BULL. AUSTRAL. MATH. SOC. 28C15
VOL. 32 (1985), 207-215.

RADON POLYMEASURES

BRIAN JEFFERIES

The Radon-Nikodým theorem and a sequential convergence result are given for integrals with respect to a Radon polymeasure on a finite product space.

The notion of a polymeasure has been introduced in relation to the study of perturbations to evolutions [2]. Vector measure techniques were used by Kluvánek [3] to treat the particular case of integration with respect to bimeasures, or polymeasures in two variables. Earlier work seems only to be concerned with the integral of product functions, perhaps because the principal motivation for considering bimeasures was to obtain an analogue of the Riesz representation theorem for bilinear mappings [6].

While these techniques may be generalized to polymeasures with a finite number of variables, a systematic treatment of integrals with respect to sufficiently general polymeasures is still lacking. The aim of this note is to at least partially fill this gap for the class of *Radon polymeasures* on finite product spaces.

Such polymeasures typically arise from the finite dimensional distributions of an evolution process [2], so roughly speaking, integration with respect to this class of set functions is relevant to the determination of a perturbation to an evolution which is specified at finitely many instants of time. Integration with respect to polymeasures in infinitely many variables (corresponding to continuous perturbations) is of greater interest, but this important topic is not touched upon here.

Received 11 January 1985.

Let $\mathcal{A}$ be an algebra of subsets of a set Ω. The *variation* $|n| : \mathcal{A} \to [0,\infty]$ of an additive set function $n : \mathcal{A} \to \mathbb{C}$ is defined by

$$|n|(A) = \sup\{\sum_{B\in\pi} |n(B)| : \pi \text{ is a finite } \mathcal{A}\text{-partition of } A\},\ A \in \mathcal{A}\ .$$

Let Σ be a Hausdorff topological space. Denote the Borel σ-algebra of Σ by $\mathcal{B}(\Sigma)$. An extended real valued measure $\mu : \mathcal{B}(\Sigma) \to [0,\infty]$ is said to be a *Radon measure* on Σ if every point of Σ is contained in an open set of finite measure, and for every $A \in \mathcal{B}(\Sigma)$

$$\mu(A) = \sup\{\mu(K) : K \subset A\ ,\ K \text{ compact}\}\ .$$

A complex valued measure $m : \mathcal{B}(\Sigma) \to \mathbb{C}$ is called a Radon measure on Σ if the variation $|m|$ of m is a Radon measure.

Now suppose that $\Sigma_1,\ldots,\Sigma_n$ are Hausdorff topological spaces. The family of finite unions of product sets $A_1 \times\ldots\times A_n$, $A_i \in \mathcal{B}(\Sigma_i)$, $i = 1,\ldots,n$ is an algebra of subsets of $\Sigma_1 \times\ldots\times \Sigma_n$ denoted by $\mathcal{B}(\Sigma_1) \times\ldots\times \mathcal{B}(\Sigma_n)$, or more concisely, by $\mathcal{E}$.

DEFINITION 1. An additive set function $m : \mathcal{B}(\Sigma_1) \times\ldots\times \mathcal{B}(\Sigma_n) \to \mathbb{C}$ is said to be a *Radon polymeasure* if the following two conditions hold.

R_1 : For each $j = 1,\ldots,n$ the set function

$$A_j \to m(A_1 \times\ldots\times A_{j-1} \times A_j \times A_{j+1} \times\ldots\times A_n)\ ,\ A_j \in \mathcal{B}(\Sigma_j)\ ,$$

is a Radon measure for every $A_i \in \mathcal{B}(\Sigma_i)$, $i \neq j$, $i = 1,\ldots,n$.

R_2 : The variation $|m|$ of m is the restriction of a Radon measure $|\tilde{m}|$ on $\Sigma_1 \times\ldots\times \Sigma_n$ to the algebra $\mathcal{B}(\Sigma_1) \times\ldots\times \mathcal{B}(\Sigma_n)$.

PROPOSITION 1. *Let $m : \mathcal{E} \to \mathbb{C}$ be an additive set function for which condition R_1 holds. Then R_2 holds if and only if for each $x \in \Sigma_1 \times\ldots\times \Sigma_n$ there exists an open subset $U_1 \times\ldots\times U_n$ of $\Sigma_1 \times\ldots\times \Sigma_n$ containing x such that $|m|(U_1 \times\ldots\times U_n) < \infty$. In this case, the Radon measure $|\tilde{m}|$ is unique.*

Proof. If R_2 holds, then the given condition follows from the fact that products of open sets form a neighbourhood base for the product topology of $\Sigma_1 \times \ldots \times \Sigma_n$.

