(x)+\int_{X\backslash wZ}|w(\xi)(x)-\xi(x)|\,d\mu(x)\\
&<\sum_{s\in S}\int_{u(s)Z_s}|u(s)(\xi)(x)-\xi(x)|\,d\mu(x)+2|S|\delta<3|S|\delta<\varepsilon
\end{aligned}$$

holds for $\xi\in\Phi$.

For $t\in\Lambda\subset[T]$ and $x\in u(s)Z_s\cap u(s)t^{-1}Z_s$, $w^{-1}x=u(s)^{-1}x\in Z_s\cap t^{-1}Z_s$. Hence, $tw^{-1}x=u(s)^{-1}x\in tZ_s\cap Z_s$, and $wtw^{-1}x=u(s)tu(s)^{-1}x$ holds.

Then,

$$\begin{aligned}
\nu(E_s\triangle(u(s)Z_s\cap u(s)t^{-1}Z_s))&\le\nu(E_s\backslash u(s)Z_s)+\nu(E_s\triangle u(s)t^{-1}Z_s)\\
&<\delta+\nu(E_s\triangle u(s)t^{-1}E_s)+\nu(u(s)t^{-1}(E_s\backslash Z_s))\quad\text{(by condition (2))}\\
&<3\delta\quad\text{(by conditions (4) and (7)).}
\end{aligned}$$

Let us assume $d_\nu(\widehat{u(s)}(t), t) < \delta$. Hence, $A_{s,t} := \{\widehat{u(s)}(t) = t\}$ satisfies $\nu(X \backslash A_{s,t}) < \delta$. Thus,

$$\begin{aligned}\nu\bigg(X \backslash \bigcup_{s \in S} (u(s) Z_s \cap u(s) t^{-1} Z_s \cap A_{s,t})\bigg) \\ \leq \sum_{s \in S} \nu(E_s \triangle (u(s) Z_s \cap u(s) t^{-1} Z_s)) + \sum_{s \in S} \nu(X \backslash A_{s,t}) \\ \leq 4\delta |S| < \varepsilon\end{aligned}$$

and we have $\nu(\{\widehat{w}(t) \neq t\}) < \varepsilon$, equivalently $d_\nu(\widehat{w}(t), t) < \varepsilon$. □

Remark. In Lemma 5.4, we can choose δ and S so that $\delta < \delta'$ and $S' \subset S$ for any given $\delta' > 0$ and $S' \Subset G$.

Now we can classify ultrafree actions.

THEOREM 5.5. *Let α and β be ultrafree actions of G into $N[T]$ with* $\mathrm{mod}(\alpha_g) = \mathrm{mod}(\beta_g)$. *Then they are strongly cocycle conjugate.*

Proof. Let $\{\xi_i\}_{i=0}^\infty$ be a countable dense subset of $M_1(X, \mu)$ with $\xi_0 = \mu$. Take $\varepsilon_n > 0$ and $K_n \Subset G$ such that $\sum_{n=0}^\infty \varepsilon_n < \infty$, $\varepsilon_n > \varepsilon_{n+1}$, $e \in K_n$, $K_n \subset K_{n+1}$, $\bigcup_{n=0}^\infty K_n = G$. Then choose $S_n \Subset G$, $\delta_n > 0$ for K_n, ε_n as in Lemma 5.4. We can assume $S_n \subset S_{n+1}$ and $\delta_{n+1} < \delta_n$. (See the remark after Lemma 5.4.)

Set $\gamma_g^{(0)} := \alpha_g$, $\gamma_g^{(-1)} := \beta_g$, and construct actions $\gamma_g^{(n)}$ of G, $v_n(g), \bar{v}_n(g), w_n, \theta_n \in [T]$, $\Phi_n \Subset M_1(X, \mu)$, and $\Lambda_n \Subset [T]$ as follows:

(1.n) $\gamma_g^{(n)} = \bar{v}_n(g) w_n \gamma_g^{(n-2)} w_n^{-1}$;

(2.n) $\theta_n = w_n \theta_{n-2}$;

(3.n) $v_n(g) = \bar{v}_n(g) \widehat{w_n}(v_{n-2}(g))$;

