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p-adic L-functions for GL_2

Daniel Barrera Salazar and Chris Williams

Abstract. Since Rob Pollack and Glenn Stevens used overconvergent modular symbols to construct p-adic L-functions for non-critical slope rational modular forms, the theory has been extended to construct p-adic L-functions for non-critical slope automorphic forms over totally real and imaginary quadratic fields by the first and second authors, respectively. In this paper, we give an analogous construction over a general number field. In particular, we start by proving a control theorem stating that the specialisation map from overconvergent to classical modular symbols is an isomorphism on the small slope subspace. We then show that if one takes the modular symbol attached to a small slope cuspidal eigenform, then one can construct a ray class distribution from the corresponding overconvergent symbol, which moreover interpolates critical values of the L-function of the eigenform. We prove that this distribution is independent of the choices made in its construction. We define the p-adic L-function of the eigenform to be this distribution.

1 Introduction

The study of L-functions has proved extremely fruitful in number theory for almost two centuries, and there are a wealth of research papers relating their critical values to important arithmetic information. A much more recent branch of the theory is the construction and study of p-adic L-functions that are natural p-adic analogues of classical (complex) L-functions. These p-adic L-functions are distributions on certain ray class groups that interpolate the algebraic parts of critical classical L-values. Such p-adic L-functions have been constructed in a number of cases; for example, one can attach p-adic L-functions to Dirichlet characters, number fields, and rational elliptic curves. Where they exist, these objects have had a number of interesting applications. For example, the *Iwasawa main conjectures* are a wide-ranging series of conjectures predicting deep links between p-adic L-functions and Selmer groups attached to Galois representations. The Iwasawa main conjecture has been proved by Skinner and Urban for a large class of elliptic curves [SU14]. If the main conjecture holds for an elliptic curve E, then the order of vanishing of the p-adic L-function of E is directly related to the rank of the p-Selmer group of E. Under finiteness of $Ш(E/\mathbb{Q})$, this is enough to deduce a p-adic analogue of the Birch and Swinnerton-Dyer conjecture (see [MTT86, Dis16] for details of the conjecture). Moreover, the

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Iwasawa main conjecture has been used to prove the p-part of the leading term formula in the (classical) Birch and Swinnerton–Dyer conjecture in analytic ranks 0 and 1 [JSW15, Cas17, CÇSS17].

Mazur and Swinnerton-Dyer [MSD74] gave the first constructions of p-adic L-functions for classical modular forms, and their work has been followed by a variety of other constructions. In particular, in 2011, Pollack and Stevens gave an alternative construction using overconvergent modular symbols [PS11]. Until recently, p-adic L-functions of automorphic forms for GL_2 over more general number fields had been constructed only in isolated cases. For the most general results previously known, see [Har87, Dep16], where such p-adic L-functions are constructed for weight 2 (also known as parallel weight 0) forms that are ordinary at p.

Pollack and Stevens's construction of p-adic L-functions for *small slope* classical modular forms is both beautiful and computationally effective. The first author generalised their approach to the case of Hilbert modular forms [BS13], whilst the second author generalised their approach to Bianchi modular forms (that is, modular forms for GL_2 over imaginary quadratic fields) [Wil17]. These two generalisations use very different methods, owing to the different difficulties that arise in the respective cases. In this paper, we generalise these results further to construct p-adic L-functions for small-slope automorphic forms for GL_2 over any number field.

1.1 Summary of the Results

The construction of these p-adic L-functions is essentially completed via a blend of the methods used previously by the authors in their respective Ph.D. theses. We now give a quick summary of the argument. Throughout the paper, we take Φ to be a cohomological cuspidal automorphic eigenform of weight λ and level $\Omega_1(\mathfrak{n})$ over a number field F, where λ and $\Omega_1(\mathfrak{n})$ are defined as in Section 2.1. We write $d = r_1 + 2r_2$ for the degree of F, where r_1 (resp. r_2) denotes the number of real (resp. complex) places of F.

Let $q = r_1 + r_2$. The space of *modular symbols* of level $\Omega_1(\mathfrak{n})$ and weight λ is the compactly supported cohomology space $\mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{V}_\lambda)$, where $Y_1(\mathfrak{n})$ is the locally symmetric space associated with $\Omega_1(\mathfrak{n})$ and $\mathcal{V}_\lambda$ is a suitable sheaf of polynomials on $Y_1(\mathfrak{n})$ depending on the weight. The *Eichler–Shimura isomorphism* gives a Hecke-equivariant isomorphism between this cohomology group and the direct sum of certain spaces of automorphic forms. In particular, with each automorphic form Φ as above—an inherently analytic object—one can associate a canonical modular symbol (up to scaling) in a way that preserves Hecke data. In passing from an analytic to an algebraic object, we obtain something that is in some ways easier to study.

Using *evaluation maps*, which were described initially by Dimitrov for totally real fields [Dim13] and which we have generalised to the case of arbitrary number fields, we relate this modular symbol to critical values of the L-function of the automorphic form. We show that these results have an algebraic analogue; that is, we can pass to a cohomology class with coefficients in a sufficiently large number field, and then relate this to the algebraic part of the critical L-values of Φ. In particular, we sketch a proof of the following result (see Theorem 6.7 for a more precise formulation):

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Theorem 1.1 *For each Hecke character φ of F, there is a map*

$$\mathrm{Ev}_\varphi \colon \mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{V}_\lambda(A)) \longrightarrow A$$

such that if Φ is a cuspidal automorphic form of weight λ with associated A-valued modular symbol ϕ_A (for A either $\mathbb{C}$ or a sufficiently large number field), then

$$\mathrm{Ev}_\varphi(\phi_A) = (*)L(\Phi, \varphi),$$

where $L(\Phi, \cdot)$ is the L-function attached to Φ and $(*)$ is an explicit factor.

All of this is rather classical in nature, and makes explicit results that are, in theory, well known, although the authors could not find the results in the generality they require in the existing literature. At this point, we start using new p-adic methods. Henceforth, assume that $(p)|\mathfrak{n}$, and take L to be a (sufficiently large) finite extension of $\mathbb{Q}_p$. We define the space of *overconvergent modular symbols of level $\Omega_1(\mathfrak{n})$ and weight λ* to be the compactly supported cohomology of $Y_1(\mathfrak{n})$ with coefficients in an (infinite-dimensional) space of p-adic distributions equipped with an action of $\Omega_1(\mathfrak{n})$ that depends on λ.

For each prime $\mathfrak{p}|p$ in F, we have the Hecke operator $U_\mathfrak{p}$ at $\mathfrak{p}$ on both automorphic forms and (classical and overconvergent) modular symbols, induced from the action of the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \pi_\mathfrak{p} \end{smallmatrix}\right)$, where $\pi_\mathfrak{p} \in L$ is a fixed uniformiser at $\mathfrak{p}$. There is a natural *specialisation map* from overconvergent to classical modular symbols that is equivariant with respect to these operators.

In Section 8, we prove that for any $h_\mathfrak{p} \in \mathbb{Q}$, the space of overconvergent modular symbols admits a slope $\leq h_\mathfrak{p}$ decomposition (Definition 8.2) with respect to the $U_\mathfrak{p}$ operator.

Definition 1.2 Let M be an L-vector space with an action of a collection of operators $\{U_\mathfrak{p} : \mathfrak{p}|p\}$. Where it exists, we denote the slope $\leq h_\mathfrak{p}$ subspace with respect to the $U_\mathfrak{p}$ operator by $M^{\leq h_\mathfrak{p}, U_\mathfrak{p}}$. If $\mathrm{h} := (h_\mathfrak{p})_{\mathfrak{p}|p}$ is a collection of rationals indexed by the primes above p, we define $M^{\leq \mathbf{h}} := \bigcap_{\mathfrak{p}|p} M^{\leq h_\mathfrak{p}, U_\mathfrak{p}}$ to be the slope $\leq \mathbf{h}$-subspace at p.

Definition 1.3 Let $p\mathcal{O}_F = \prod \mathfrak{p}^{e_\mathfrak{p}}$ be the decomposition of p in F, and for each $\mathfrak{p}|p$ let $h_\mathfrak{p} \in \mathbb{Q}$. Let Σ denote the set of complex embeddings of F, and write the weight λ as $\lambda = ((k_\sigma), (\nu_\sigma)) \in \mathbb{Z}[\Sigma]^2$. For each $\sigma \in \Sigma$, there is a unique prime $\mathfrak{p}(\sigma)|p$ corresponding to σ, and to denote this we write $\sigma \sim \mathfrak{p}$. Define $k^0_\mathfrak{p} := \min\{k_\sigma : \sigma \sim \mathfrak{p}\}$ and $\nu_\mathfrak{p}(\lambda) := \sum_{\sigma \sim \mathfrak{p}} \nu_\sigma$.

We say that the slope $\mathbf{h} := (h_\mathfrak{p})_{\mathfrak{p}|p}$ is *small* if $h_\mathfrak{p} < (k^0_\mathfrak{p} + \nu_\mathfrak{p}(\lambda) + 1)/e_\mathfrak{p}$ for each $\mathfrak{p}|p$.

There is a surjective Hecke-equivariant specialisation map ρ from the space of overconvergent modular symbols to the space of classical modular symbols of fixed weight. In Section 9, we prove the following *control theorem*.

Theorem 1.4 *Let $\mathbf{h} \in \mathbb{Q}^{\{\mathfrak{p}|p\}}$ be a small slope. Then the restriction of the specialisation map ρ to the slope $\leq \mathbf{h}$ subspaces of the spaces of modular symbols is an isomorphism.*

In particular, to a small slope cuspidal eigenform, *i.e.*, an eigenform whose associated modular symbol lives in some small-slope subspace of the space of classical

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modular symbols, one can attach a *unique* small-slope overconvergent eigenlift of its associated modular symbol.

Let Ψ be an overconvergent eigensymbol. We can use a slightly different version of the evaluation maps from previously to construct a distribution μ_Ψ on the narrow ray class group $\mathrm{Cl}_F^+(p^\infty)$ attached to Ψ, closely following the work of the first author in [BS13]. We prove that the distribution we define is independent of the choice of class group representatives. Via compatibility between classical and overconvergent evaluation maps, this distribution then interpolates the critical values of the L-function of Φ, and we hence define the p-adic L-function to be this distribution. To summarise, the main result of this paper is the following.

Theorem 1.5 *Let Φ be a small slope cuspidal eigenform over F. Let ϕ_Φ be the (p-adic) classical modular symbol attached to Φ, and let Ψ_Φ be its (unique) small-slope overconvergent eigenlift. Let μ_Φ be the distribution on $\mathrm{Cl}_F^+(p^\infty)$ attached to Ψ_Φ.*

If φ is a critical Hecke character, then we can define a canonical locally algebraic character $\varphi_{p\text{-}fin}$ on $\mathrm{Cl}_F^+(p^\infty)$ associated with φ. Then $\mu_\Phi(\varphi_{p\text{-}fin}) = (*)L(\Phi, \varphi)$, where $(*)$ is an explicit factor.

Definition 1.6 We define the *p-adic L-function of* Φ to be the distribution μ_Φ on $\mathrm{Cl}_F^+(p^\infty)$.

For a precise notion of which characters are critical, and the factor $(*)$, see Theorem 12.1.

In the case that F is totally real or imaginary quadratic, given slightly tighter conditions on the slope, one can prove that the distribution we obtain is *admissible*, that is, it satisfies a growth property that then determines the distribution uniquely. In the general situation, it is rather more difficult to define the correct notion of admissibility; we discuss this further in Section 13. We instead settle for proving that our construction is independent of choice, so that it is indeed reasonable to define the p-adic L-function in this manner.

1.2 Structure of the Paper

Sections 2 to 6 of the paper focus on the classical side of the theory. The main results of this part of the paper come in Sections 5 and 6, where we relate modular symbols to L-values using evaluation maps. Sections 7 to 9 focus on proving the control theorem, allowing us to lift small slope classical eigensymbols to overconvergent symbols. Section 10 then uses evaluation maps to define a distribution attached to an overconvergent eigensymbol. In Section 11, we prove compatibility results between overconvergent and classical evaluation maps that allow us to prove interpolation properties of this distribution. Our results are summarised fully in Section 12.

2 Notation, Hecke Characters and Automorphic Forms

2.1 Notation

This section will serve as an index for the notation we will use in this paper. Let p be a prime, and fix, once and for all, embeddings $\mathrm{inc}_\infty\colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\mathrm{inc}_p\colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Let F be

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a number field of degree $d = r_1 + 2r_2$, where r_1 is the number of real embeddings and r_2 the number of pairs of complex embeddings of F. Write $q = r_1 + r_2$. We write Σ for the set of all infinite embeddings of F. Let $\Sigma(\mathbb{R})$ denote the set of real places of F and let $\Sigma(\mathbb{C})$ be the set containing a (henceforth fixed) choice of embedding from each pair of complex places, so that $\Sigma = \Sigma(\mathbb{R}) \cup \Sigma(\mathbb{C}) \cup c\Sigma(\mathbb{C})$, where c denotes complex conjugation. We write $\mathfrak{D}$ for the different of F and D for the discriminant of F. For each finite place v in F, fix (once and for all) a uniformiser π_v in the completion F_v.

Let $\mathbb{A}_F = F_\infty \times \mathbb{A}_F^f$ denote the adele ring of F, with infinite adeles $F_\infty \cong F \otimes_{\mathbb{Q}} \mathbb{R}$ and finite adeles $\mathbb{A}_F^f$. Let $\widehat{\mathcal{O}}_F \cong \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_F$ denote the integral (finite) adeles. Let $F_\infty^+ \cong \mathbb{R}_{>0}^{r_1} \times (\mathbb{C}^\times)^{r_2}$ be the connected component of the identity in $F_\infty^\times$.

Let $\mathfrak{n} \subset \mathcal{O}_F$ be an ideal with $(p)|\mathfrak{n}$. This will be our level; write

$$\Omega_1(\mathfrak{n}) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\widehat{\mathcal{O}}_F) : c \equiv 0 \pmod{\mathfrak{n}}, d \equiv 1 \pmod{\mathfrak{n}} \right\}.$$

This is an open compact subgroup of $\mathrm{GL}_2(\mathbb{A}_F^f)$. Let $K_\infty^+ := \mathrm{SO}_2(\mathbb{R})^{r_1} \times \mathrm{SU}_2(\mathbb{C})^{r_2}$, a subgroup of the standard maximal compact subgroup K_∞ of $\mathrm{GL}_2(F_\infty)$, and let $Z_\infty := Z(\mathrm{GL}_2(F_\infty)) \cong (F \otimes_{\mathbb{Q}} \mathbb{R})^\times$ (with Z_∞^+ the connected component of Z_∞ including the identity). Then the locally symmetric space associated with $\Omega_1(\mathfrak{n})$ is

$$Y_1(\mathfrak{n}) := \mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbb{A}_F) / \Omega_1(\mathfrak{n}) K_\infty^+ Z_\infty.$$

For an ideal $\mathfrak{f} \subset \mathcal{O}_F$, we define $U(\mathfrak{f})$ to be the set of elements of $\widehat{\mathcal{O}_F}$ that are congruent to $1 \pmod{\mathfrak{f}}$, and denote the narrow ray class group modulo $\mathfrak{f}$ by

$$\mathrm{Cl}_F^+(\mathfrak{f}) := F^\times \backslash \mathbb{A}_F^\times / U(\mathfrak{f}) F_\infty^+.$$

When $\mathfrak{f} = \mathcal{O}_F$, we write simply Cl_F^+ (the narrow class group of F). Write h for the narrow class number of F and choose fixed representatives $I_1, \dots, I_h$ of the narrow class group, coprime to $\mathfrak{n}$, represented by ideles $a_1, \dots, a_h$, with $(a_i)_v = 1$ for all $v|\mathfrak{n}\infty$.