Now suppose that R_1 holds. Let $A = A_1 \times \ldots \times A_n$ be a set in E such that $|m|(A) < \infty$. Given $\varepsilon > 0$, let B_j , $j = 1,\ldots,p$ be an E-partition of A such that $|m|(A) - \sum_{j=1}^{p} |m(B_j)| < \varepsilon/2$.

The sets B_j , $j = 1,\ldots,p$ can be assumed to be product sets of the form $B_j = A_1^j \times \ldots \times A_n^j$, $j = 1,\ldots,p$. By R_1 there exist compact subsets K_i^j of A_i^j , $i = 1,\ldots,n$, such that

$$|m(A_1^j \times \ldots \times A_n^j) - m(A_1^j \times \ldots \times A_{n-1}^j \times K_n^j)| < \varepsilon/(2np)$$

$$|m(A_1^j \times \ldots \times A_{n-1}^j \times K_n^j) - m(A_1^j \times \ldots \times A_{n-2}^j \times K_{n-1}^j \times K_n^j)| < \varepsilon/(2np)$$

$$\vdots$$

$$|m(A_1^j \times K_2^j \times \ldots \times K_n^j) - m(K_1^j \times \ldots \times K_n^j)| < \varepsilon/(2np)$$

for $j = 1,\ldots,p$. Consequently $\sum_{j=1}^{p} |m(B_j) - m(K_1^j \times \ldots \times K_n^j)| < \varepsilon/2$. Put $K = \cup_{j=1}^{p} K_1^j \times \ldots \times K_n^j$. Then K is a compact subset of A and we have

$$|m|(A) - |m|(K) \leqslant |m|(A) - \sum_{j=1}^{p} |m(B_j \cap K)|$$

$$\leqslant |m|(A) - \sum_{j=1}^{p} |m(B_j)| + \varepsilon/2 < \varepsilon .$$

Thus for every $A \in E$ with $|m|(A) < \infty$,

$$|m|(A) = \sup\{|m|(K) : K \subset A , K \in E , K \text{ compact}\} .$$

The additive set function $|m| : E \to [0,\infty]$ is therefore σ-additive, and finite on compact subsets of $\Sigma_1 \times \ldots \times \Sigma_n$. The Carathéodory-Hahn extension procedure now shows that $|m|$ is the restriction of a unique Radon measure $|\tilde{m}|$ on $\Sigma_1 \times \ldots \times \Sigma_n$. The details may be gleaned from Fremlin [1;71A].

Thus a *bounded* additive set function satisfying R_1 may be extended uniquely to a Radon measure on $\Sigma_1 \times \ldots \times \Sigma_n$. In particular, if the spaces

$\Sigma_1, \ldots, \Sigma_n$ are compact, then Radon polymeasures are the restrictions of finite Radon measures.

Suppose that $m : E \to \mathbb{R}$ is a Radon polymeasure. Define $m^{\pm} : E \to [0,\infty]$ by $m^{\pm}(A) = (|m|(A) \pm m(A))/2$ for each $A \in E$. Proposition 1 shows that m^{+} and m^{-} are the restrictions to E of uniquely defined Radon measures $\tilde{m}^{+}$ and $\tilde{m}^{-}$ respectively. Integration with respect to Radon measures is taken in the sense of Schwartz [5], so that the integral of a function with respect to Radon measure is again a Radon measure.

DEFINITION 2. A function $f : \Sigma_1 \times \ldots \times \Sigma_n \to \mathbb{C}$ is said to be *m-integrable* if for each compact subset K of $\Sigma_1 \times \ldots \times \Sigma_n$ the function $f\chi_K$ is $|\tilde{m}|$-integrable, and there exists a Radon polymeasure $fm : E \to \mathbb{C}$ such that

$$fm(K) = \tilde{m}^{+}(f\chi_K) - \tilde{m}^{-}(f\chi_K)$$

for each compact set $K \in E$.

The values of a Radon polymeasure are determined by its values on compact sets, so if fm exists it is unique *within the class of Radon polymeasures*. It is termed the *indefinite integral* of f with respect to m. The *definite integral* is defined by $m(f) = fm(\Sigma_1 \times \ldots \times \Sigma_n)$.

An additive set function $p : E \to \mathbb{R}$ is said to be *locally absolutely continuous* with respect to the additive set function $q : E \to \mathbb{R}$ (denoted $p \ll_{\ell} q$) if for every compact set $K \in E$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that $|p|(A) < \varepsilon$ whenever $A \in E \cap K$ and $|q|(A) < \delta$. Clearly if f is m-integrable, then $fm \ll_{\ell} m$. The converse is given by the Radon-Nikodým theorem.