(4.n) $\|\gamma_g^{(n)}(\xi) - \gamma_g^{(n-1)}(\xi)\| < \varepsilon_n$, $g \in K_n$, $\xi \in \Phi_{n-1}$;

(5.n) $d_\mu(\widehat{\gamma_g^{(n)}}(t), \widehat{\gamma_g^{(n-1)}}(t)) < \varepsilon_n$, $g \in K_n$, $t \in \Lambda_{n-1}$;

(6.n) $\|\gamma_g^{(n)}(\xi) - \gamma_g^{(n-1)}(\xi)\| < \delta_{n-1}/2$, $g \in S_{n-1}$, $\xi \in \bigcup_{g \in S_{n-1}} \gamma_{g^{-1}}^{(n-1)}(\Phi_{n-1})$;

(7.n) $d_\nu(\widehat{\gamma_g^{(n)}}(t), \widehat{\gamma_g^{(n-1)}}(t)) < \delta_{n-1}/2$, $g \in S_{n-1}$, $t \in \bigcup_{s \in S_{n-1}} \gamma_{g^{-1}}^{(n-1)}(\Lambda_{n-1})$, $\nu \in \Phi_{n-1}$;

(8.n) $d_\nu(\bar{v}_n(g), \mathrm{id}) < \varepsilon_{n-2}$, $g \in K_{n-2}$, $\nu \in \Phi_{n-2}$, $(n \geq 2)$;

(9.n) $\|w_n(\xi) - \xi\| < \varepsilon_{n-2}$, $\xi \in \Phi_{n-2}$, $(n \geq 2)$;

(10.n) $d_\nu(\widehat{w_n}(t), t) < \varepsilon_{n-2}$, $\nu \in \Phi_{n-2}$, $t \in \Lambda_{n-2}$, $(n \geq 2)$;

(11.n) $\Phi_n = \{\xi_i\}_{i=0}^n \cup \{\theta_n(\xi_i)\}_{i=0}^n \cup \{v_n(g)(\mu)\}_{g \in K_n}$;

(12.n) $\Lambda_n = \{T^i\}_{i=-n}^n \cup \{\theta_n(T^i)\}_{i=-n}^n \cup \{v_n(g), v_n(g)^{-1}\}_{g \in K_n}$.

1st step. Let $\theta_{-1} = \theta_0 = \mathrm{id}$, $v_{-1}(g) = v_0(g) = \mathrm{id}$. By Theorem 5.3, take a 1-cocycle $u_1(\cdot)$ for $\gamma^{(-1)}$ such that:

(a.1) $\|{}_{u_1}\gamma_g^{(-1)}(\xi) - \gamma_g^{(0)}(\xi)\| < \varepsilon_1$, $g \in K_1$, $\xi \in \Phi_0$;

(b.2) $d_\mu(\widehat{{}_{u_1}\gamma_g^{(-1)}}(t), \widehat{\gamma_g^{(0)}}(t)) < \varepsilon_1$, $g \in K_1$, $t \in \Lambda_0$;

(c.2) $\|{}_{u_1}\gamma_g^{(-1)}(\xi) - \gamma_g^{(0)}(\xi)\| < \delta_0/2$, $g \in S_0$, $\xi \in \bigcup_{g \in S_0} \gamma_{g^{-1}}^{(0)}(\Phi_0)$;

(d.1) $d_\nu(\widehat{{}_{u_1}\gamma_g^{(-1)}}(t), \widehat{\gamma_g^{(0)}}(t)) < \delta_0/2$, $g \in S_0$, $t \in \bigcup_{g \in S_0} \gamma_{g^{-1}}^{(0)}(\Lambda_0)$, $\nu \in \Phi_0$.