Throughout, $\lambda = (\mathbf{k}, \mathbf{v}) \in \mathbb{Z}[\Sigma]^2$, with $\mathbf{k} \geq 0$, will denote an admissible weight (see Definition 2.8). If $\mathbf{r} \in \mathbb{Q}[\Sigma]$ is parallel, then we write $[\mathbf{r}]$ for the unique rational such that $\mathbf{r} = [\mathbf{r}]\mathbf{t}$, where $\mathbf{t} = (1, \dots, 1) \in \mathbb{Z}[\Sigma]$.

For a ring A and an integer k, we define $V_k(A)$ to be the ring of homogeneous polynomials in two variables of degree k over A. This has a natural left $\mathrm{GL}_2(A)$-action given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot f(X, Y) = f(bY + dX, aY + cX)$. For $\mathbf{k} \in \mathbb{Z}[\Sigma]$, we write $V_{\mathbf{k}}(A) := \otimes_v V_{k_v}(A)$. This has a natural $\mathrm{GL}_2(A)^d$-action induced from that on each component. For $\lambda = (\mathbf{k}, \mathbf{v})$ as above, we also write $V_\lambda(A)$ for the module $V_{\mathbf{k}}(A)$ with $\mathrm{GL}_2(A)^d$ action twisted by $\det^{\mathbf{v}}$, that is, given by

$$\gamma \cdot_\lambda f(\mathbf{X}, \mathbf{Y}) = \left(\prod_{v \in \Sigma} \det(\gamma_v)^{v_v} \right) \gamma \cdot f(\mathbf{X}, \mathbf{Y}), \quad \gamma = (\gamma_v)_{v \in \Sigma} \in \mathrm{GL}_2(A)^d.$$

2.2 Hecke characters

A *Hecke character for F* is a continuous homomorphism $\varphi: F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$. For a place v of F, we write φ_v for the restriction of φ to $F_v^\times$, where F_v denotes the completion of F at v. We will typically write $\mathfrak{f}$ for the conductor of φ. For an ideal $I \subset \mathcal{O}_F$, write $\varphi_I := \prod_{v|I} \varphi_v$. We write $\varphi_f := \prod_{v \nmid \infty} \varphi_v$ and $\varphi_\infty := \prod_{v|\infty} \varphi_v$.

We can identify a Hecke character φ with a function on ideals of F that has support on those that are coprime to the conductor in a natural way. Concretely, if $\mathfrak{q}$ is a prime

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ideal coprime to the conductor, define $\varphi(\mathfrak{q}) = \varphi(\pi_\mathfrak{q})$ (which is independent of the choice of uniformiser $\pi_\mathfrak{q}$), and if $\mathfrak{q}$ is not coprime to the conductor, define $\varphi(\mathfrak{q}) = 0$. In an abuse of notation, we also write φ for this function.

2.2.1 Admissible Infinity Types

Let φ be a Hecke character. There is a canonical decomposition $F_\infty^\times = \{\pm 1\}^{\Sigma(\mathbb{R})} \times F_\infty^+$, and we write $\varphi_\infty^+ := \varphi|_{F_\infty^+}$. We say φ is *arithmetic* if φ_∞^+ takes the form

$$\mathbf{z} = (z_v)_{v|\infty} \longmapsto \mathbf{z}^\mathbf{r} = \prod_{v|\infty} z_v^{r_v}$$

for some $\mathbf{r} \in \mathbb{Z}[\Sigma]$, and we say $\mathbf{r}$ is the *infinity-type* of φ. Henceforth, all Hecke characters will be assumed to be arithmetic.

Define a character ε_φ of the *Weyl group* $\{\pm 1\}^{\Sigma(\mathbb{R})}$ attached to φ by $\varepsilon_\varphi(\iota) := \varphi_\infty(\iota)\iota^\mathbf{r}$, where we consider $\iota \in \{\pm 1\}^{\Sigma(\mathbb{R})}$ as an infinite idele by setting its entries at complex places to 1. In the sequel, we will (in an abuse of notation) write ε_φ for both this character of $\{\pm 1\}^{\Sigma(\mathbb{R})}$ and for the character of the ideles given by

$$\varepsilon_\varphi(x) = \varepsilon_\varphi((\operatorname{sign}(x_v))_{v\in\Sigma(\mathbb{R})}).$$

Note then that $\varphi_\infty\varepsilon_\varphi$ is the unique algebraic character of $F_\infty^\times$ that restricts to φ_∞^+ on F_∞^+; namely, it is the character of $F_\infty^\times$ given by $\mathbf{z} \mapsto \mathbf{z}^\mathbf{r}$. Note that if $F = \mathbb{Q}$ and $\varphi = |\cdot|$ is the norm character on $\mathbb{A}_\mathbb{Q}^\times$, then $\varepsilon_\varphi(-1) = -1$, even though φ itself takes only positive values. Not all elements of $\mathbb{Z}[\Sigma]$ can be realised as the infinity type of a Hecke character. See [Hid94, Chapter 3] for a description of the set $\Xi \subset \mathbb{Z}[\Sigma]$ of admissible types. A necessary (but not sufficient) condition for $\mathbf{r} \in \Xi$ is that $\mathbf{r} + c\mathbf{r}$ is parallel. This motivates the following piece of notation required in the sequel.

Definition 2.1 Let $\mathbf{r} \in \mathbb{Z}[\Sigma]$ be admissible, that is, let $\mathbf{r} \in \Xi$. Then define $[\mathbf{r}] \in \mathbb{R}$ to be the unique number such that $\mathbf{r} + c\mathbf{r} = 2[\mathbf{r}]\mathbf{t}$. Note that, in particular, for any $\zeta \in F^\times$, we have $N((\zeta))^{[\mathbf{r}]} = |\zeta|^\mathbf{r}$, which we will use later.

Proposition 2.2 ([Wei56]) *An element $\mathbf{r} \in \mathbb{Z}[\Sigma]$ can be realised as the infinity type of a Hecke character of F if and only if $\mathbf{r} \in \Xi$, that is, $\mathbf{r}$ is admissible.*

For example, if F is totally real (or more generally has any real embedding), then the only admissible infinity types are parallel. If F is imaginary quadratic, then *any* pair $(r, s) \in \mathbb{Z}[\Sigma]$ is admissible.

2.2.2 Hecke Characters on Ray Class Groups

With a Hecke character φ of conductor $\mathfrak{f}|p^\infty$ we can associate a locally analytic function $\varphi_{p\text{-fin}}$ on the p-adic analytic group

$$\mathrm{Cl}_F^+(p^\infty) := F^\times\backslash\mathbb{A}_F^\times/U(p^\infty)F_\infty^+,$$

where $U(p^\infty)$ is the group of elements of $\widehat{\mathcal{O}}_F^\times$ that are congruent to 1 $\pmod{p^n}$ for all integers n (that is, elements of $\widehat{\mathcal{O}}_F^\times$ such that their components at primes above p are all equal to 1). By class field theory, $\mathrm{Cl}_F^+(p^\infty)$ is isomorphic to the Galois group of the

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maximal abelian extension of F unramified outside p and ∞. The p-adic L-function of an automorphic form over F should be a distribution on this space, and to this end we discuss the structure of this space in the sequel.

Let φ be a Hecke character with infinity type $\mathbf{r}$ and associated character ε_φ on $\{\pm 1\}^{\Sigma(\mathbb{R})}$, as above. Then there is a unique algebraic homomorphism $w^{\mathbf{r}}\colon F^\times \to \overline{\mathbb{Q}}^\times$ given by $w^{\mathbf{r}}(\gamma) = \prod_{v\in\Sigma} \sigma_v(\gamma)^{r_v}$, where σ_v is the complex embedding corresponding to the infinite place v. This then induces maps

$$w_\infty^{\mathbf{r}}\colon (F\otimes_{\mathbb{Q}}\mathbb{R})^\times \longrightarrow \mathbb{C}^\times \quad \text{and} \quad w_p^{\mathbf{r}}\colon (F\otimes_{\mathbb{Q}}\mathbb{Q}_p)^\times \longrightarrow \overline{\mathbb{Q}}_p^\times \subset \mathbb{C}_p^\times.$$

Note that $w_\infty^{\mathbf{r}}$ is equal to $\varepsilon_\varphi\varphi_\infty$, the unique algebraic character of F_∞ that agrees with φ_∞ on F_∞^+.

The finite part of any Hecke character takes algebraic values [Wei56]. In particular, under our fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$, we can see φ_f as taking values in $\mathbb{C}_p^\times$. In particular, the following function is well defined.

Definition 2.3 We define $\varphi_{p\text{-fin}}$ to be the function

$$\begin{aligned}\varphi_{p\text{-fin}}\colon \mathbb{A}_F^\times &\longrightarrow \mathbb{C}_p^\times\\ x &\longmapsto \varepsilon_\varphi\varphi_f(x)w_p^{\mathbf{r}}(x_p).\end{aligned}$$

Proposition 2.4 *Let φ be a Hecke character of conductor $\mathfrak{f}\,|\,(p^\infty)$. Then the function $\varphi_{p\text{-fin}}$ gives a well-defined function on the narrow ray class group* $\mathrm{Cl}_F^+(p^\infty)$.

Proof By definition, $\varphi_{p\text{-fin}}$ is trivial on F_∞^+. Since $w_\infty^{\mathbf{r}}$ and $w_p^{\mathbf{r}}$ are both induced from the same function on $F^\times$, we see that $\varphi_{p\text{-fin}} = \varphi = 1$ on $F^\times$. As φ has conductor $\mathfrak{f}$, it is trivial on $U(\mathfrak{f})$, and hence on $U(p^\infty)$. Finally, if $x \in U(p^\infty)$, then $x_p = x_\infty = 1$, so that $w_p^{\mathbf{r}}(x_p) = w_\infty^{\mathbf{r}}(x_\infty) = 1$. This completes the proof. ∎

2.2.3 Gauss Sums

Let φ be a Hecke character of conductor $\mathfrak{f}$. We can attach a *Gauss sum* to φ that has many of the desirable properties that Gauss sums of Dirichlet characters enjoy. We first introduce a more general exponential map on the adeles of F.

Definition 2.5 Let e_F be the unique continuous homomorphism $e_F\colon \mathbb{A}_F/F \to \mathbb{C}^\times$ that satisfies $x_\infty \mapsto e^{2\pi i \mathrm{Tr}_{F/\mathbb{Q}}(x_\infty)}$, where x_∞ is an infinite adele. We can describe e_F explicitly as

$$e_F(\mathbf{x}) = \prod_{v\in\Sigma(\mathbb{C})} e^{2\pi i \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(x_v)} \prod_{v\in\Sigma(\mathbb{R})} e^{2\pi i x_v} \prod_{\lambda\mid \ell \text{ finite}} e_\ell(-\mathrm{Tr}_{F_\lambda/\mathbb{Q}_\ell}(x_\lambda)),$$

where $e_\ell\big(\sum_j c_j \ell^j\big) = e^{2\pi i \sum_{j<0} c_j \ell^j}$.

Let d be a (finite) idele representing the different $\mathfrak{D}$.

Definition 2.6 Define the *Gauss sum attached to* φ to be

$$\tau(\varphi) := \varphi(d^{-1}) \sum_{b\in(\mathcal{O}_F/\mathfrak{f})^\times} \varphi_{\mathfrak{f}}(b) e_F(bd^{-1}(\pi_{\mathfrak{f}}^{-1})_{v\mid\mathfrak{f}}),$$

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where $(\pi_{\mathfrak{f}}^{-1})_{v|\mathfrak{f}}$ is the adele given by

$$((\pi_{\mathfrak{f}}^{-1})_{v|\mathfrak{f}})_w := \begin{cases} \pi_w^{-v_w(\mathfrak{f})} & \text{if } w|\mathfrak{f}, \\ 0 & \text{otherwise.} \end{cases}$$

Remark This definition, which is independent of the choice of d, is a natural one; in fact, it is the product of the ϵ-factors over $v|\mathfrak{f}$, as defined by Deligne [Del72]. For this particular iteration of the definition, we have followed [Hid94, p. 480], although we have phrased the definition slightly differently by choosing more explicit representatives.

Proposition 2.7 *For $\zeta \in \mathcal{O}_F$ non-zero, we have*

$$\varphi(d^{-1}) \sum_{b \in (\mathcal{O}_F/\mathfrak{f})^\times} \varphi_{\mathfrak{f}}(b) e_F(\zeta b d^{-1} (\pi_{\mathfrak{f}}^{-1})_{v|\mathfrak{f}}) = \begin{cases} \varphi_{\mathfrak{f}}(\zeta)^{-1} \tau(\varphi) & \text{if } ((\zeta), \mathfrak{f}) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where the notation $((\zeta), \mathfrak{f}) = 1$ means that the two ideals are coprime.

Proof See [Del72], or, for an English translation, [Tat79]. There is also an account of Gauss sums and their properties in [Nar04]. ∎

2.3 Automorphic Forms

We now give a brief summary of the theory of automorphic forms for GL_2, fixing as we do so the notation and conventions we will use during the rest of the paper. For a more comprehensive survey, see [Hid94, Chapters 2, 3], or for a more detailed account of the general theory, see [Wei71].

Definition 2.8 An element $\lambda = (\mathbf{k}, \mathbf{v}) \in \mathbb{Z}[\Sigma] \times \mathbb{Z}[\Sigma]$ is an *admissible weight* if we have $\mathbf{k} = c\mathbf{k} \geq 0$, and $\mathbf{k} + 2\mathbf{v}$ is parallel.

Let $\lambda = (\mathbf{k}, \mathbf{v})$ be an admissible weight as above. Recall the definition of $Y_1(\mathfrak{n})$ from Section 2.1; we now define a representation ρ of $K_\infty^+ \times Z_\infty$ that will give us the appropriate *weight λ automorphy condition*. We do this individually at each place.

- Suppose $v \in \Sigma(\mathbb{C})$. Note that for any non-negative integer n, the space $V_n(\mathbb{C})$ (as defined in Section 2.1) is an irreducible right $\mathrm{SU}_2(\mathbb{C})$-module; write

$$\widetilde{\rho}(n) \colon \mathrm{SU}_2(\mathbb{C}) \longrightarrow \mathrm{GL}(V_n(\mathbb{C}))$$

for the corresponding antihomomorphism. Then define

$$\begin{aligned} \rho_v \colon \mathrm{SU}_2(\mathbb{C}) \times \mathbb{C}^\times &\longrightarrow \mathrm{GL}(V_{2k_v+2}(\mathbb{C})) \\ (u, z) &\longmapsto \widetilde{\rho}(2k_v + 2)(u)|z|^{-k_v - 2v_v}. \end{aligned}$$

- Suppose $v \in \Sigma(\mathbb{R})$. Define

$$\begin{aligned} \rho_v \colon \mathrm{SO}_2(\mathbb{R}) \times \mathbb{R}^\times &\longrightarrow \mathbb{C}^\times \\ (r(\theta), x) &\longmapsto e^{ik\theta} x^{-k_v - 2v_v}, \end{aligned}$$

where $r(\theta) := \left(\begin{smallmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{smallmatrix}\right)$.

