SOME BESSEL FUNCTION INTEGRALS

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The basic formula to be proved is

$$\int_0^\infty E(p\,;\alpha_r:q\,;\rho_s:z\,\text{sech}^2\,u)(\sinh u)^{2n-1}\cosh u\,du=\tfrac{1}{2}\Gamma(n)z^n\,E(p\,;\alpha_r-n:q\,;\rho_s-n:z),\quad \ldots(1)$$

where $p \geqq q+1$, $z\neq 0$, $|\text{amp}\,z|<\pi$, $R(n)>0$, $R(\alpha_r-n)>0$, $r=1, 2, \ldots, p$. For other values of p and q the result holds if the integral converges. From this formula some results, involving Bessel functions and Confluent Hypergeometric functions, will be deduced.

In the formula

$$\int_0^\infty e^{-\mu/z}\mu^{n-1}d\mu=\Gamma(n)z^n,$$

where $R(z)>0$, $R(n)>0$, replace μ by $\lambda-1$ and it can be written

$$\int_1^\infty E(\ :\ :z/\lambda)(\lambda-1)^{n-1}\,d\lambda=\Gamma(n)z^n\,E(\ :\ :z)\,;\quad \ldots\ldots\ldots\ldots(2)$$

and, on generalising, this gives

$$\int_1^\infty E(p\,;\alpha_r:q\,;\rho_s:z/\lambda)(\lambda-1)^{n-1}\,d\lambda=\Gamma(n)z^n\,E(p\,;\alpha_r-n:q\,;\rho_s-n:z),\quad \ldots\ldots(3)$$

where $p \geqq q+1$, $z\neq 0$, $|\text{amp}\,z|<\pi$, $R(n)>0$, $R(\alpha_r-n)>0$, $r=1, 2, \ldots, p$. For other values of p and q the result holds if the integral converges. Formula (1) is obtained by putting $\lambda=\cosh^2 u$.

Note 1. In the process of increasing q while p remains fixed the formula used is

$$\frac{1}{2\pi i}\int e^{\zeta}\zeta^{-\rho}\,d\zeta=\frac{1}{\Gamma(\rho)},\quad \ldots\ldots\ldots\ldots(4)$$

where the integral starts from $-\infty$ on the ξ-axis, passes round the origin in the positive direction, and returns to $-\infty$ on the ξ-axis, and amp $\zeta=0$ to the right of the origin. In deriving the case, $p=0$, $q=1$, from (2) z should be taken to be real and positive and the contour should be replaced by a line parallel to and to the right of the imaginary axis. The integral (4) then converges if $R(\rho)>0$.

Note 2. If λ is replaced by $1/\lambda$ in (3), the resulting formula is a particular case of the formula given on page 118 of Volume I of these *Proceedings*, or of Ragab's more general formula (2) on page 77 of the present volume. By replacing the variables of integration in these formulae by $\text{sech}^2\,u$, more general integrals of the same type as (1) can be obtained.

Now in (3) put $p=0$, $q=1$, $\rho_1=\frac{3}{2}$ and replace n by $\frac{1}{2}-n$, λ by ξ^2 and z by $4/x^2$; so obtaining the known formula *

$$\int_1^\infty \frac{\sin(x\xi)}{(\xi^2-1)^{n+\frac{1}{2}}}\,d\xi=\tfrac{1}{2}\Gamma(\tfrac{1}{2})\,\Gamma(\tfrac{1}{2}-n)\left(\frac{x}{2}\right)^n J_n(x),\quad \ldots\ldots\ldots\ldots(5)$$

where x is real and positive and $-\frac{1}{2}<R(n)<\frac{1}{2}$.

* Titchmarsh, E.C., *Introduction to the Theory of Fourier Integrals*, p. 200.

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In what follows the formulae

$$\cos m\pi\, E(\tfrac{1}{2}+m, \tfrac{1}{2}-m : : 2z) = \surd(2\pi z) e^z K_m(z), \quad \text{.........................(6)}$$

$$E(\tfrac{1}{2}-k+m, \tfrac{1}{2}-k-m : : z) = \Gamma(\tfrac{1}{2}-k+m)\, \Gamma(\tfrac{1}{2}-k-m) z^{-k} e^{\frac{1}{2}z} W_{k,m}(z) \quad \text{...........(7)}$$

will be required.

In (1) put $p=2$, $q=0$, $\alpha_1=\frac{1}{2}+m$, $\alpha_2=\frac{1}{2}-m$, and replace n by k and z by $2z$; then, from (6) and (7),

$$\int_0^\infty e^{z \operatorname{sech}^2 u} K_m(z \operatorname{sech}^2 u)(\sinh u)^{2k-1}\, du$$

$$= \frac{\cos m\pi}{\surd(2\pi z)} \tfrac{1}{2}\, \Gamma(k)\, \Gamma(\tfrac{1}{2}-k+m)\, \Gamma(\tfrac{1}{2}-k-m) e^z W_{k,m}(2z), \quad \text{.....................(8)}$$

where $z \neq 0$, $|\operatorname{amp} z| < \pi$, $R(k)>0$, $R(k \pm m) < \frac{1}{2}$.

Again, in (1) put $p=2$, $q=0$, $\alpha_1=\frac{1}{2}+k+m$, $\alpha_2=\frac{1}{2}+k-m$ and replace z by $2z$ and n by k; then

$$\int_0^\infty e^{z \operatorname{sech}^2 u} W_{-k,m}(2z \operatorname{sech}^2 u)(\tanh u)^{2k-1}\, du = \frac{\Gamma(k)\, \surd(2\pi z)\, e^z K_m(z)}{2 \cos m\pi\, \Gamma(\frac{1}{2}+k+m)\, \Gamma(\frac{1}{2}+k-m)}, \quad \text{...(9)}$$

where $z \neq 0$, $|\operatorname{amp} z| < \pi$, $R(k)>0$, $R(\frac{1}{2} \pm m)>0$.

Finally, let $p=0$, $q=1$, $\rho_1=m+1$, and replace z by $4/x^2$; then, if x is real and positive,

$$\int_0^\infty J_m(x \cosh u)(\sinh u)^{2n-1}(\cosh u)^{1-m}\, du = 2^{n-1}\, \Gamma(n) x^{-n} J_{m-n}(x), \quad \text{...........(10)}$$

provided that $R(n)>0$, $R(m-2n)>-\frac{1}{2}$.

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