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Set $w_1 = \mathrm{id}$, $\bar{v}_1(g) = u_1(g)$, and define

$$\begin{aligned}\gamma_g^{(1)} &:= \bar{v}_1(g) w_1 \gamma_g^{(-1)} w_1^{-1} = {}_{u_1}\gamma_g^{(-1)},\\ \theta_1 &:= w_1\theta_{-1} = \mathrm{id},\\ v_1(g) &:= \bar{v}_1(g)\widehat{w_1}(v_{-1}(g)) = u_1(g)\end{aligned}$$

as in conditions (1.1), (1.2), and (1.3), respectively. By conditions $(a.1)$, $(b.1)$, $(c.1)$, and $(d.1)$, we get conditions (4.1), (5.1), (6.1), and (7.1), respectively. Define Φ_1 and Λ_1 as in conditions (11.1) and (12.1), respectively. Then we have finished the 1st step of the induction.

Assume that we have done up to the nth step. By Theorem 5.3, let us take a $\gamma^{(n-1)}$-cocycle $u_{n+1}(\cdot)$ such that:

$(a.n+1)$ $\|{}_{u_{n+1}}\gamma_g^{(n-1)}(\xi) - \gamma_g^{(n)}(\xi)\| < \varepsilon_{n+1}, g \in K_{n+1}, \xi \in \Phi_n$;

$(a.n+2)$ $d_\mu(\widehat{{}_{u_{n+1}}\gamma_g^{(n-1)}}(t), \widehat{\gamma_g^{(n)}}(t)) < \varepsilon_{n+1},\ g \in K_{n+1}, t \in \Lambda_n$;

$(c.n+1)$ $\|{}_{u_{n+1}}\gamma_g^{(n-1)}(\xi) - \gamma_g^{(n)}(\xi)\| < \delta_n/2, g \in S_n, \xi \in \bigcup_{g\in S_n} \gamma_{g^{-1}}^{(n)}(\Phi_n)$;

$(d.n+1)$ $d_\nu(\widehat{{}_{u_{n+1}}\gamma_g^{(n-1)}}(t), \widehat{\gamma_g^{(n)}}(t)) < \delta_n/2, g \in S_n, t \in \bigcup_{g\in S_n} \gamma_{g^{-1}}^{(n)}(\Lambda_n), \nu \in \Phi_n$;

$(e.n+1)$ $\|{}_{u_{n+1}}\gamma_g^{(n-1)}(\xi) - \gamma_g^{(n)}(\xi)\| < \delta_{n-1}/2, g \in S_{n-1}, \xi \in \bigcup_{g\in S_{n-1}} \gamma_{g^{-1}}^{(n-1)}(\Phi_{n-1})$;

$(f.n+1)$ $d_\nu(\widehat{{}_{u_{n+1}}\gamma_g^{(n-1)}}(t), \widehat{\gamma_g^{(n)}}(t)) < \delta_{n-1}/2, g \in S_{n-1}, t \in \bigcup_{g\in S_{n-1}} \widehat{\gamma_{g^{-1}}^{(n-1)}}(\Lambda_{n-1})$, $\nu \in \Phi_{n-1}$.

By conditions $(6.n)$ and $(e.n+1)$, we have

$$\|u_{n+1}(g)\gamma_g^{(n-1)}(\xi) - \gamma_g^{(n-1)}(\xi)\| < \delta_{n-1}, \quad g \in S_{n-1}, \xi \in \bigcup_{g\in S_{n-1}} \gamma_{g^{-1}}^{(n-1)}(\Phi_{n-1})$$

and hence

$$\|u_{n+1}(g)(\xi) - \xi\| < \delta_{n-1}, \quad g \in S_{n-1}, \xi \in \Phi_{n-1}.$$

By conditions $(7.n)$ and $(f.n+1)$,

$$d_\nu(\widehat{u_{n+1}(g)}\widehat{\gamma_g^{(n-1)}}(t), \widehat{\gamma_g^{(n-1)}}(t)) < \delta_{n-1}, \ \ g \in S_{n-1}, t \in \bigcup_{g\in S_{n-1}} \widehat{\gamma_{g^{-1}}^{(n-1)}}(\Lambda_{n-1}), \nu \in \Phi_{n-1},$$

and hence

$$d_\nu(\widehat{u_{n+1}(g)}(t), t) < \delta_{n-1}, \quad g \in S_{n-1}, t \in \Lambda_{n-1}, \nu \in \Phi_{n-1}.$$