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Define $\mathbf{k}^* \in \mathbb{Z}[\Sigma]$ by

$$k_\nu^* := \begin{cases} 2k_\nu + 2 & \text{if } \nu \in \Sigma(\mathbb{C}) \cup c\Sigma(\mathbb{C}), \\ 0 & \nu \in \Sigma(\mathbb{R}). \end{cases}$$

Now define $\rho: K_\infty^+ \times Z_\infty \to \mathrm{GL}(V_{\mathbf{k}^*}(\mathbb{C}))$ by $\rho := \otimes_{\nu \in \Sigma(\mathbb{C}) \cup \Sigma(\mathbb{R})} \rho_\nu$.

Definition 2.9 We say that a function $\Phi: \mathrm{GL}_2(\mathbb{A}_F) \to V_{\mathbf{k}^*}(\mathbb{C})$ is a *cusp form of weight λ and level $\Omega_1(\mathfrak{n})$* if it satisfies the following.

(i) (Automorphy condition) $\Phi(zgu) = \Phi(g)\rho(u, z)$ for $u \in K_\infty^+$ and $z \in Z_\infty \cong (F \otimes_{\mathbb{Q}} \mathbb{R})^\times$.

(ii) (Level condition) Φ is right invariant under $\Omega_1(\mathfrak{n})$.

(iii) Φ is left invariant under $\mathrm{GL}_2(F)$.

(iv) (Harmonicity/holomorphy condition) If we write Φ_∞ for the restriction of Φ to $\mathrm{GL}_2(F_\infty^+)$, where F_∞^+ is the connected component of the identity in F_∞, then Φ_∞ is an eigenfunction of the operators δ_ν for all places ν, with

$$\delta_\nu(\Phi_\infty) = \left(\frac{k_\nu^2}{2} + k_\nu\right)\Phi_\infty,$$

where δ_ν is a component of the Casimir operator in the Lie algebra $\mathfrak{sl}_2(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{F}_\nu$ (see [Hid93, §1.3]).

(v) (Growth condition) Let $B = \left(\begin{smallmatrix} t & z \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{A}_F)\}$. Then Φ is *B-moderate* in the sense that there exists $N \geq 0$ such that for every compact subset S of B, we have

$$\|\Phi[\left(\begin{smallmatrix} t & z \\ 0 & 1 \end{smallmatrix}\right)]\| = O(|t|^N + |t|^{-N})$$

(for any fixed norm $\|\cdot\|$) uniformly over $\left(\begin{smallmatrix} t & z \\ 0 & 1 \end{smallmatrix}\right) \in S$.

(vi) (Cuspidal condition) We have $\int_{F\backslash\mathbb{A}_F} \Phi(ug)\, du = 0$, where $\mathbb{A}_F \hookrightarrow \mathrm{GL}_2(\mathbb{A}_F)$ via $u \mapsto \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$, and du is the Lebesgue measure on $\mathbb{A}_F$.

We write $S_\lambda(\Omega_1(\mathfrak{n}))$ for the space of cusp forms of weight λ and level $\Omega_1(\mathfrak{n})$.

There is a good theory of Hecke operators on the space of automorphic forms, indexed by ideals of $\mathcal{O}_F$ and given by double coset operators. We do not go into details here; see [Wei71, Chapter VI], [Hid88, §2]. Many of the nice properties that Hecke operators satisfy for classical modular forms, such as algebraicity of Hecke eigenvalues, also hold in the general case. By a *Hecke eigenform* we mean an eigenvector of all of the Hecke operators.

3 *L*-functions and Periods

In the following section, we attach L-functions to automorphic forms, and state some algebraicity results for their critical values.

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Let Φ be a cuspidal eigenform over F of weight $\lambda = (\mathbf{k}, \mathbf{v})$ and level $\Omega_1(\mathfrak{n})$, with T_I-eigenvalue λ_I for each non-zero ideal $I \subset \mathcal{O}_F$.

Definition 3.1 Let φ be a Hecke character of F. The L-function of Φ twisted by φ is defined to be

$$L(\Phi, \varphi, s) = \sum_{0 \neq I \subset \mathcal{O}_F} \lambda_I \varphi(I) N(I)^{-s}, \quad \text{for all } s \in \mathbb{C}.$$

This converges absolutely for $\mathrm{Re}(s) \gg 0$ [Wei71, Chapter II]. In fact, one can show that it has an analytic continuation to all of $\mathbb{C}$ by writing down an integral formula for $L(\Phi, \varphi, s)$. With this analytic continuation taken as a given, we also define

$$L(\Phi, \varphi) := L(\Phi, \varphi, 1).$$

As in the case of classical normalised eigenforms, we can make sense of this L-function in terms of Fourier coefficients, for a suitable Fourier expansion of Φ. For details of this approach, see [Hid94, §6]. We make one more definition for convenience. The L-function has been built using local data at finite primes; here we complete it by adding in Deligne's Γ-factors at infinity.

Definition 3.2 Let $\Lambda(\Phi, \varphi) := \left[\prod_{v \in \Sigma} \frac{\Gamma(j_v+1)}{(-2\pi i)^{j_v+1}} \right] L(\Phi, \varphi)$, where φ has infinity type $\mathbf{j} + \mathbf{v}$.

3.1 Periods and Algebraicity

To p-adically interpolate L-values, we need to renormalise so that they are algebraic. The following is a result proved by Hida [Hid94, Theorem 8.1]. Earlier, Shimura proved this result over $\mathbb{Q}$ and later over totally real fields [Shi77, Shi78].

Theorem 3.3 *Let Φ be a cuspidal eigenform over F of weight $\lambda = (\mathbf{k}, \mathbf{v})$ and level $\Omega_1(\mathfrak{n})$, with associated L-function $L(\Phi, \cdot)$. Let φ be a Hecke character of infinity type $\mathbf{j} + \mathbf{v}$, where $0 \leq \mathbf{j} \leq \mathbf{k}$, and let $\varepsilon = \varepsilon_\varphi$ be its associated character on $\{\pm 1\}^{\Sigma(\mathbb{R})}$ (as in Section 2.2.1). Let K be a number field containing the normal closure of F and the Hecke eigenvalues of Φ. Then there is a period $\Omega_\Phi^\varepsilon \in \mathbb{C}^\times$, depending only on Φ and ε, such that $\frac{\Lambda(\Phi, \varphi)}{\Omega_\Phi^\varepsilon \tau(\varphi)} \in K(\varphi)$, where $K(\varphi)$ is the number field generated over K by adjoining the values of φ.*

Remarks • We are assuming that all Hecke characters are arithmetic; if we dropped this assumption, then $K(\varphi)$ need not be finite over K [Hid94, §8].

• There are many choices of such a period, differing by elements of $K^\times$. Throughout the rest of the paper, we shall assume that we fix a period for each character ε.

• Note that the period depends on the character $\varepsilon_\varphi(\iota) := \varphi|_{\{\pm 1\}^{\Sigma(\mathbb{R})}}(\iota)\iota^{\mathbf{j}+\mathbf{v}}$ of the Weyl group, and *not* on the character $\varphi|_{\{\pm 1\}^{\Sigma(\mathbb{R})}}$.

Thus we have a collection of 2^{r_1} periods attached to Φ, and each corresponds to a different collection of L-values, depending on the parity of the corresponding Hecke characters.

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4 Classical Modular Symbols

Modular symbols are algebraic objects attached to automorphic forms that retain Hecke data. As we discard analytic conditions, they are frequently easier to work with than automorphic forms themselves. In this section, we give a brief description of how one associates a p-adic modular symbol with an automorphic form. We start with an essential piece of notation.

Definition 4.1 Let A be a ring. Define $V_\lambda(A)^* := \mathrm{Hom}(V_\lambda(A), A)$ to be the topological dual of the weight λ polynomials over A. This inherits a right action of $\mathrm{GL}_2(A)^d$ via $(P|\gamma)(f) = P(\gamma \cdot f)$.

4.1 Local Systems

We will need to study the interplay between complex and p-adic coefficients. We give two ways of defining local systems on $Y_1(\mathfrak{n})$.

Definition 4.2 For all modules M below, we suppose that the centre of

$$\mathrm{GL}_2(F) \cap \Omega_1(\mathfrak{n}),$$

which is isomorphic to $\{\varepsilon \in \mathcal{O}_F^\times : \varepsilon \equiv 1 \pmod{\mathfrak{n}}\}$, acts trivially on M. If this were not the case, the following local systems would not be well defined.

(i) Suppose M is a right $\mathrm{GL}_2(F)$-module. Then define $\mathcal{L}_1(M)$ to be the locally constant sheaf on $Y_1(\mathfrak{n})$ given by the fibres of the projection

$$\mathrm{GL}_2(F)\backslash(\mathrm{GL}_2(\mathbb{A}_F) \times M)/\Omega_1(\mathfrak{n})K_\infty^+ Z_\infty \longrightarrow Y_1(\mathfrak{n}),$$

where the action is given by $\gamma(g, m)ukz = (\gamma gukz, m|\gamma^{-1})$.

(ii) Suppose M is a right $\Omega_1(\mathfrak{n})$-module. Then define $\mathcal{L}_2(M)$ to be the locally constant sheaf on $Y_1(\mathfrak{n})$ given by the fibres of the projection

$$\mathrm{GL}_2(F)\backslash(\mathrm{GL}_2(\mathbb{A}_F) \times M)/\Omega_1(\mathfrak{n})K_\infty^+ Z_\infty \longrightarrow Y_1(\mathfrak{n}),$$

where the action is given by $\gamma(g, m)ukz = (\gamma gukz, m|u)$.

Remarks 4.3 Note that if M is a right $\mathrm{GL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$- or a right $\mathrm{GL}_2(F \otimes_{\mathbb{Q}} \mathbb{Q}_p)$-module, then M can be given a $\mathrm{GL}_2(F)$-module structure by restriction in the natural way, giving a sheaf $\mathcal{L}_1(M)$ as in (i) above.

Similarly, for any right $\mathrm{GL}_2(F \otimes_{\mathbb{Q}} \mathbb{Q}_p)$-module, we have an action of $\Omega_1(\mathfrak{n})$ on M via the projection $\mathrm{Pr}\colon \mathrm{GL}_2(\mathbb{A}_F) \to \mathrm{GL}_2(F \otimes_{\mathbb{Q}} \mathbb{Q}_p)$, and we get a sheaf $\mathcal{L}_2(M)$ as above. In this case, the sheaves $\mathcal{L}_1(M)$ and $\mathcal{L}_2(M)$ are naturally isomorphic via the map

$$(g, m) \longmapsto (g, m|g_p)$$

of local systems, where g_p is the image of g under the map Pr above.

Note that, for a number field K containing the normal closure of F, the space $V_\lambda(K)^*$ is naturally a $\mathrm{GL}_2(F)$-module via the embedding of $\mathrm{GL}_2(F)$ in $\mathrm{GL}_2(F \otimes_{\mathbb{Q}} \mathbb{R})$, whilst if $L/\mathbb{Q}_p$ is a finite extension containing $\mathrm{inc}_p(K)$, then $V_\lambda(L)^*$ is naturally a $\mathrm{GL}_2(F \otimes_{\mathbb{Q}} \mathbb{Q}_p)$-module. So our above comments apply and we get sheaves attached to $V_\lambda(A)^*$ for suitable A.

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It will usually be clear which sheaf we must take. However, when the coefficient system is $V_\lambda(L)^*$ (for a sufficiently large finite extension $L/\mathbb{Q}_p$), we can associate two *different* (though isomorphic) local systems. As we will later need (in Lemma 11.1) to keep track of precisely what this isomorphism does to cohomology elements, throughout the paper we will retain the subscript for clarity.

4.2 Operators on Cohomology Groups

4.2.1 Hecke Operators

Recall $q := r_1 + r_2$. We can define actions of the Hecke operators on the cohomology groups $\mathrm{H}^q_c(Y_1(\mathfrak{n}), V_\lambda(A)^*)$. This is described fully in [Hid88, pp. 346–347] and [Dim05, p. 518]. We give a very brief description of the definition, following Dimitrov.

For each prime ideal $\mathfrak{p}$ of $\mathcal{O}_F$, we have a Hecke operator $T_\mathfrak{p}$ induced by the double coset $[\Omega_1(\mathfrak{n})a_\mathfrak{p}\Omega_1(\mathfrak{n})]$, where $a_\mathfrak{p} \in \mathrm{GL}_2(\mathbb{A}_F)$ is defined by

$$(a_\mathfrak{p})_v = \begin{cases} \left(\begin{smallmatrix} 1 & 0 \\ 0 & \pi_\mathfrak{p} \end{smallmatrix}\right) & \text{if } v = \mathfrak{p}, \\ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) & \text{otherwise.} \end{cases}$$

When $\mathfrak{p}|\mathfrak{n}$, we write $U_\mathfrak{p}$ in place of $T_\mathfrak{p}$ in the usual manner.

4.2.2 Action of the Weyl group

We also have an action of the Weyl group $\{\pm 1\}^{\Sigma(\mathbb{R})}$ on the cohomology, again described by Dimitrov. To describe this, recall that we took $I_1, \dots, I_h$ to be a complete set of representatives for the class group, with idelic representatives a_i, and define

$$g_i = \left(\begin{smallmatrix} a_i & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{A}_F).$$

Note that via strong approximation [Hid94, (3.4b)], there is a decomposition

$$Y_1(\mathfrak{n}) = \bigsqcup_{i=1}^{h} Y_1^i(\mathfrak{n}), \tag{4.1}$$

where

$$Y_1^i(\mathfrak{n}) = \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(F) g_i \Omega_1(\mathfrak{n}) \mathrm{GL}_2^+(F_\infty)/\Omega_1(\mathfrak{n}) K_\infty^+ Z_\infty^+ = \Gamma_1^i(\mathfrak{n})\backslash \mathcal{H}_F.$$

Here $\Gamma_1^i(\mathfrak{n}) := \mathrm{SL}_2(F) \cap g_i \Omega_1(\mathfrak{n}) \mathrm{GL}_2^+(F_\infty) g_i^{-1}$ and $\mathcal{H}_F := \mathcal{H}^{\Sigma(\mathbb{R})} \times \mathcal{H}_3^{\Sigma(\mathbb{C})}$, where $\mathcal{H}$ is the standard upper half-plane and $\mathcal{H}_3 := \{(z, t) \in \mathbb{C} \times \mathbb{R}_{>0}\}$ is the upper half-space.

Now let $\iota = (\iota_v)_{v \in \Sigma(\mathbb{R})} \in \{\pm 1\}^{\Sigma(\mathbb{R})}$. Then ι acts on $\mathcal{H}_F$ by

$$\iota \cdot \mathbf{z} = \left[(\iota_v \cdot z_v)_{v \in \Sigma(\mathbb{R})}, (z_v)_{v \in \Sigma(\mathbb{C})}\right],$$

where for $v \in \Sigma(\mathbb{R})$ we define

$$\iota_v \cdot z_v := \begin{cases} z_v & \text{if } \iota_v = 1, \\ -\overline{z_v} & \text{if } \iota_v = -1. \end{cases}$$

This action induces an action of $\{\pm 1\}^{\Sigma(\mathbb{R})}$ on $Y_1^i(\mathfrak{n})$ for each i and hence on $Y_1(\mathfrak{n})$. The action of $\{\pm 1\}^{\Sigma(\mathbb{R})}$ on $\mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(\mathbb{C})^*))$ is then induced by the map of

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local systems $\iota \cdot (g, P) \mapsto (\iota \cdot g, P)$. We write this action on the right by $\phi \mapsto \phi|\iota$. The actions of the Hecke operators and the Weyl group commute.