THEOREM 1. *Let* $m : E \to \mathbb{R}$, $n : E \to \mathbb{R}$ *be Radon polymeasures such that* $n \ll_{\ell} m$. *Then there exists an m-integrable function* $f : \Sigma_1 \times \ldots \times \Sigma_n \to \mathbb{R}$ *such that* $n = fm$.

If g *is another m-integrable function such that* $n = gm$, *then* $f = g$ $|\tilde{m}|$*-almost everywhere.*

https://doi.org/10.1017/S0004972700009904 Published online by Cambridge University Press

Proof. A standard argument shows that in the terminology of [5], every $|\tilde{m}|$-negligible set is also $|\tilde{n}|$-negligible. An application of the Lebesgue-Nikodým theorem [5; p.47] yields a positive, real valued $|\tilde{m}|$-measurable and locally $|\tilde{m}|$-integrable function h such that $|\tilde{n}| = h|\tilde{m}|$.

By virtue of the same theorem, for each compact set $K \in E$ there exists a function f_K on $\Sigma_1 \times \ldots \times \Sigma_n$ vanishing outside of K such that

$$n(K \cap A) = f_K\tilde{m}^+(K \cap A) - f_K\tilde{m}^-(K \cap A)$$

for every $A \in E$. Moreover, if $K_1, K_2 \in E$ are compact sets such that $K_1 \subset K_2$, then $f_{K_2} \mid K_1 = f_{K_1}$ $|\tilde{m}|$-almost everywhere by the essential uniqueness of densities. Let f_K^+, f_K^- be the positive and negative parts respectively of f_K for $K \in E$ compact. The families $\{f_K^+ : K \in E$ compact$\}$, $\{f_K^- : K \in E$ compact$\}$ of (equivalence classes of) functions is bounded above in the space $L^0(|\tilde{m}|)$ of (equivalence classes of) $|\tilde{m}|$-measurable functions by h . According to Schwartz [5;p.46] there exists an $|\tilde{m}|$-concassage of $|\tilde{m}|$; in the terminology of Fremlin [1;64G], this means that the associated measure space is decomposable. Therefore the space $L^0(|\tilde{m}|)$ is Dedekind complete by [1;64H,64B]. Denote by f^+, f^- a version of the least upper bound in $L^0(|\tilde{m}|)$ of the families $\{f_K^+ : K \in E$ compact$\}$, $\{f_K^- : K \in E$ compact $\}$ respectively.

Let $f = f^+ - f^-$. Then it is easily seen that $f\chi_K = f_K$ $|\tilde{m}|$-almost everywhere for each compact $K \in E$. Because n is a polymeasure, f is m-integrable by Definition 2. The essential uniqueness of f follows from the fact that two densities must agree almost everywhere on each compact set $K \in E$.

Radon polymeasures satisfy a weak property of interchangeability of limits and integrals. Let m be a Radon polymeasure on $\Sigma_1 \times \Sigma_2 \times \ldots \times \Sigma_n$.

THEOREM 2. *Let* f_k , $k = 1,2,\ldots$ *be a sequence of real valued* m*-integrable functions. Let* f *be a locally* $|\tilde{m}|$*-integrable function, and suppose that for each compact subset* K *of* $\Sigma_1 \times \ldots \times \Sigma_n$,

https://doi.org/10.1017/S0004972700009904 Published online by Cambridge University Press

$\lim_{k\to\infty} |\tilde{m}|(f_k\chi_K) = |\tilde{m}|(f\chi_K)$.

If for each $A \in E$, *the sequence* $f_k m(A)$, $k = 1,2,\ldots$ *converges, then* f *is m-integrable and* $fm(A) = \lim_{k\to\infty} f_k m(A)$ *for each* $A \in E$.