By Lemma 5.4, there exists $w_{n+1} \in [T]$ such that

$$d_\nu(u_{n+1}(g)\widehat{\gamma_g^{(n-1)}}(w_{n+1})w_{n+1}^{-1}, \mathrm{id}) < \varepsilon_{n-1}, \quad g \in K_{n-1}, \nu \in \Phi_{n-1},$$

$$\|w_{n+1}(\xi) - \xi\| < \varepsilon_{n-1}, \quad d_\nu(\widehat{w_{n+1}}(t), t) < \varepsilon_{n-1}, \quad \xi, \nu \in \Phi_{n-1}, t \in K_{n-1}.$$

Set

$$\begin{aligned}\bar{v}_{n+1}(g) &:= u_{n+1}(g)\widehat{\gamma_g^{(n-1)}}(w_{n+1})w_{n+1}^{-1},\\ \gamma_g^{(n+1)} &:= {}_{u_{n+1}}\gamma_g^{(n-1)} = \bar{v}_{n+1}(g)w_{n+1}\gamma_g^{(n-1)}w_{n+1}^{-1},\\ \theta_{n+1} &:= w_{n+1}\theta_{n-1}.\end{aligned}$$

We clearly have conditions $(1.n+1)$, $(2.n+1)$, $(3.n+1)$, $(8.n+1)$, $(9.n+1)$, and $(10.n+1)$. From conditions $(a.n+1)$, $(b.n+1)$, $(c.n+1)$, and $(d.n+1)$, we obtain conditions $(4.n+1)$, $(5.n+1)$, $(6.n+1)$, and $(7.n+1)$, respectively. We define Φ_{n+1} and Λ_{n+1} as in conditions $(11.n+1)$ and $(12.n+1)$, respectively. Then we have finished the $(n+1)$st step, and the induction is complete.

By the construction, we have

$$\gamma_g^{(2n)} = v_{2n}(g)\theta_{2n}\alpha_g\theta_{2n}^{-1}, \quad \gamma_g^{(2n+1)} = v_{2n+1}(g)\theta_{2n+1}\beta_g\theta_{2n+1}^{-1}.$$

We will show that sequences $\{\theta_{2n}\}_n$, $\{\theta_{2n+1}\}_n$, $\{v_{2n}(g)\}_n$, and $\{v_{2n+1}(g)\}_n$ will converge. Fix $k \in \mathbb{N}$ and take $\xi \in \{\xi_i\}_{i=1}^k$, $t \in \{T^l\}_{|l|\le k}$. For $n > k+2$, we have $\xi, \theta_{n-2}(\xi) \in \Phi_{n-2}$, $\widehat{\theta_{n-2}}(t) \in \Lambda_{n-2}$. Then

$$\|\theta_n(\xi) - \theta_{n-2}(\xi)\| = \|w_n(\theta_{n-2}(\xi)) - \theta_{n-2}(\xi)\| < \varepsilon_{n-2},$$
$$\|\theta_n^{-1}(\xi) - \theta_{n-2}^{-1}(\xi)\| = \|w_n^{-1}(\xi) - \xi\| < \varepsilon_{n-2},$$

and

$$d_\mu(\widehat{\theta_n}(t), \widehat{\theta_{n-2}}(t)) = d_\mu(\widehat{w_n}(\widehat{\theta_{n-2}}(t)), \widehat{\theta_{n-2}}(t)) < \varepsilon_{n-2}$$

hold by conditions $(9.n)$ and $(10.n)$. It follows that $\{\theta_{2n}\}_n$ and $\{\theta_{2n+1}\}_n$ are both Cauchy sequences with respect to the metric d on $N[T]$. (See §2.1 for the definition of d.) Hence, both $\{\theta_{2n}\}_n$ and $\{\theta_{2n+1}\}_n$ converge to some $\sigma_0, \sigma_1 \in \overline{[T]}$, respectively, in the u-topology.