4.3 The Eichler–Shimura Isomorphism

The major step in the construction of a modular symbol attached to an automorphic form is the *Eichler–Shimura isomorphism*.

Theorem 4.4 (Eichler–Shimura) *There is a Hecke-equivariant injection*

$$S_\lambda(\Omega_1(\mathfrak{n})) \hookrightarrow \mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(\mathbb{C})^*)).$$

Proof An explicit recipe is given in [Hid94]. Note that we have composed the classical version of the theorem with the canonical inclusion of cuspidal into compactly supported cohomology. ∎

Under the decomposition of equation (4.1), we see that for sufficiently large extensions A of $\mathbb{Q}$ or $\mathbb{Q}_p$, there is a (non-canonical) decomposition

$$\mathrm{H}^q_c\big(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(A)^*)\big) \cong \bigoplus_{i=1}^{h} \mathrm{H}^q_c\big(Y_1^i(\mathfrak{n}), \mathcal{L}_1(V_\lambda(A)^*)\big).$$

4.4 Modular Symbols

Let $L/\mathbb{Q}_p$ be a finite extension.

Definition 4.5 The space of *modular symbols of weight λ and level $\Omega_1(\mathfrak{n})$ with values in L* is the compactly supported cohomology space $\mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_2(V_\lambda(L)^*))$.

Let $\Phi \in S_\lambda(\Omega_1(\mathfrak{n}))$ be a Hecke eigenform. Then via Theorem 4.4 we can attach to Φ an element $\phi_{\mathbb{C}} \in \mathrm{H}^q_c\big(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(\mathbb{C})^*)\big)$. We want to pass from a cohomology class with complex coefficients to one with p-adic coefficients. To do this, we use the theory of periods described earlier in Section 3.1.

Definition 4.6 Let ε be a character of the Weyl group $\{\pm 1\}^{\Sigma(\mathbb{R})}$. Then define

$$\mathrm{H}^q_c\big(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(\mathbb{C})^*)\big)[\varepsilon] \subset \mathrm{H}^q_c\big(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(\mathbb{C})^*)\big)$$

to be the subspace on which $\{\pm 1\}^{\Sigma(\mathbb{R})}$ acts by ε.

Proposition 4.7 *Let K be a number field containing the normal closure of F and the Hecke eigenvalues of Φ, and let ε be as above. Let Ω^ε_Φ be the period appearing in Theorem 3.3. Define $\phi^\varepsilon_{\mathbb{C}} := 2^{-r_1} \sum_{\iota \in \{\pm 1\}^{\Sigma(\mathbb{R})}} \varepsilon(\iota)\phi_{\mathbb{C}}|\iota$. Then*

$$\phi^\varepsilon_{\mathbb{C}} \in \mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(\mathbb{C})^*))[\varepsilon]$$

and

$$\phi^\varepsilon_K := \phi^\varepsilon_{\mathbb{C}}/\Omega^\varepsilon_\Phi \in \mathrm{H}^q_c\big(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(K)^*)\big)[\varepsilon].$$

Proof See [Hid94, Chapter 8]. ∎

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Definition 4.8 Define $\theta_K := \sum_\varepsilon \phi_K^\varepsilon \in \mathrm{H}_c^q(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(K)^*))$, where the sum is over all possible characters of the Weyl group $\{\pm 1\}^{\Sigma(\mathbb{R})}$.

Now let $L/\mathbb{Q}_p$ be a finite extension containing $\mathrm{inc}_p(K)$ (for our fixed embedding $\mathrm{inc}_p\colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$). Then inc_p induces an inclusion

(4.2)
$$\mathrm{H}_c^q\big(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(K)^*)\big) \hookrightarrow \mathrm{H}_c^q\big(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(L)^*)\big) \cong \mathrm{H}_c^q(Y_1(\mathfrak{n}), \mathcal{L}_2(V_\lambda(L)^*)).$$

Finally, there is a canonical inclusion

$$\mathrm{H}_c^q\big(Y_1(\mathfrak{n}), \mathcal{L}_2(V_\lambda(L)^*)\big) \hookrightarrow \mathrm{H}_c^q\big(Y_1(\mathfrak{n}), \mathcal{L}_2(V_\lambda(L)^*)\big). \tag{4.3}$$

Definition 4.9 Let Φ be an eigenform of weight λ and level $\Omega_1(\mathfrak{n})$, and let L be as above. The *modular symbol attached to* Φ *with values in* L is the image

$$\theta_L \in \mathrm{H}_c^q\big(Y_1(\mathfrak{n}), \mathcal{L}_2(V_\lambda(L)^*)\big)$$

of the symbol θ_K under the inclusion of equations (4.2) and (4.3).

5 Automorphic Cycles, Evaluation Maps, and L-values

Let Φ be a cuspidal automorphic form over F. In this section, we give a connection between the cohomology class $\phi_{\mathbb{C}}$ associated with Φ via the Eichler–Shimura isomorphism and critical values of its L-function. We do so via *automorphic cycles*. The cycles we define here are a generalisation of the objects Dimitrov used in the totally real case [Dim13]. As a consequence of this section, we also get an integral formula for the L-function of Φ, generalising the results of [Hid94, §7], where such a formula is obtained for Hecke characters with trivial conductor.

5.1 Automorphic Cycles

Let $\mathfrak{f}$ be an integral ideal of F. We begin with some essential definitions.

Definition 5.1 Recall $F_\infty^+ \subset (F \otimes_{\mathbb{Q}} \mathbb{R})^\times$ is the connected component of the identity in the subgroup of infinite ideles, and let F_∞^1 be the subset defined by

$$F_\infty^1 := \{x \in F_\infty^+ : |x_v|_v = 1 \text{ for all } v|\infty\}.$$

Definition 5.2 Recall the definition of $U(\mathfrak{f}) \subset \mathbb{A}_{F,f}^\times$ from Section 2.1, and define a global equivalent $E(\mathfrak{f}) := \{x \in \mathcal{O}_{F,+}^\times : x \equiv 1 \pmod{\mathfrak{f}}\} = U(\mathfrak{f}) \cap F^\times$.

We define the *automorphic cycle of level* $\mathfrak{f}$ to be $X_\mathfrak{f} := F^\times \backslash \mathbb{A}_F^\times / U(\mathfrak{f}) F_\infty^1$.

Remark There is a natural decomposition $X_\mathfrak{f} = \bigsqcup_{\mathbf{y} \in \mathrm{Cl}_F^+(\mathfrak{f})} X_\mathbf{y}$, where

$$X_\mathbf{y} = \{[x] \in X_\mathfrak{f} : x \text{ represents } \mathbf{y} \text{ in } \mathrm{Cl}_F^+(\mathfrak{f})\}.$$

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There is a natural embedding $\eta_{\mathfrak{f}}\colon X_{\mathfrak{f}} \hookrightarrow Y_1(\mathfrak{n})$ induced by

$$\eta\colon \mathbb{A}_F^\times \hookrightarrow \mathrm{GL}_2(\mathbb{A}_F)$$
$$x \longmapsto \begin{pmatrix} x & (x\pi_{\mathfrak{f}}^{-1})_{v|\mathfrak{f}} \\ 0 & 1 \end{pmatrix},$$

where $(\pi_{\mathfrak{f}}^{-1})_{v|\mathfrak{f}}$ is the idele defined in Definition 2.6. This map is shown to be well defined in Proposition 5.3 below.

Recall that we have a decomposition $Y_1(\mathfrak{n}) = \bigsqcup_{i=1}^{h} Y_1^i(\mathfrak{n})$, where $Y_1^i(\mathfrak{n})$ is as defined in equation (4.1). In particular, $Y_1^i(\mathfrak{n})$ can be described as

$$\{[g] \in Y_1(\mathfrak{n}) : \det(g) \text{ represents } i \text{ in } \mathrm{Cl}_F^+\}.$$

Proposition 5.3 *The map $\eta_{\mathfrak{f}}$ induces a well-defined map $\eta_{\mathfrak{f}}\colon X_{\mathfrak{f}} \to Y_1(\mathfrak{n})$. Moreover, the restriction of $\eta_{\mathfrak{f}}$ to $X_{\mathbf{y}}$ has image in $Y_1^{i_{\mathbf{y}}}(\mathfrak{n})$, where $i_{\mathbf{y}}$ denotes the element of the narrow class group given by the image of $\mathbf{y}$ under the natural projection $\mathrm{Cl}_F^+(\mathfrak{f}) \to \mathrm{Cl}_F^+$.*

Proof Suppose $\gamma x u r$ is a different representative of $[x] \in X_{\mathfrak{f}}$. Then

$$\begin{aligned}(5.1)\quad [\eta_{\mathfrak{f}}(\gamma x u r)] &= \left[\begin{pmatrix} \gamma x u r & (\gamma x u r \pi_{\mathfrak{f}}^{-1})_{v|\mathfrak{f}} \\ 0 & 1 \end{pmatrix}\right] \\ &= \left[\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & (x\pi_{\mathfrak{f}}^{-1})_{v|\mathfrak{f}} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u & ((u-1)\pi_{\mathfrak{f}}^{-1})_{v|\mathfrak{f}} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}\right] \\ &= [\eta_{\mathfrak{f}}(x)] \in Y_1(\mathfrak{n}),\end{aligned}$$

showing that the induced map is well defined. To see that the restriction to $X_{\mathbf{y}}$ lands in $Y_1^{i_{\mathbf{y}}}(\mathfrak{n})$, note that $\det(\eta_{\mathfrak{f}}(x)) = x$, so that if x represents $\mathbf{y} \in \mathrm{Cl}_F^+(\mathfrak{f})$, we see that $\eta_{\mathfrak{f}}(x)$ represents $i_{\mathbf{y}} \in \mathrm{Cl}_F^+$, and in particular, $\eta_{\mathfrak{f}}$ induces a map

$$\{x \in \mathbb{A}_F^\times : [x] = \mathbf{y} \in \mathrm{Cl}_F^+(\mathfrak{f})\} \longrightarrow Y_1^{i_{\mathbf{y}}}(\mathfrak{n}),$$

which then descends as claimed. ∎

5.2 Evaluation Maps

We now use these automorphic cycles to define evaluation maps

$$\mathrm{Ev}\colon \mathrm{H}_{\mathrm{c}}^q\left(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(\mathbb{C})^*)\right) \longrightarrow \mathbb{C}.$$

This will be done in several stages.

5.2.1 Pulling Back to $X_{\mathfrak{f}}$

First, we pullback under the inclusion $\eta_{\mathfrak{f}}\colon X_{\mathfrak{f}} \hookrightarrow Y_1(\mathfrak{n})$. The corresponding sheaf $\mathcal{L}_{\mathfrak{f},1}(V_\lambda(\mathbb{C})^*) := \eta_{\mathfrak{f}}^*\mathcal{L}_1(V_\lambda(\mathbb{C})^*)$ can be seen, via equation (5.1), to be given by the sections of the natural map $F^\times\backslash(\mathbb{A}_F^\times \times V_\lambda(\mathbb{C})^*)/U(\mathfrak{f})F_\infty^1 \to X_{\mathfrak{f}}$, where the action is given by

$$f(x,P)ur = \left(fxur, P|\begin{pmatrix} f^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right).$$

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5.2.2 Passing to Individual Components

We can explicitly write $X_{\mathbf{y}} := F^\times \backslash F^\times a_{\mathbf{y}} U(\mathfrak{f}) F_\infty^+ / U(\mathfrak{f}) F_\infty^1$, for $\{a_{\mathbf{y}} : \mathbf{y} \in \mathrm{Cl}_F^+(\mathfrak{f})\}$ a (henceforth fixed) set of class group representatives. Note here that there is an isomorphism

$$\begin{aligned} E(\mathfrak{f}) F_\infty^1 \backslash F_\infty^+ &\xrightarrow{\sim} X_{\mathbf{y}}, \\ r &\longmapsto a_{\mathbf{y}} r \end{aligned}$$

Pulling back under this isomorphism composed with the inclusion $X_{\mathbf{y}} \subset X_{\mathfrak{f}}$, we see that the corresponding sheaf $\mathcal{L}_{\mathfrak{f},\mathbf{y},1} := \tau_{a_{\mathbf{y}}}^* \mathcal{L}_{\mathfrak{f},1}(V_\lambda(\mathbb{C})^*)$ is given by the sections of

$$E(\mathfrak{f}) F_\infty^1 \backslash (F_\infty^+ \times V_\lambda(\mathbb{C})^*) \longrightarrow E(\mathfrak{f}) F_\infty^1 \backslash F_\infty^+,$$

where now the action is by

$$es(r, P) = \left(esr, P \middle| \begin{pmatrix} e^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right).$$

5.2.3 Evaluating

Let $\mathbf{j} \in \mathbb{Z}[\Sigma]$ be such that there is a Hecke character φ of conductor $\mathfrak{f}$ and infinity type $\mathbf{j} + \mathbf{v}$. Note that in this case, for all $e \in E(\mathfrak{f})$, we have $e^{\mathbf{j}+\mathbf{v}} = 1$; indeed, $e^{\mathbf{j}+\mathbf{v}} = \varphi_\infty(e) = \varphi_f(e)^{-1} = 1$, since $e \equiv 1 \pmod{\mathfrak{f}}$. Now let $\rho_{\mathbf{j}}$ denote the map $\rho_{\mathbf{j}} \colon V_\lambda(\mathbb{C})^* \to \mathbb{C}$ given by evaluating at the polynomial $\mathbf{X}^{\mathbf{k}-\mathbf{j}}\mathbf{Y}^{\mathbf{j}}$. Then $\rho_{\mathbf{j}}$ induces a map $(\rho_{\mathbf{j}})_*$ of local systems on $E(\mathfrak{f}) F_\infty^1 \backslash F_\infty^+$, as

$$P\left| \begin{pmatrix} e^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right. (\mathbf{X}^{\mathbf{k}-\mathbf{j}}\mathbf{Y}^{\mathbf{j}}) = (e^{\mathbf{j}+\mathbf{v}})^{-1} P(\mathbf{X}^{\mathbf{k}-\mathbf{j}}\mathbf{Y}^{\mathbf{j}}) = P(\mathbf{X}^{\mathbf{k}-\mathbf{j}}\mathbf{Y}^{\mathbf{j}}).$$

We see that the sheaf $(\rho_{\mathbf{j}})_* \mathcal{L}_{\mathfrak{f},\mathbf{y},1}(V_\lambda(\mathbb{C})^*)$ is the constant sheaf attached to $\mathbb{C}$ over $E(\mathfrak{f}) F_\infty^1 \backslash F_\infty^+$. But note that this space is a connected orientable real manifold of dimension q, and hence that there is an isomorphism $\mathrm{H}_{\mathrm{c}}^q\left(E(\mathfrak{f}) F_\infty^1 \backslash F_\infty^+, \mathbb{C}\right) \cong \mathbb{C}$, given by integration over $E(\mathfrak{f}) F_\infty^1 \backslash F_\infty^+$.