Proof. Let S^+, S^- be the supports of the Radon measures $\tilde{m}^+, \tilde{m}^-$ respectively. Then $|\tilde{m}|(S^+ \cap S^-) = 0$ and $\chi_{S^+}|\tilde{m}| = \tilde{m}^+$, $\chi_{S^-}|\tilde{m}| = \tilde{m}^-$. The sets $S^\pm$ are closed, so if $K \in E$ is compact, then so are the sets $K^\pm = K \cap S^\pm$ and we have

$$\lim_{k\to\infty} \tilde{m}^\pm(f_k\chi_K) = \lim_{k\to\infty} |\tilde{m}|(f_k\chi_{K^\pm}) = |\tilde{m}|(f\chi_{K^\pm})$$
$$= \tilde{m}^\pm(f\chi_K) .$$

Let $n(A) = \lim_{k\to\infty} f_k m(A)$ for each $A \in E$. Then $n(K) = \tilde{m}^+(f\chi_K) - \tilde{m}^-(f\chi_K)$. It remains to show that n is a Radon polymeasure. Let $A_1 \times \ldots \times A_n \in E$. Property R_1 is verified with respect to the first variable A_1 ; the same argument holds for the others. There exists an increasing family C^i , $i = 1,2,\ldots$ of compact subsets of A_1 such that for each $k = 1,2,\ldots$ we have

$$\lim_{i\to\infty} f_k m\, C^i \times A_2 \times \ldots \times A_n) = f_k m(A_1 \times \ldots \times A_n) ,$$

because $f_k m$ is a Radon polymeasure. The Vitali-Hahn-Saks theorem ensures that the convergence as $i \to \infty$ is uniform for $k = 1,2,\ldots$ and the set function $n(\cdot \times A_2 \times \ldots \times A_n)$ is a measure. It follows that $n(\cdot \times A_2 \times \ldots \times A_n)$ is a Radon measure on Σ_1 ; by taking the Hahn decomposition of the measure, it is easily seen that the variation is a finite Radon measure. Thus condition R_1 is satisfied by n . Because f is assumed to be locally integrable, Proposition 1 shows that n is a Radon polymeasure.

COROLLARY 1. (i) *Let* f_k , $k = 1,2,\ldots$ *be a sequence of non-negative m-integrable functions such that* $f_k \uparrow f$ $|\tilde{m}|$*-almost everywhere. Suppose* *that* f *is locally* $|\tilde{m}|$*-integrable. If for each* $A \in E$ *, the sequence* $f_k m(A)$, $k = 1,2,\ldots$ *converges, then* f *is* m*-integrable and*

$fm(A) = \lim_{k\to\infty} f_k m(A)$ *for each* $A \in E$.

(ii) *Let* f_k , $k = 1,2,\ldots$ *be a sequence of* m*-integrable functions such that* $f_k \to f$ $|\tilde{m}|$*-almost everywhere. Furthermore, suppose that there exists a locally* $|\tilde{m}|$*-integrable function* g *such that* $|f_k| \leq |g|$ $|\tilde{m}|$*-almost everywhere,* $k = 1,2,\ldots$. *If for each* $A \in E$ *, the sequence* $f_k m(A)$, $k = 1,2,\ldots$ *converges, then* f *is* m*-integrable and* $fm(A) = \lim_{k\to\infty} f_k m(A)$ *for each* $A \in E$.

Theorem 2 has the following topological interpretation when $\Sigma_1,\ldots,\Sigma_n$ are locally compact. Let $L^1_{\text{loc}}(|\tilde{m}|)$ be the space of locally $|\tilde{m}|$-integrable functions with the topology of convergence in the mean on compact subsets of $\Sigma_1 \times \ldots \times \Sigma_n$, and let $RL^1(m)$ be the subspace of $L^1_{\text{loc}}(|\tilde{m}|)$ consisting of m-integrable functions. Then the integration map $f \mapsto fm$, $f \in RL^1(m)$ is sequentially closed from the space $L^1_{\text{loc}}(|\tilde{m}|)$ into the space of additive set functions on E with the topology of setwise convergence. It cannot be expected that the integration map is continuous in this sense, because it would imply that m is σ-additive on E which is not true in general.

The point to be made is that it is possible to integrate with respect to unbounded set functions and still retain useful convergence properties.

EXAMPLE 1. Let $\ell \geq 1$ be an integer. Put $\Sigma_j = \mathbb{R}^\ell$ for $j = 1,2$. Let $\phi \in L^2(\mathbb{R}^\ell) \setminus L^1(\mathbb{R}^\ell)$, $\psi \in L^2(\mathbb{R}^\ell) \setminus L^1(\mathbb{R}^\ell)$ and

$$p(x) = (\pi i)^{-\frac{1}{2}}\exp(i|x|^2) , \ x \in \mathbb{R}^\ell$$

For every set $S \in B(\mathbb{R}^\ell)$ with finite Lebesgue measure, let

$$M(S)(y) = \int_S p(y-x)\phi(x)\,dx , \ y \in \mathbb{R}^\ell .$$

Then M can be uniquely extended to a measure $M : B(\mathbb{R}^\ell) \to L^2(\mathbb{R}^\ell)$. Now define

https://doi.org/10.1017/S0004972700009904 Published online by Cambridge University Press

$$\beta(A\times B) = \int_B M(A)(y)\overline{\psi(y)}\,dy \ , \ A,B \in \mathcal{B}(\mathbb{R}^\ell) \ .$$