Fix $l \in \mathbb{N}$ and take any $g \in K_l$. Then for $n > l+2$, we have $v_{n-2}(g), v_{n-2}(g)^{-1} \in \Lambda_{n-2}$, $v_{n-2}(g)(\mu) \in \Phi_{n-2}$. Thus,

$$\begin{aligned}
&d_\mu(v_n(g), v_{n-2}(g)) \\
&\quad \le d_\mu(\bar{v}_n(g)\widehat{w_n}(v_{n-2}(g)), \bar{v}_n(g)v_{n-2}(g)) + d_\mu(\bar{v}_n(g)v_{n-2}(g), v_{n-2}(g)) \\
&\quad = d_\mu(\widehat{w_n}(v_{n-2}(g)), v_{n-2}(g)) + d_{v_{n-2}(g)(\mu)}(\bar{v}_n(g), \mathrm{id}) \\
&\quad < 2\varepsilon_{n-2}
\end{aligned}$$

and

$$\begin{aligned}
&d_\mu(v_n(g)^{-1}, v_{n-2}(g)^{-1}) \\
&\quad \le d_\mu(\widehat{w_n}(v_{n-2}(g)^{-1})\bar{v}_n(g)^{-1}, \widehat{w_n}(v_{n-2}(g)^{-1})) + d_\mu(\widehat{w_n}(v_{n-2}(g)^{-1}), v_{n-2}(g)^{-1}) \\
&\quad = d_\mu(\bar{v}_n(g)^{-1}, \mathrm{id}) + d_\mu(\widehat{w_n}(v_{n-2}(g)^{-1}), v_{n-2}(g)^{-1}) \\
&\quad < 2\varepsilon_{n-2}
\end{aligned}$$

by conditions $(8.n)$ and $(10.n)$. Thus, both $\{v_{2n}(g)\}_n$ and $\{v_{2n+1}(g)\}_n$ are Cauchy sequences with respect to d_μ, and hence converge to some $z_0(g), z_1(g) \in [T]$ uniformly, respectively.

Summarizing these results, we have

$$\lim_{n\to\infty} \gamma_g^{(2n)} = \lim_{n\to\infty} v_{2n}(g)\theta_{2n}\alpha_g\theta_{2n}^{-1} = z_0(g)\sigma_0\alpha_g\sigma_0^{-1},$$
$$\lim_{n\to\infty} \gamma_g^{(2n+1)} = \lim_{n\to\infty} v_{2n+1}(g)\theta_{2n+1}\beta_g\theta_{2n+1}^{-1} = z_1(g)\sigma_1\beta_g\sigma_1^{-1}.$$

By conditions $(4.n)$ and $(5.n)$, we have $z_0(g)\sigma_0\alpha_g\sigma_0^{-1} = z_1(g)\sigma_1\beta_g\sigma_1^{-1}$. Hence, α and β are cocycle conjugate. □

Proof of Theorem 2.4. Let $N := N_\alpha = N_\beta$, $Q := G/N$, and $\pi : G \to Q$ be the quotient map. Fix a section $s : Q \to G$ such that $s(e) = e$. Then $\alpha_{s(p)}$ is an ultrafree cocycle crossed action of Q. By Theorem 4.4, there exists $v \in C^1(Q, [T])$ such that $\bar{\alpha}_p := v(p)\alpha_{s(p)}$ is a genuine action of Q. Here define $v(g) := v(p)\alpha_n^{-1} \in [T]$, where $g = ns(p)$ with $p = \pi(g)$ and $n \in N$. Then $v(g)\alpha_g = v(p)\alpha_{s(p)} = \bar{\alpha}_{\pi(g)}$, and $\bar{\alpha}_{\pi(g)}$ is an action of G. Thus, α_g is strongly cocycle conjugate to $\bar{\alpha}_{\pi(g)}$ for some ultrafree action $\bar{\alpha}$ of Q. In the same way, β_g is strongly cocycle conjugate to $\bar{\beta}_{\pi(g)}$ for some ultrafree action $\bar{\beta}$ of Q. Since $\mathrm{mod}(\bar{\alpha}_p) = \mathrm{mod}(\bar{\beta}_p)$, $\bar{\alpha}$ and $\bar{\beta}$ are strongly cocycle conjugate as actions of Q by Theorem 5.5, and hence also are as actions of G. Therefore, the two actions α and β of G are strongly cocycle conjugate. □

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