Definition 5.4 Define $\mathrm{Ev}_{\mathfrak{f},\mathbf{j},1}^{a_{\mathbf{y}}} \colon \mathrm{H}_{\mathrm{c}}^q(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(\mathbb{C})^*)) \to \mathbb{C}$ to be the composition of the maps

$$\begin{aligned} \mathrm{H}_{\mathrm{c}}^q(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(\mathbb{C})^*)) &\xrightarrow{\eta_{\mathfrak{f}}^*} \mathrm{H}_{\mathrm{c}}^q(X_{\mathfrak{f}}, \mathcal{L}_{\mathfrak{f},1}(V_\lambda(\mathbb{C})^*)) \xrightarrow{\tau_{a_{\mathbf{y}}}^*} \cdots \\ &\mathrm{H}_{\mathrm{c}}^q(E(\mathfrak{f}) F_\infty^1 \backslash F_\infty^+, \mathcal{L}_{\mathfrak{f},\mathbf{y},1}(V_\lambda(\mathbb{C})^*)) \xrightarrow{(\rho_{\mathbf{j}})_*} \mathrm{H}_{\mathrm{c}}^q(E(\mathfrak{f}) F_\infty^1 \backslash F_\infty^+, \mathbb{C}) \cong \mathbb{C}. \end{aligned}$$

Remarks (i) Note that this definition is not restricted to polynomials with coefficients in $\mathbb{C}$. Indeed, the evaluation maps are well defined for cohomology with coefficients in a number field or an extension of $\mathbb{Q}_p$. We will distinguish between the various cases by using a subscript on the cohomology class (for example, $\phi_{\mathbb{C}}$ is a complex modular symbol).

(ii) The subscript 1 in $\mathrm{Ev}_{\mathfrak{f},\mathbf{j},1}^{a_{\mathbf{y}}}$ dictates that this is an evaluation map from the cohomology with coefficients in $\mathcal{L}_1(V_\lambda(\mathbb{C})^*)$. Later, we will define an evaluation map $\mathrm{Ev}_{\mathfrak{f},\mathbf{j},2}^{a_{\mathbf{y}}}$.

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5.3 An Integral Formula for the L-function

Let φ be a Hecke character of conductor $\mathfrak{f}$ and infinity type $\mathbf{j}+\mathbf{v}$ for some $0 \le \mathbf{j} \le \mathbf{k}$. The following is a generalisation of a result of Hida.

Theorem 5.5 *Let $F/\mathbb{Q}$ be a number field, let Φ be a cuspidal eigenform over F of weight $\lambda = (\mathbf{k}, \mathbf{v}) \in \mathbb{Z}[\Sigma]^2$, where $\mathbf{k}+2\mathbf{v}$ is parallel, and let φ be a Hecke character of conductor $\mathfrak{f}$ and infinity type $\mathbf{j}+\mathbf{v}$, where $0 \le \mathbf{j} \le \mathbf{k}$. Let $\Lambda(\Phi, \cdot)$ be the normalised L-function attached to Φ defined in Definition 3.2. Then there is an integral formula*

$$\sum_{\mathbf{y} \in \mathrm{Cl}_F^+(\mathfrak{f})} \varphi(a_{\mathbf{y}})\, \mathrm{Ev}_{\mathfrak{f},\mathbf{j},1}^{a_{\mathbf{y}}}(\phi_{\mathbb{C}}) = (-1)^{R(\mathbf{j},\mathbf{k})} \left[\frac{|D|\tau(\varphi)}{2^{r_2}} \right] \cdot \Lambda(\Phi, \varphi),$$

where

(i) $\{a_{\mathbf{y}}\}$ *is a (fixed) set of adelic representatives for* $\mathrm{Cl}_F^+(\mathfrak{f})$ *with* $(a_{\mathbf{y}})_v = 1$ *for v infinite,*
(ii) $R(\mathbf{j},\mathbf{k}) := \sum_{v \in \Sigma(\mathbb{C})} k_v + \sum_{v \in \Sigma(\mathbb{R})} k_v + j_v$,
(iii) $\tau(\varphi)$ *is the Gauss sum attached to φ defined in Definition 2.6,*
(iv) D *is the discriminant of the number field F,*
(v) $\mathrm{Ev}_{\mathfrak{f},\mathbf{j},1}^{a_{\mathbf{y}}}$ *is the classical evaluation map from Definition 5.4,*
(vi) $\phi_{\mathbb{C}}$ *is the modular symbol attached to Φ under the Eichler–Shimura isomorphism.*

Proof (Sketch). The proof is standard but long, messy, and technical, and we omit the details. A full and detailed proof can be found in [Wil16, Chapter 12.1.4].

The proof relies on explicit computations using the Fourier expansion of the automorphic form. It can be split broadly into several stages, as follows.

Step 1: First, we explicitly compute the differential $\delta_{\mathbf{y}} := \tau_{a_{\mathbf{y}}}^* \eta_{\mathfrak{f}}^* \phi_{\mathbb{C}}$. This uses the isomorphism between Betti and de Rham cohomology at the level of complex coefficients, and was done for trivial conductor $\mathfrak{f}$ in [Hid94, §2.5].

Step 2: Write $\delta_{\mathbf{y}} = \sum_{0 \le \mathbf{j} \le \mathbf{k}} \delta_{\mathbf{y}}^{\mathbf{j}}(z) \mathcal{X}^{\mathbf{k}-\mathbf{j}} \mathcal{Y}^{\mathbf{j}}$. We then introduce an auxiliary varible s and consider the integral $C_{\mathbf{y}}^{\mathbf{j}}(s) := \int_{E(\mathfrak{f})F_\infty^1 \backslash F_\infty^+} \delta_{\mathbf{y}}^{\mathbf{j}}(\mathbf{y}) |\mathbf{y}|_{\mathbb{A}_F}^s$, where $\mathbf{y}$ denotes an element of F_∞^+.

Step 3: For $\mathrm{Re}(s) \gg 0$, we explicitly compute $C_{\mathbf{y}}^{\mathbf{j}}(s)$, broadly following [Hid94, §7]. To do this, we substitute the Fourier expansion of our automorphic form into the expression, and rearrange the result into a product of local integrals at the archimedean places, which are easily computed. We are left with a sum over ideals that are equivalent to $a_{\mathbf{y}}\mathcal{O}_F$ in $\mathrm{Cl}_F^+(\mathfrak{f})$.

Step 4: By summing over $\mathbf{y} \in \mathrm{Cl}_F^+(\mathfrak{f})$, we get a sum over *all* ideals of $\mathcal{O}_F$, and this collapses via a Gauss sum to give the value of the L-function at $\varphi|\cdot|^s$. We deduce that there is an analytic continuation of $L(\Phi, \varphi, s)$ to the whole complex plane, and that setting $s = 0$, we see the (critical) L-value at the character φ.

Step 5: We conclude by noting that $C_{\mathbf{y}}^{\mathbf{j}}(0) = \mathrm{Ev}_{\mathfrak{f},\mathbf{j},1}^{a_{\mathbf{y}}} \phi_{\mathbb{C}}$, from which we deduce the theorem. ∎

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For later use, it is convenient to record a variant of this theorem here. In particular, in the sequel, we will only be able to consider evaluations at conductors $\mathfrak{f}$ divisible by every prime above p. We want to use such evaluations to obtain L-values at characters whose conductors do not necessarily satisfy this (for example, the trivial character). To do so, we need a compatibility result between evaluation maps for different conductors. By examining the Gauss sum in the proof of the integral formula, we obtain the following.

Theorem 5.6 *Suppose $\phi_{\mathbb{C}}$ is an eigensymbol for all the Hecke operators. Let φ be a Hecke character of conductor $\mathfrak{f}$ and infinity type $\mathbf{j}+\mathbf{v}$, and let $\mathfrak{p}$ be a prime that divides the level $\mathfrak{n}$, but does* not *divide $\mathfrak{f}$. Then*

$$\sum_{\mathbf{x}\in \mathrm{Cl}_F^+(\mathfrak{f}\mathfrak{p})} \varphi(a_{\mathbf{x}})\,\mathrm{Ev}^{a_{\mathbf{x}}}_{\mathfrak{f}\mathfrak{p},\mathbf{j},1}(\phi_{\mathbb{C}}) = (\varphi(\mathfrak{p})\lambda_{\mathfrak{p}}-1)\sum_{\mathbf{y}\in \mathrm{Cl}_F^+(\mathfrak{f})} \varphi(a_{\mathbf{y}})\,\mathrm{Ev}^{a_{\mathbf{y}}}_{\mathfrak{f},\mathbf{j},1}(\phi_{\mathbb{C}}),$$

where $\lambda_{\mathfrak{p}}$ is the Hecke eigenvalue at $\mathfrak{p}$.

Corollary 5.7 *Suppose $(p)|\mathfrak{n}$, and let φ be a Hecke character of conductor $\mathfrak{f}|(p^\infty)$ and infinity type $\mathbf{j}+\mathbf{v}$. Let B be the set of primes above p for which φ is not ramified, and define $\mathfrak{f}' := \mathfrak{f}\prod_{\mathfrak{p}\in B}\mathfrak{p}$. Then $\mathfrak{f}'$ is divisible by every prime above p and we have*

$$\sum_{\mathbf{y}\in \mathrm{Cl}_F^+(\mathfrak{f}')} \varphi(a_{\mathbf{y}})\,\mathrm{Ev}^{a_{\mathbf{y}}}_{\mathfrak{f}',\mathbf{j},1}(\phi_{\mathbb{C}}) = \Big(\prod_{\mathfrak{p}\in B}(\varphi(\mathfrak{p})\lambda_{\mathfrak{p}}-1)\Big)\sum_{\mathbf{y}\in \mathrm{Cl}_F^+(\mathfrak{f})} \varphi(a_{\mathbf{y}})\,\mathrm{Ev}^{a_{\mathbf{y}}}_{\mathfrak{f},\mathbf{j},1}(\phi_{\mathbb{C}}).$$

6 Algebraicity Results

So far all of our work has been done over $\mathbb{C}$. We will now refine these results to connect the algebraic modular symbol to the critical L-values above.

Definition 6.1 Let $A_{\mathfrak{f}} = \{a_{\mathbf{y}} : \mathbf{y}\in \mathrm{Cl}_F^+(\mathfrak{f})\}$ denote a fixed set of representatives for $\mathrm{Cl}_F^+(\mathfrak{f})$, with components at infinity that are not necessarily trivial. For a Hecke character φ of conductor $\mathfrak{f}$ and infinity type $\mathbf{j}+\mathbf{v}$, where $0\le \mathbf{j}\le \mathbf{k}$, define a function $\mathrm{Ev}_\varphi^{A_{\mathfrak{f}}}\colon \mathrm{H}^q_c\big(Y_1(\mathfrak{n}),\mathcal{L}_1(V_\lambda(\mathbb{C})^*)\big)\to\mathbb{C}$ by $\mathrm{Ev}_\varphi^{A_{\mathfrak{f}}}(\phi) = \sum_{\mathbf{y}\in \mathrm{Cl}_F^+(\mathfrak{f})}\varepsilon_\varphi\varphi_f(a_{\mathbf{y}})\,\mathrm{Ev}^{a_{\mathbf{y}}}_{\mathfrak{f},\mathbf{j},1}(\phi)$, where as previously we write ε_φ as a function on the ideles by composing it with the natural sign map $\mathbb{A}_F^\times\to\{\pm1\}^{\Sigma(\mathbb{R})}$.

Lemma 6.2 *The function $\mathrm{Ev}_\varphi^{A_{\mathfrak{f}}}$ is independent of class group representatives.*

Proof Let $a'_{\mathbf{y}}$ be an alternative representative corresponding to $\mathbf{y}\in \mathrm{Cl}_F^+(\mathfrak{f})$. Then $a_{\mathbf{y}} = f a_{\mathbf{y}} u r$, where $f\in F^\times$, $u\in U(\mathfrak{f})$, and $r\in F_\infty^+$. Looking at the description of the evaluation maps, we see that $\mathrm{Ev}^{a'_{\mathbf{y}}}_{\mathfrak{f},\mathbf{j},1}(\phi) = f^{\mathbf{j}+\mathbf{v}}\,\mathrm{Ev}^{a_{\mathbf{y}}}_{\mathfrak{f},\mathbf{j},1}(\phi)$. But

$$\varepsilon_\varphi\varphi_f(a'_{\mathbf{y}}) = \varepsilon_\varphi\varphi_f(f a_{\mathbf{y}} u r) = \varepsilon_\varphi\varphi_f(f)\varepsilon_\varphi\varphi_f(a_{\mathbf{y}}) = f^{-\mathbf{j}-\mathbf{v}}\varepsilon_\varphi\varphi_f(a_{\mathbf{y}}),$$

since $\varepsilon_\varphi\varphi_f$ is trivial on $U(\mathfrak{f})F_\infty^+$ and by our earlier comment, we have

$$\varepsilon_\varphi\varphi_f(f) = \varphi(f)/\varphi_\infty^{\mathrm{alg}}(f) = f^{-\mathbf{j}-\mathbf{v}}.$$

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Putting this together, we find that

$$\varepsilon_\varphi \varphi_f(a'_{\mathbf{y}}) \operatorname{Ev}^{a'_{\mathbf{y}}}_{\mathfrak{f},\mathbf{j},1}(\phi) = \varepsilon_\varphi \varphi_f(a_{\mathbf{y}}) \operatorname{Ev}^{a_{\mathbf{y}}}_{\mathfrak{f},\mathbf{j},1}(\phi),$$

which is the required result. ∎

Definition 6.3 Define $\operatorname{Ev}_\varphi$ to be the map $\operatorname{Ev}^{A_\mathfrak{f}}_\varphi$ for any choice of class group representatives $A_\mathfrak{f}$. This is well defined by the above lemma.

We will combine this with the following to deduce the result we desire.

Proposition 6.4 *Let $\iota \in \{\pm 1\}^{\Sigma(\mathbb{R})}$. Then for any idele a, we have*

$$\operatorname{Ev}^{\iota a}_{\mathfrak{f},\mathbf{j},1}(\phi|\iota) = \operatorname{Ev}^{a}_{\mathfrak{f},\mathbf{j},1}(\phi).$$

Proof Recall that the definition of the action of $\iota \in \{\pm 1\}^{\Sigma(\mathbb{R})}$ on the cohomology of $Y_1(\mathfrak{n})$ was described in Section 4.2.2. There is a well-defined action of $\{\pm 1\}^{\Sigma(\mathbb{R})}$ on the local system corresponding to $\mathcal{L}_{\mathfrak{f},1}(V_\lambda(\mathbb{C})^*)$ given by $\iota \cdot (x, P) = (\iota x, P)$, where here we have considered ι to be an idele by setting $\iota_v = 1$ for all complex and finite places v. A simple check shows that if $\phi \in \mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(\mathbb{C})^*))$, then we have

$$\eta^*_\mathfrak{f}(\phi|\iota) = \eta^*_\mathfrak{f}(\phi)|\iota$$

coming from the commutative diagram

$$\begin{array}{ccc} (g,P) & \xrightarrow{\eta^*_\mathfrak{f}} & (x,P) \\ \downarrow{\scriptstyle |\iota} & & \downarrow{\scriptstyle |\iota} \\ (\iota\cdot g,P) & \xrightarrow{\eta^*_\mathfrak{f}} & (\iota x,P) \end{array}$$

of local systems. Continuing to work at the level of local systems, suppose x is an idele that, under the natural quotient map, lies in the component of $X_\mathfrak{f}$ corresponding to $a_{\mathbf{y}}$. Then the image of ιx lies in the component corresponding to $\iota a_{\mathbf{y}}$. (Here we note that if $\{a_{\mathbf{y}} : \mathbf{y} \in \mathrm{Cl}^+_F(\mathfrak{f})\}$ is a complete set of representatives for $\mathrm{Cl}^+_F(\mathfrak{f})$, then so is the set $\{\iota a_{\mathbf{y}} : \mathbf{y} \in \mathrm{Cl}^+_F(\mathfrak{f})\}$). Thus we see that there is a commutative diagram of maps of local systems

$$\begin{array}{ccccc} (x,P) & \xrightarrow{\tau^*_{a_{\mathbf{y}}}} & (r,P) & \xrightarrow{\text{ev. at } \mathbf{X}^{\mathbf{k}-\mathbf{j}}\mathbf{Y}^{\mathbf{j}}} & (r,c) \\ \downarrow{\scriptstyle |\iota} & & \downarrow & & \downarrow \\ (\iota x,P) & \xrightarrow{\tau^*_{\iota a_{\mathbf{y}}}} & (r,P) & \xrightarrow{\text{ev. at } \mathbf{X}^{\mathbf{k}-\mathbf{j}}\mathbf{Y}^{\mathbf{j}}} & (r,c) \end{array}$$

where the local system on the far right-hand side defines the constant sheaf given by sections of $(E(\mathfrak{f})F^1_\infty \backslash F^+_\infty) \times \mathbb{C} \to E(\mathfrak{f})F^1_\infty \backslash F^+_\infty$. The result follows. ∎