Then $\beta : \mathcal{B}(\mathbb{R}^\ell) \times \mathcal{B}(\mathbb{R}^\ell) \to \mathbb{C}$ is a bimeasure. The variation $|\beta|$ of β is

$$|\beta|(A\times B) = \int_A |\phi|(x)\,dx \int_B |\psi|(x)\,dx \ , \ A,B \in \mathcal{B}(\mathbb{R}^\ell),$$

so it is σ-finite. Signed Borel measures on $\mathbb{R}^\ell$ are automatically Radon measures, so β is a Radon bimeasure.

The bimeasure is itself first defined on compact product sets and then extended to the whole algebra $\mathcal{B}(\mathbb{R}^\ell) \times \mathcal{B}(\mathbb{R}^\ell)$. Consequently, Definition 2 is the natural way to view integrals with respect to β . Polymeasures of this type arise in connection with Schrödinger's equation [2].

As pointed out in [3], many examples of bimeasures which do not extend to measures on the generated σ-algebras can be constructed by exploiting the difference between the Bochner and Pettis integrals. A similar idea gives an example of a polymeasure which is not a Radon polymeasure.

EXAMPLE 2. Let x_k , $k = 1,2,\ldots$ be an unconditionally summable sequence in $\ell^1(\mathbb{N})$ which is not absolutely summable. The Dvoretsky-Rogers theorem guarantees the existence of such a sequence in any infinite-diminsional Banach space; an explicit example is given in [4].

Let $A_k = [1/(k+1), 1/k]$, $k = 1,2,\ldots$. The Lebesgue measure on $\mathbb{R}$ is denoted by λ . Then for every $A \in \mathcal{B}(\mathbb{R})$,

$$\sum_{j=1}^{\infty} \Big| \sum_{k=1}^{\infty} x_k(j)\lambda(A\cap A_k)/\lambda(A_k) \Big| < \infty \ .$$

Define an additive map $m : \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \to \mathbb{R}$ by

$$m(A\times B) = \sum_{j=1}^{\infty} \{ \sum_{k=1}^{\infty} x_k(j)\lambda(A\cap A_k)/\lambda(A_k)\}\delta_{1/j}(B) \ , \ A,B \in \mathcal{B}(\mathbb{R}).$$

Here δ_a denotes the unit point mass at $a \in \mathbb{R}$.

It is easily verified that m is a bimeasure and that condition R_1 holds. However the variation of m on the compact subset $[0,1] \times [0,1]$ of $\mathbb{R} \times \mathbb{R}$ is $\sum_{k=1}^{\infty} \sum_{k=1}^{\infty} |x_k(j)| = \sum_{k=1}^{\infty} \|x_k\|_1 = \infty$. Consequently, m is not a

Radon polymeasure on $\mathbb{R} \times \mathbb{R}$.

There are several directions in which the approach presented here may be extended. Similar arguments would work for infinite product spaces, but unfortunately the most interesting examples would not satisfy the assumptions [2]; the variation of a polymeasure arising from an evolution process typically takes only the values 0 or ∞. In the abstract setting, the family of compact sets could be replaced by a compact family of sets, and it could then be assumed that the variation of the polymeasure extends uniquely to some form of localizable measure; apparently *some* compactness conditions are needed [3].

References

[1] D.H. Fremlin, *Topological Riesz Spaces and Measure Theory* (Cambridge University Press, Cambridge, 1974).

[2] B. Jefferies, "Evolution processes and the Feynman-Kac formula", preprint.

[3] I. Kluvánek, "Remarks on bimeasures", *Proc. Amer. Math. Soc.* 81 (1981), 233-239.

[4] M.S. Macphail, "Absolute and unconditional convergence", *Bull. Amer. Math. Soc.* 53 (1947), 121-123.

[5] L. Schwartz, *Radon Measures on Arbitrary Topological Spaces and Cylindrical Measures* (Oxford University Press, 1973).

[6] K. Ylinen, "On vector bimeasures", *Annali Mat. Pura Appl.* 117 (1978), 115-138.

Department of Mathematics,
The Ohio State University,
231 West 18th Avenue,
Columbus, Ohio 43210
U.S.A.

https://doi.org/10.1017/S0004972700009904 Published online by Cambridge University Press