Corollary 6.5 *We have the relation $\operatorname{Ev}_\varphi(\phi|\iota) = \varepsilon_\varphi(\iota) \operatorname{Ev}_\varphi(\phi)$.*

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Proof Considering ι as an idele in the usual way, we have

$$\begin{aligned}\mathrm{Ev}_\varphi(\phi|\iota) &= \sum_{\mathbf{y}\in \mathrm{Cl}_F^+(\mathfrak{f})} \varepsilon_\varphi \varphi_f(\iota a_\mathbf{y})\, \mathrm{Ev}^{\iota a_\mathbf{y}}_{\mathbf{j},\mathfrak{f},1}(\phi|\iota) = \varepsilon_\varphi(\iota) \sum_{\mathbf{y}\in \mathrm{Cl}_F^+(\mathfrak{f})} \varepsilon_\varphi \varphi_f(a_\mathbf{y})\, \mathrm{Ev}^{a_\mathbf{y}}_{\mathfrak{f},\mathbf{j},1}(\phi) \\ &= \varepsilon_\varphi(\iota)\, \mathrm{Ev}_\varphi(\phi),\end{aligned}$$

as required. ∎

Corollary 6.6 *We have*

$$\mathrm{Ev}_\varphi(\phi^\varepsilon_\mathbb{C}) = \begin{cases} \mathrm{Ev}_\varphi(\phi_\mathbb{C}) & \text{if } \varepsilon = \varepsilon_\varphi, \\ 0 & \text{otherwise.}\end{cases}$$

Proof By definition,

$$\mathrm{Ev}_\varphi(\phi^\varepsilon_\mathbb{C}) = \mathrm{Ev}_\varphi\Big(2^{-r_1} \sum_{\iota\in\{\pm1\}^{\Sigma(\mathbb{R})}} \varepsilon(\iota)\phi_\mathbb{C}|\iota\Big) = \Big[2^{-r_1} \sum_{\iota\in\{\pm1\}^{\Sigma(\mathbb{R})}} \varepsilon(\iota)\varepsilon_\varphi(\iota)\Big]\, \mathrm{Ev}_\varphi(\phi_\mathbb{C}),$$

using linearity of the evaluation maps and Corollary 6.5. The result then follows from orthogonality of characters, since $\varepsilon_\varphi^2 = 1$. ∎

Recall that in Definition 4.8, we set $\theta_K := \sum_\varepsilon \phi_K^\varepsilon$. Note that here θ_K is an element of the cohomology with *algebraic* coefficients in the number field K.

Theorem 6.7 *Let φ be a Hecke character of conductor $\mathfrak{f}$ and infinity type $\mathbf{j}+\mathbf{v}$, where $0 \le \mathbf{j} \le \mathbf{k}$, and write ε_φ for the associated character of $\{\pm1\}^{\Sigma(\mathbb{R})}$ defined in Section 2.2.1. Let Ev_φ be as in Definition 6.3. We have*

$$\mathrm{Ev}_\varphi(\theta_K) = (-1)^{R(\mathbf{j},\mathbf{k})}\Big[\frac{|D|\tau(\varphi)}{2^{r_2}\Omega_\Phi^{\varepsilon_\varphi}}\Big] \cdot \Lambda(\Phi,\varphi),$$

where $R(\mathbf{j},\mathbf{k}) = \sum_{\nu\in\Sigma(\mathbb{R})} j_\nu + k_\nu + \sum_{\nu\in\Sigma(\mathbb{C})} k_\nu$.

Proof We use Theorem 5.5. In particular, note that we choose $(a_\mathbf{y})_\infty = 1$, so that $\varepsilon_\varphi\varphi_f(a_\mathbf{y}) = \varphi(a_\mathbf{y})$. Thus the sum we obtained in the statement of this theorem is exactly $\mathrm{Ev}_\varphi(\phi_\mathbb{C})$. The result follows. ∎

To summarise: we have now defined an *algebraic* cohomology class that encompasses the algebraic parts of all of the critical L-values that we hope to interpolate. In particular, by embedding K into a sufficiently large finite extension $L/\mathbb{Q}_p$, we get a p-adic modular symbol θ_L that sees all of these critical values.

7 Distributions and Overconvergent Cohomology

In this section, we define the distribution modules that we will use as coefficient modules for the spaces of overconvergent modular symbols. This closely follows the analogous section of [BS13].

Throughout this section, L is a finite extension of $\mathbb{Q}_p$ containing the image of

$$\mathrm{inc}_p \circ \sigma\colon F \hookrightarrow \overline{\mathbb{Q}}_p$$

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for each embedding σ of F into $\overline{\mathbb{Q}}$. First, we give some motivation by reformulating the definition of the space $V_\lambda(L)$. We previously defined this to be the d-fold tensor product of the polynomial spaces $V_{k_\nu}(L)$, with an action of $\mathrm{GL}_2(L)$ depending on λ. Note that $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ embeds naturally in $\overline{\mathbb{Q}}_p^d$, and in particular, we can see an element of $V_\lambda(L)$ as a function on $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ in a natural way. We see that the following definition agrees with the definition we gave in Section 2.

Definition 7.1 Let $L/\mathbb{Q}_p$ be a finite extension and let $\lambda = (\mathbf{k}, \mathbf{v}) \in \mathbb{Z}[\Sigma]$ be admissible (so that, in particular, $\mathbf{k} \geq 0$). Define $V_\lambda(L)$ to be the space of functions on $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ that are polynomial of degree at most $\mathbf{k}$ with coefficients in L, with a left action of $\mathrm{GL}_2(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ given by

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot P(x) = (ad - bc)^{\mathbf{v}}(a + cx)^{\mathbf{k}} P\left(\tfrac{b+dx}{a+cx}\right).$$

We have passed to a non-homogeneous version here. This definition is more easily seen to be compatible with the rest of this section. In particular, it is compatible with the following.

Definition 7.2 Let $\mathcal{A}(L)$ be the space of locally analytic functions on $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ that are defined over L.

We would like to define an action of $\mathrm{GL}_2(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ on this space, analogously to above. Unfortunately, the action above does not extend to the full space $\mathcal{A}(L)$. We can, however, define an action of a different semigroup.

Definition 7.3 Let $\Sigma_0(p)$ be the semigroup

$$\Sigma_0(p) := \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p) : c \in p\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p, a \in (\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times, ad - bc \neq 0\right\}.$$

Define $\mathcal{A}_\lambda(L)$ to be the space $\mathcal{A}(L)$ equipped with a left 'weight λ action' of $\Sigma_0(p)$ given by

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot f(z) = (ad - bc)^{\mathbf{v}}(a + cz)^{\mathbf{k}} P\left(\tfrac{b+dz}{a+cz}\right).$$

Note in particular that this semigroup contains the image of $\Gamma_1(\mathfrak{n})$ under the natural embedding $M_2(\mathcal{O}_F) \subset M_2(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ as well as the matrices that we will need to define a Hecke action at primes above p. It is not a subset of $\mathrm{GL}_2(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)$, but the action of this different semigroup also extends naturally to $V_\lambda(L)$, since both live inside $\mathrm{GL}_2(F \otimes_{\mathbb{Q}} \mathbb{Q}_p)$.

We are now in a position to define the distribution spaces.

Definition 7.4 Define $\mathcal{D}_\lambda(L) := \mathrm{Hom}_{\mathrm{cts}}(\mathcal{A}_\lambda(L), L)$ to be the topological dual of $\mathcal{A}_\lambda$, with a right action of $\Sigma_0(p)$ defined by $(\mu|\gamma)(f) := \mu(\gamma \cdot f)$.

Note that $\Omega_1(\mathfrak{n})$ acts on $\mathcal{D}_\lambda(L)$ via its projection to $\mathrm{GL}_2(\mathbb{Q}_p)$, giving rise to a local system $\mathcal{L}_2(\mathcal{D}_\lambda(L))$ on $Y_1(\mathfrak{n})$.

Definition 7.5 The space of *overconvergent modular symbols* is the compactly supported cohomology group $\mathrm{H}^q_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L)))$.

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By dualising the inclusion $V_\lambda(L) \subset \mathcal{A}_\lambda(L)$, we get a $\Sigma_0(p)$-equivariant surjection $\mathcal{D}_\lambda(L) \to V_\lambda(L)^*$. This gives rise to a $\Sigma_0(p)$-equivariant *specialisation map*, a map

$$\rho\colon \mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L))) \longrightarrow \mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_2(V_\lambda(L)^*)).$$

The space of overconvergent modular symbols is, in a sense, a p-adic deformation of the space of classical modular symbols. It was introduced by Glenn Stevens [Ste94].

We conclude this section with a result that will be crucial in the following section, where we prove that the space of overconvergent modular symbols admits a slope decomposition with respect to the Hecke operators. For the relevant definitions, see [Urb11, §2.3.12]. The space $\mathcal{D}_\lambda(L)$ is naturally a nuclear Fréchet space;[1] indeed, let $\mathcal{A}_{n,\lambda}(L)$ be the space of functions that are *locally analytic of order n*, that is, functions that are analytic on each open set of the form $a + p^n\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Each $\mathcal{A}_{n,\lambda}(L)$ is a Banach space, and the inclusions $\mathcal{A}_{n,\lambda}(L) \hookrightarrow \mathcal{A}_{n+1,\lambda}(L)$ are compact [Urb11, Lemma 3.2.2]. We write $\mathcal{D}_{n,\lambda}(L)$ for the topological dual of $\mathcal{A}_{n,\lambda}(L)$. Then $\mathcal{D}_\lambda(L) \cong \varprojlim \mathcal{D}_{n,\lambda}(L)$ is equipped with a family of norms coming from the Banach spaces $\mathcal{D}_{n,\lambda}(L)$.

Definition 7.6 Let $M \cong \varprojlim M_n$ be a nuclear Fréchet space. We say that an endomorphism U of M is *compact* if it is continuous and there are continuous maps U'_n making the following commute:

$$\begin{array}{ccc} M & \longrightarrow & M_{n-1} \\ \downarrow{\scriptstyle U} & & \downarrow{\scriptstyle U'_n} \\ M & \longrightarrow & M_n \end{array},$$

where the horizontal maps are the natural projections.

Lemma 7.7 *Let $\eta \in \mathrm{GL}_2(F) \cap \Sigma_0(p)$, which acts naturally on $\mathcal{D}_\lambda(L)$. This action is compact. In particular, the action of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$ is compact on $\mathcal{D}_\lambda(L)$.*

Proof See [Urb11, Lemma 3.2.8]. ∎

8 Slope Decompositions

We start by recalling the relevant definitions about slope decompositions.

Definition 8.1 Let L be a finite extension of $\mathbb{Q}_p$, and let $h \in \mathbb{Q}$. We say a polynomial $Q(X) \in L[X]$ has *slope* $\leq h$ if $Q(0) \in \mathcal{O}_L^\times$ and if $\alpha \in \overline{L}$ is a root of $Q^*(X) := X^{\deg(Q)}Q(1/X)$, then $v_p(\alpha) \leq h$.

Definition 8.2 Let M be an L-vector space equipped with the action of an L-linear endomorphism U. We say that M has a *slope $\leq h$ decomposition with respect to U* if there is a decomposition $M \cong M_1 \oplus M_2$ such that M_1 is finite-dimensional, the polynomial $\det(1 - UX)|_{M_1}$ has slope $\leq h$, and for all polynomials $P \in L[X]$ with

[1] That is, an inverse limit of Banach spaces in which the projection maps are compact. In [Urb11], Urban calls this a *compact Fréchet space*. We instead follow the terminology utilised in [Sch02].

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slope $\leq h$, the polynomial $P^*(U)$ acts invertibly on M_2. We write $M^{\leq h,U} := M_1$ for the elements of *slope* $\leq h$ in M. Where the operator U is clear, we drop it from the notation and just write $M^{\leq h}$.

The crucial theorem we require is the following:

Theorem 8.3 *Let $\lambda = (\mathbf{k}, \mathbf{v})$ be an admissible weight. Then for each $i \in \mathbb{N}$ and any $h \in \mathbb{Q}$, the L-vector space $\mathrm{H}^i_c(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L)))$ admits a slope $\leq h$ decomposition with respect to the Hecke operator U_p.*

Sketch of Proof To prove this theorem we follow the arguments given in [Urb11, BS15], where the same statement is proved in the cases of the cohomology without compact support and GL_2 over a totally real field, respectively. Both of these rely on general results from earlier in [Urb11], where Urban proved that any nuclear Fréchet space M equipped with a compact endomorphism U admits a slope decomposition with respect to U. Given this, the key step is to construct a complex whose cohomology is $\mathrm{H}^\bullet_c(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L)))$ and such that each term of the complex is isomorphic to finitely many copies of $\mathcal{D}_\lambda(L)$. We can find a lift of the Hecke operators on the cohomology to this complex, and then we use the fact that the action of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$ on $\mathcal{D}_\lambda(L)$ is compact to deduce that this lift acts compactly on the complex. Using Urban's results, we deduce the theorem. ∎

9 A Control Theorem

In this section, we prove a control theorem, showing that the restriction of the specialisation map from overconvergent to classical modular symbols to the small slope subspaces is an isomorphism. We actually need a slightly finer definition of slope decomposition; namely, we define the slope decomposition with respect to a finite set of operators rather than just one.

To this end, let I be a finite set, and suppose that for each $i \in I$, we have an endomorphism U_i on the L-vector space M. Write $A := L[U_i, i \in I]$ for the algebra of polynomials in the variables U_i. Then A acts on M, and for $\mathbf{h} = (h_i) \in \mathbb{Q}^I$ we define the *slope* $\leq \mathbf{h}$ *subspace with respect to* A to be $M^{\leq \mathbf{h},A} := \bigcap_{i \in I} M^{\leq h_i, U_i}$. Where the choice of operators is clear, we will drop the A from the notation and just write $M^{\leq \mathbf{h}}$.

9.1 Preliminary Results

We start by stating some properties of slope decompositions that will be required in the proof.

Lemma 9.1 (i) *Let M, N, and P be L-vector spaces equipped with an action of A, and suppose that M, N, and P each admit a slope $\leq \mathbf{h}$ decomposition with respect to A. If $0 \to M \to N \to P \to 0$ is an exact sequence of A-modules, then we have an exact sequence $0 \to M^{\leq \mathbf{h}} \to N^{\leq \mathbf{h}} \to P^{\leq \mathbf{h}} \to 0$.*

(ii) *Let $M \cong \varprojlim M_n$ be a nuclear Fréchet space equipped with a compact endomorphism U that induces compact operators U_n on M_n for each n. Then for each n there is an isomorphism $M^{\leq \mathbf{h},U} \cong M_n^{\leq \mathbf{h},U_n}$. This fact holds as well for compact*

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maps between complexes of nuclear Fréchet spaces and the induced slope decomposition on their cohomology.

(iii) *Let $(M, \|\cdot\|)$ be an L-Banach space equipped with an action of A, where $\|\cdot\|$ denotes the norm on M, and suppose that there is a $\mathcal{O}_L$-submodule*

$$\mathcal{M} \subset \{m \in M : \|m\| \geq 0\}$$

that is stable under the action of A. Let $\mathbf{h} = (h_i)_{i\in I}$ with $h_{i_0} < 0$ for some $i_0 \in I$. Then $M^{\leq \mathbf{h}} = 0$.

Proof Part (i) is simple [Urb11, Corollary 2.3.5]. Part (ii) was proved in [Urb11, Lemma 2.3.13]. For part (iii), suppose that $M^{\leq \mathbf{h}} \neq 0$. Then, after possibly replacing L with a finite extension, we can find $\alpha \in L$ and $x \in \mathcal{M}$ such that $v_p(\alpha) < 0$ and $U_{i_0} x = \alpha x$. Then there exists $n \in \mathbb{Z}$ such that $\alpha^n x \notin \mathcal{M}$. This is a contradiction because $\alpha^n x = U_{i_0}^n x \in \mathcal{M}$ by A-stability of $\mathcal{M}$. ∎

In particular, we have Corollary 9.4.

Definition 9.2 For each $\sigma \in \Sigma$, denote by $\mathfrak{p}(\sigma)$ the unique prime $\mathfrak{p}|p$ such that the embedding $\sigma\colon F \hookrightarrow \overline{\mathbb{Q}} \subset \mathbb{C}$ extends to an embedding $F_{\mathfrak{p}} \hookrightarrow \overline{\mathbb{Q}}_p \subset \mathbb{C}_p$ that is compatible with the fixed embedding $\mathrm{inc}_p\colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. If σ corresponds to $\mathfrak{p}$ under this identification, we write $\sigma \sim \mathfrak{p}$.

Definition 9.3 Let $\nu = (\mathbf{k}, \mathbf{v}) \in \mathbb{Z}[\Sigma]^2$ be an admissible weight. Define

$$\nu_{\mathfrak{p}}(\nu) := \sum_{\sigma\sim\mathfrak{p}} \nu_\sigma.$$

Corollary 9.4 (i) *Let $\nu = (\mathbf{k}, \mathbf{v}) \in \mathbb{Z}[\Sigma]^2$ be a weight with $\mathbf{k} + 2\mathbf{v}$ parallel (but allowing for negative values of k_σ). Let $\mathbf{h} \in \mathbb{Q}^{\{\mathfrak{p}|p\}}$ be such that $h_{\mathfrak{p}} < \frac{\nu_{\mathfrak{p}}(\nu)}{e_{\mathfrak{p}}}$ for some prime $\mathfrak{p}$ above p. Then for all r we have $\mathrm{H}^r_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\nu(L)))^{\leq \mathbf{h}} = \{0\}$.*

(ii) *Under the same hypotheses, the same result holds if we replace $\mathcal{D}_\nu(L)$ with any $\Sigma_0(p)$-stable submodule or by quotients by such submodules.*

Proof From Section 7, we know that $\mathcal{D}_\lambda(L) \cong \varprojlim \mathcal{D}_{\lambda,n}(L)$, where $\mathcal{D}_{\lambda,n}(L)$ is the L-Banach space of distributions that are locally analytic of order n. We also know (from results in the previous section) that the cohomology group

$$\mathrm{H}^r_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_{\nu,0}(L)))$$

is an L-Banach space, and we see that $\mathrm{H}^r_{\mathrm{c}}(Y_1, \mathcal{L}_2(\mathcal{D}_{\nu,0}(\mathcal{O}_L)))$ is a $\mathcal{O}_L$-submodule of the elements of non-negative norm. This space is *not* necessarily preserved by the Hecke operators at p, but it is preserved by the modified operators $U'_{\mathfrak{p}} := \pi_{\mathfrak{p}}^{-\nu_{\mathfrak{p}}(\nu)} U_{\mathfrak{p}}$, where we scale by $\pi_{\mathfrak{p}}^{-\nu_{\mathfrak{p}}(\nu)}$ to ensure integrality in the case $\nu_{\mathfrak{p}}(\nu)$ is large and negative. Write $A' := L[U'_{\mathfrak{p}}]$ for the algebra generated by these modified operators. Applying Lemma 9.1 (ii) and (iii), we see that if $\mathbf{h}' \in \mathbb{Q}^{\{\mathfrak{p}|p\}}$ is chosen such that $h'_{\mathfrak{p}} < 0$ for some prime $\mathfrak{p}$ above p, we have $\mathrm{H}^r_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_{\nu,0}(L)))^{\leq \mathbf{h}', A'} = \{0\}$. By Lemma 9.1 (ii), the finite slope cohomologies of $\mathcal{D}_\nu(L)$ and $\mathcal{D}_{\nu,0}(L)$ are isomorphic; hence we conclude

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that $\mathrm{H}^r_c(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_v(L)))^{\leq \mathbf{h}', A'} = \{0\}$. Now note that for any operator U on a nuclear Fréchet space M, we have a relation

$$M^{\leq h, p^k U} \cong M^{\leq h-k, U}.$$

In particular, define $\mathbf{h} \in \mathbb{Q}^{\{\mathfrak{p}|p\}}$ by $h_\mathfrak{p} := h'_\mathfrak{p} + v_\mathfrak{p}(v)/e_\mathfrak{p}$. Note that $h'_\mathfrak{p} < 0$ for some $\mathfrak{p}$ above p if and only if $h_\mathfrak{p} < v_\mathfrak{p}(v)/e_\mathfrak{p}$ for some $\mathfrak{p}$ above p, and that the space on which the Hecke operators at p act with slope $\leq \mathbf{h}$ is isomorphic to the space on which the operators $U'_\mathfrak{p}$ act with slope $\leq \mathbf{h}'$. Part (i) follows.

The proof for submodules is identical. The case of quotients then follows by taking a long exact sequence, applying Lemma 9.1 (i), and using the result for submodules. ∎

9.2 Theta Maps and Partially Overconvergent Coefficients

We now introduce modules of partially overconvergent coefficients that will play a key role in the proof.

For any $\sigma \in \Sigma$, let $\lambda_\sigma = (\mathbf{k}', \mathbf{v}')$ be the weight defined by

$$k'_\tau = \begin{cases} k_\tau & \text{if } \tau \neq \sigma, \\ -2 - k_\sigma & \text{if } \tau = \sigma, \end{cases} \qquad v'_\sigma = \begin{cases} v_\tau & \text{if } \tau \neq \sigma, \\ v_\sigma + k_\sigma + 1 & \text{if } \tau = \sigma. \end{cases} \tag{9.1}$$

Let f be a locally analytic function on $\mathcal{O}_F \otimes_\mathbb{Z} \mathbb{Z}_p$, and let $\{V\}$ be an open cover of $\mathcal{O}_F \otimes_\mathbb{Z} \mathbb{Z}_p$ such that $f|_V$ is analytic for each V. Then we can consider $f|_V$ as a power series in the d variables $\{z_\sigma : \sigma \in \Sigma\}$. We can consider the operator $(d/dz_\sigma)^{k_\sigma+1}$ on such power series in the natural way, and note that this induces a map

$$\Theta_\sigma : \mathcal{A}_\lambda(L) \longrightarrow \mathcal{A}_{\lambda_\sigma}(L).$$

For more details about this map, see [Urb11, Proposition 3.2.11]. Taking the continuous dual of this map, we obtain a map $\Theta^*_\sigma : \mathcal{D}_{\lambda_\sigma}(L) \to \mathcal{D}_\lambda(L)$.

Remark This map is equivariant with respect to the action of $\Sigma_0(p)$. Note, however, that the action of the $U_\mathfrak{p}$ operator is *different* on $\mathcal{D}_{\lambda_\sigma}(L)$ and $\mathcal{D}_\lambda(L)$, due to the scaling of $\mathbf{v}$ at σ. Indeed, we introduce a factor of the determinant of the component at σ to the power of $k_\sigma + 1$.

Now label the elements of Σ as $\sigma_1, \sigma_2, \ldots, \sigma_d$, where we can choose any ordering of the elements. We write $\Theta^*_0 : \{0\} \to \mathcal{D}_\lambda$, and for each $s = 1, \ldots, d$, we denote by Θ^*_s the map $\Theta^*_s := \sum_{i=1}^s \Theta^*_{\sigma_i} : \bigoplus_{i=1}^s \mathcal{D}_{\lambda_{\sigma_i}}(L) \to \mathcal{D}_\lambda(L)$. The cokernels of the maps Θ^*_s play a crucial role in the sequel. In particular, from the definition it is clear that $\mathrm{coker}(\Theta^*_0) = \mathcal{D}_\lambda(L)$. Consider now the map Θ^*_1. If $\mu \in \mathcal{D}_{\lambda_{\sigma_1}}(L)$, then $\Theta^*_1(\mu)$ is 0 on elements of $\mathbb{A}_\lambda(L)$ that are locally polynomial in z_{σ_1} of degree at most k_{σ_1}. Hence, for $\mu \in \mathcal{D}_\lambda(L)$, we have $\mu \notin \mathrm{Im}(\Theta^*_1)$ if and only if there exists a monomial $\mathbf{z}^\mathbf{r} := \prod_{\sigma \in \Sigma} z_\sigma^{r_\sigma}$ with $r_{\sigma_1} \leq k_{\sigma_1} + 1$ such that $\mu(\mathbf{z}^\mathbf{r}) \neq 0$. From this one can see that $\mathrm{coker}(\Theta^*_1)$ can be seen as the module of coefficients that are classical at σ_1 and overconvergent at $\sigma_2, \ldots, \sigma_d$. This motivates the following.

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Definition 9.5 Let $J \subset \Sigma$. For $\nu = (\mathbf{k}, \mathbf{v}) \in \mathbb{Z}[\Sigma]^2$, define $\mathcal{A}^J_\nu(L)$ to be the space of functions on $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ defined over L that are locally analytic in the variables z_σ for $\sigma \notin J$ and locally algebraic of degree at most $\max(k_\sigma, 0)$ in the variables z_σ for $\sigma \in J$. Define $\mathcal{D}^J_\nu(L)$ be the topological dual of $\mathcal{A}^J_\nu(L)$.

Thus we see that $\operatorname{coker}(\Theta_1^*) = \mathcal{D}_\lambda^{\{\sigma_1\}}(L)$. Continuing in the same vein, we see that $\operatorname{coker}(\Theta_s^*) = \mathcal{D}_\lambda^{J_s}(L)$, where $J_s := \{\sigma_1, \dots, \sigma_s\}$. In particular, if we write $V_{\lambda,\mathrm{loc}}(L)$ for the space of locally algebraic polynomials on $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ of degree at most $\mathbf{k}$, with the natural action of $\Sigma_0(p)$ depending on λ, then we get the following.

Proposition 9.6 *There is an exact sequence*

$$\bigoplus_{\sigma\in\Sigma} \mathcal{D}_{\lambda_\sigma}(L) \xrightarrow{\Theta_d^*} \mathcal{D}_\lambda(L) \longrightarrow V_{\lambda,\mathrm{loc}}(L) \longrightarrow 0.$$

In particular, we have $\operatorname{coker}(\Theta_d^*) = \mathcal{D}_\lambda^\Sigma(L) \cong V_{\lambda,\mathrm{loc}}(L)^*$.

These are the last terms of the locally analytic BGG resolution introduced in [Urb11, §3.3]; see [Urb11, Proposition 3.2.12] for further details of this exact sequence.

9.3 The Control Theorem

The following theorem is the main result of this part of the paper, and allows us to canonically lift small-slope classical modular symbols to overconvergent modular symbols.

Theorem 9.7 *Let $\lambda = (\mathbf{k}, \mathbf{v})$ be an admissible weight, and let $\mathbf{h} = (h_\mathfrak{p})_{\mathfrak{p}|p} \in \mathbb{Q}^{\{\mathfrak{p}|p\}}$. Let $k^0_\mathfrak{p} := \min\{k_\sigma : \sigma \sim \mathfrak{p}\}$ and recall the definition of $v_\mathfrak{p}(\lambda)$ from Definition 9.3. If for each prime $\mathfrak{p}$ above p we have*

$$h_\mathfrak{p} < \frac{k^0_\mathfrak{p} + v_\mathfrak{p}(\lambda) + 1}{e_\mathfrak{p}}, \tag{9.2}$$

then, for each r, the restriction

$$\rho\colon \mathrm{H}^r_c(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L)))^{\le \mathbf{h}} \xrightarrow{\sim} \mathrm{H}^r_c(Y_1(\mathfrak{n}), \mathcal{L}_2(V_\lambda(L)^*))^{\le \mathbf{h}}$$

of the specialisation map to the slope $\le \mathbf{h}$ subspaces with respect to the $U_\mathfrak{p}$-operators is an isomorphism.

To prove this, we make use of the following.

Lemma 9.8 *In the set-up of Theorem 9.7, if $\mathbf{h}$ satisfies equation (9.2), then for any s there is an isomorphism*

$$\mathrm{H}^r_c(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}^{J_{s-1}}_\lambda(L)))^{\le \mathbf{h}} \xrightarrow{\sim} \mathrm{H}^r_c(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}^{J_s}_\lambda(L)))^{\le \mathbf{h}}$$

induced from the natural specialisation maps.

Proof We follow [Urb11]. For any $\sigma \in \Sigma$, let $\lambda_\sigma = (\mathbf{k}', \mathbf{v}')$ be the weight defined in equation (9.1), and recall the theta maps $\Theta_s^*\colon \bigoplus_{i=1}^s \mathcal{D}_{\lambda_{\sigma_i}}(L) \to \mathcal{D}_\lambda(L)$. Recall that

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$\operatorname{coker}(\Theta_s^*) = \mathcal{D}_\lambda^{J_s}(L)$ can be viewed as a module of distributions that are classical at $\sigma_1, \dots, \sigma_s$ and overconvergent at $\sigma_{s+1}, \dots, \sigma_d$. In particular, there are natural projection maps $\mathcal{D}_\lambda^{J_{s-1}}(L) \to \mathcal{D}_\lambda^{J_s}(L)$ given by specialising from overconvergent to classical coefficients at σ_s. Moreover, from the definition of $\Theta_{\sigma_s}^*$ there is an exact sequence

$$\mathcal{D}_{\lambda_s}(L) \xrightarrow{\Theta_{\sigma_s}^*} \mathcal{D}_\lambda^{J_{s-1}}(L) \longrightarrow \mathcal{D}_\lambda^{J_s}(L) \longrightarrow 0,$$

and a closer inspection shows that the sequence

$$0 \longrightarrow \mathcal{D}_{\lambda_{\sigma_s}}^{J_{s-1}}(L) \longrightarrow \mathcal{D}_\lambda^{J_{s-1}}(L) \longrightarrow \mathcal{D}_\lambda^{J_s}(L) \longrightarrow 0 \tag{9.3}$$

is exact for the quotient $\mathcal{D}_{\lambda_{\sigma_s}}^{J_{s-1}}(L)$ of $\mathcal{D}_{\lambda_{\sigma_s}}(L)$.

Using Lemma 9.1 on the exact sequence of equation (9.3), we obtain the exact sequence

$$\begin{aligned}\cdots \longrightarrow \mathrm{H}_{\mathrm{c}}^i(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_{\lambda_{\sigma_s}}^{J_{s-1}}(L)))^{\leq \mathbf{h}} &\longrightarrow \mathrm{H}_{\mathrm{c}}^i(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda^{J_{s-1}}(L)))^{\leq \mathbf{h}} \\ \longrightarrow \mathrm{H}_{\mathrm{c}}^i(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda^{J_s}(L)))^{\leq \mathbf{h}} &\longrightarrow \mathrm{H}_{\mathrm{c}}^{i+1}(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_{\lambda_{\sigma_s}}^{J_{s-1}}(L)))^{\leq \mathbf{h}} \longrightarrow \cdots,\end{aligned}$$

where here we are taking slope decompositions with respect to the Hecke operators at p.

If $h_{\mathfrak{p}} < (k_{\mathfrak{p}}^0 + v_{\mathfrak{p}}(\lambda) + 1)/e_{\mathfrak{p}}$ for all primes above p, it follows that

$$h_{\mathfrak{p}(\sigma_s)} < \frac{k_{\sigma_s} + v_{\mathfrak{p}(\sigma_s)}(\lambda) + 1}{e_{\mathfrak{p}(\sigma_s)}} = \frac{v_{\mathfrak{p}(\sigma_s)}(\lambda_{\sigma_s})}{e_{\mathfrak{p}(\sigma_s)}}.$$

Now, by Corollary 9.4 (ii), as $\mathcal{D}_{\lambda_{\sigma_s}}^{J_{s-1}}$ is a quotient of $\mathcal{D}_{\lambda_{\sigma_s}}$, we must have

$$\mathrm{H}_{\mathrm{c}}^r(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_{\lambda_{\sigma_s}}^{J_{s-1}}(L)))^{\leq \mathbf{h}} = \{0\}$$

for all r. Then, using the long exact sequence, for all r we have

$$\mathrm{H}_{\mathrm{c}}^r(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda^{J_{s-1}}(L)))^{\leq \mathbf{h}} \cong \mathrm{H}_{\mathrm{c}}^r(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda^{J_s}(L)))^{\leq \mathbf{h}}$$

as required. ∎

Proof of Theorem 9.7 Recall that we defined $V_{\lambda,\mathrm{loc}}(L) \subset \mathcal{A}(L)$ to be the subspace of functions which are locally polynomial of degree at most $\mathbf{k}$. We see that $V_{\lambda,\mathrm{loc}}(L) \cong \varprojlim V_{\lambda,n}(L)$, where $V_{\lambda,n}(L) := \mathcal{A}_{\lambda,n}(L) \cap V_{\lambda,\mathrm{loc}}(L)$. Note that $V_\lambda(L) = V_{\lambda,0}(L)$. In particular, using part (ii) of Lemma 9.1, we have

$$\mathrm{H}_{\mathrm{c}}^q(Y_1(\mathfrak{n}), \mathcal{L}_2(V_{\lambda,\mathrm{loc}}(L)^*))^{\leq \mathbf{h}} \cong \mathrm{H}_{\mathrm{c}}^q(Y_1(\mathfrak{n}), \mathcal{L}_2(V_\lambda(L)^*))^{\leq \mathbf{h}}.$$

Hence it suffices to prove the theorem by considering the coefficients of the target space to be in $V_{\lambda,\mathrm{loc}}(L)^*$ instead of $V_\lambda(L)^*$.

We use Lemma 9.8. For this, note that $\mathcal{D}_\lambda^{\Sigma}(L) = V_{\lambda,\mathrm{loc}}(L)^*$ and $\mathcal{D}_\lambda^{\varnothing}(L) = \mathcal{D}_\lambda(L)$. A simple induction on s then shows that we have the required isomorphism. ∎

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10 Construction of the Distribution

Let Φ be a cuspidal eigenform over F that has small slope (in the sense of the previous section). Then via Eichler–Shimura, we can attach to Φ a small slope p-adic classical modular eigensymbol, and using the results of previous sections, we can lift this to a unique small slope overconvergent eigensymbol. In the work of Pollack and Stevens [PS11, PS12] and the work of the second author in [Wil17], once one has such a symbol, one can evaluate it at the cycle $\{0\} - \{\infty\}$ to obtain the p-adic L-function we desire. This, however, relies on the identification of $\mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L)))$ with the space $\mathrm{Hom}_\Gamma(\mathrm{Div}^0(\mathbb{P}^1(F)), \mathcal{D}_\lambda(L))$, an identification that exists only for $q = 1$, that is, for $F = \mathbb{Q}$ or an imaginary quadratic field. To generalise this to the totally real case, in [BS13] the first author used automorphic cycles, as introduced in Section 5.1, writing down overconvergent analogues of the evaluation maps we used with classical coefficients. Here, we generalise his results to the case of general number fields. The notation we use here was fixed in Section 5.1.

10.1 Evaluating Overconvergent Classes

Suppose $\Psi \in \mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L)))$. Here recall that we consider the local system given by fibres of $\mathrm{GL}_2(F)\backslash(\mathrm{GL}_2(\mathbb{A}_F) \times \mathcal{D}_\lambda(L))/\Omega_1(\mathfrak{n})K^+_\infty Z_\infty \to Y_1(\mathfrak{n})$, where the action is by $\gamma(x,\mu)uk = (\gamma x u k, \mu * u)$. In this setting, slightly different versions of the evaluation maps will allow us to associate a distribution with such a class.

Step 1. Pulling back to $X_\mathfrak{f}$: First we pullback along the map $\eta_\mathfrak{f}\colon X_\mathfrak{f} \to Y_1(\mathfrak{n})$. We have $\eta_\mathfrak{f}^*\Psi \in \mathrm{H}^q_c(X_\mathfrak{f}, \eta_\mathfrak{f}^*\mathcal{L}_2(\mathcal{D}_\lambda(L)))$. We can see (by examining equation (5.1)) that here the local system corresponding to $\mathcal{L}'_{\mathfrak{f},2}(\mathcal{D}_\lambda(L)) \coloneqq \eta_\mathfrak{f}^*\mathcal{L}_2(\mathcal{D}_\lambda(L))$ is given by the fibres of $F^\times\backslash(\mathbb{A}_F^\times \times \mathcal{D}_\lambda(L))/U(\mathfrak{f})F^1_\infty \to X_\mathfrak{f}$, with action

$$\gamma(x,\mu)ur = \left(\gamma x u r, \mu * \begin{pmatrix} u & ((u-1)\pi_\mathfrak{f}^{-1})_{v|p} \\ 0 & 1 \end{pmatrix}\right).$$

Step 2. Twisting the action: Unlike in the complex case described earlier, the action describing the local system above is not a nice action, so we twist to get a nicer action of units. To this end, the matrix

$$\begin{pmatrix} 1 & -1 \\ 0 & (\pi_\mathfrak{f})_{v|p} \end{pmatrix} \in \mathrm{GL}_2\Big(\prod_{\mathfrak{p}|p} F_\mathfrak{p}\Big) = \mathrm{GL}_2(F \otimes_\mathbb{Q} \mathbb{Q}_p)$$

lies in $\Sigma_0(p)$. So we twist our local system by this; denote this twist on distributions by

$$\begin{aligned} \zeta\colon \mathcal{D}_\lambda(L) &\longrightarrow \mathcal{D}_\lambda(L), \\ \mu &\longmapsto \mu * \begin{pmatrix} 1 & -1 \\ 0 & (\pi_\mathfrak{f})_{v|p} \end{pmatrix}, \end{aligned}$$

and consider $\zeta_*\eta_\mathfrak{f}^*\Psi \in \mathrm{H}^q_c(X_\mathfrak{f}, \mathcal{L}_{\mathfrak{f},2}(\mathcal{D}_\lambda(L)))$, where the local system $\mathcal{L}_{\mathfrak{f},2}(\mathcal{D}_\lambda(L))$ now is given by

$$F^\times\backslash(\mathbb{A}_F^\times \times \mathcal{D}_\lambda(L))/U(\mathfrak{f})F^1_\infty \longrightarrow X_\mathfrak{f},$$

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$$\gamma(x,\mu)ur = \left(\gamma xur, \mu * \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\right).$$

Step 3. Passing to individual components: In identical fashion to Section 5.2.2, we pull back under the isomorphism $\tau_{a_\mathbf{y}} : E(\mathfrak{f})F_\infty^1 \backslash F_\infty^+ \xrightarrow{\sim} X_\mathbf{y} \hookrightarrow X_\mathfrak{f}$ given by multiplication by $a_\mathbf{y}$. Then we have

$$\tau_{a_\mathbf{y}}^* \zeta_* \eta_\mathfrak{f}^* \Psi \in \mathrm{H}_\mathrm{c}^q(E(\mathfrak{f})F_\infty^1 \backslash F_\infty^+, \mathcal{L}_{\mathfrak{f},\mathbf{y},2}(\mathcal{D}_\lambda(L))),$$

where the local system $\mathcal{L}_{\mathfrak{f},\mathbf{y},2}(\mathcal{D}_\lambda(L))$ is given by

$$E(\mathfrak{f})F_\infty^1 \backslash (F_\infty^+ \times \mathcal{D}_\lambda(L)) \longrightarrow E(\mathfrak{f})F_\infty^1 \backslash F_\infty^+,$$

$$er(z,\mu) = \left(erz, \mu * \begin{pmatrix} e^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right).$$

(Note here that whilst $u \in U(\mathfrak{f})$ acts as $\left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)$, in this step we now have an inverse. This because u is considered as an element of the finite ideles whilst we instead see e as a diagonal infinite idele, which is equivalent under multiplication by $F^\times$ to e^{-1} as a diagonal finite idele and thus an element of $U(\mathfrak{f})$).

Step 4. Restricting the coefficient system: We would like a constant local system. This would allow us to evaluate the cohomology class easily. We see that if we restrict to a quotient of $\mathcal{D}_\lambda(L)$ such that, for all $e \in E(\mathfrak{f})$, the matrix $\left(\begin{smallmatrix} e & 0 \\ 0 & 1 \end{smallmatrix}\right)$ acts trivially, then we have precisely this. With this in mind, we make the following definitions.

Definition 10.1 Define $\mathcal{A}_\lambda^{\mathfrak{f},+}(L)$ to be the subspace of $\mathcal{A}_\lambda(L)$ given by

$$\mathcal{A}_\lambda^{\mathfrak{f},+}(L) := \left\{ f \in \mathcal{A}_\lambda(L) : \left(\begin{smallmatrix} e & 0 \\ 0 & 1 \end{smallmatrix}\right) * f = f \ \forall e \in E(\mathfrak{f}) \right\}.$$

Note that equivalently this is the set of all $f \in \mathcal{A}_\lambda(L)$ such that $f(ez) = e^{\mathbf{k}+\mathbf{v}} f(z)$.

Define $\mathcal{D}_\lambda^{\mathfrak{f},+}(L)$ to be the topological dual of $\mathcal{A}_\lambda^{\mathfrak{f},+}(L)$. Note that $\mathcal{D}_\lambda^{\mathfrak{f},+}(L)$ is a quotient of $\mathcal{D}_\lambda(L)$. (Henceforth, we will drop $\mathfrak{f}$ from the notation, as the level will be clear from context).

Now, if we pushforward via the map

$$\begin{aligned} \nu : \mathcal{D}_\lambda(L) &\longrightarrow \mathcal{D}_\lambda^+(L), \\ \mu &\longmapsto \mu|_{\mathcal{A}_\lambda^+(L)}, \end{aligned}$$

then the resulting local system is constant. We see that

$$\nu_* \tau_{a_\mathbf{y}}^* \zeta_* \eta_\mathfrak{f}^* \Psi \in \mathrm{H}_\mathrm{c}^q(E(\mathfrak{f})F_\infty^1 \backslash F_\infty^+, \mathcal{D}_\lambda^+(L)) \cong \mathcal{D}_\lambda^+(L),$$

where the isomorphism is given by integrating over $E(\mathfrak{f})F_\infty^1 \backslash F_\infty^+$.

10.1.1 Definition of the Evaluation Map

Definition 10.2 We write $\mathrm{Ev}_{\mathfrak{f},\dagger}^{a_\mathbf{y}}$ for the composition

$$\mathrm{Ev}_{\mathfrak{f},\dagger}^{a_\mathbf{y}} : \mathrm{H}_\mathrm{c}^q(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L))) \longrightarrow \mathcal{D}_\lambda^+(L)$$

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of the maps

$$\mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L))) \xrightarrow{\zeta_* \eta_{\mathfrak{f}}^*} \mathrm{H}^q_c(X_{\mathfrak{f}}, \mathcal{L}_{\mathfrak{f},2}(\mathcal{D}_\lambda(L))) \xrightarrow{\tau^*_{a_{\mathbf{y}}}} \cdots$$
$$\mathrm{H}^q_c(E(\mathfrak{f})F^1_\infty \backslash F^+_\infty, \mathcal{L}_{\mathfrak{f},\mathbf{y},2}(\mathcal{D}_\lambda(L)) \xrightarrow{\nu_*} \mathrm{H}^q_c(E(\mathfrak{f})F^1_\infty \backslash F^+_\infty, \mathcal{D}^+_\lambda(L)) \cong \mathcal{D}^+_\lambda(L).$$

In particular, we have maps $\mathrm{Ev}^{a_{\mathbf{y}}}_{\mathfrak{f},\dagger}$ for each $\mathbf{y} \in \mathrm{Cl}^+_F(\mathfrak{f})$. Note that these maps are dependent on the choice of representatives. In any case, for a fixed choice of representatives $\{a_{\mathbf{y}} \in \mathbb{A}_F^\times : \mathbf{y} \in \mathrm{Cl}^+_F(\mathfrak{f})\}$, we have now defined a map

$$\mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L))) \xrightarrow{\oplus_{\mathbf{y}} \mathrm{Ev}^{a_{\mathbf{y}}}_{\mathfrak{f},\dagger}} \bigoplus_{\mathrm{Cl}^+_F(\mathfrak{f})} \mathcal{D}^+_\lambda(L).$$

10.2 Locally Analytic Functions on $\mathrm{Cl}^+_F(p^\infty)$

Let L be a (not necessarily finite) extension of $\mathbb{Q}_p$ contained in $\mathbb{C}_p$, the completion of an algebraic closure of $\mathbb{Q}_p$. Denote by $\mathcal{A}(\mathrm{Cl}^+_F(p^\infty), L)$ the space of locally analytic functions on $\mathrm{Cl}^+_F(p^\infty)$ defined over L, and denote by $\mathcal{D}(\mathrm{Cl}^+_F(p^\infty), L)$ its topological dual over L. The p-adic L-function should be an element of this space of distributions; we now give some properties of locally analytic functions that will be required in the sequel.

10.2.1 The Geometry of $\mathrm{Cl}^+_F(p^\infty)$

We first recall the geometry of $\mathrm{Cl}^+_F(p^\infty)$, which is defined as follows: $\mathrm{Cl}^+_F(p^\infty) := F^\times \backslash \mathbb{A}_F^\times / U(p^\infty) F^+_\infty$. Letting $\mathfrak{f}$ range over all ideals dividing $(p)^\infty$ and taking the inverse limit of the series of exact sequences $\mathcal{O}^\times_{F,+} \to (\mathcal{O}_F/\mathfrak{f})^\times \to \mathrm{Cl}^+_F(\mathfrak{f}) \to \mathrm{Cl}^+_F \to 0$, we see that we have an exact sequence $\overline{\mathcal{O}^\times_{F,+}} \to (\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \to \mathrm{Cl}^+_F(p^\infty) \to \mathrm{Cl}^+_F \to 0$, so that, after picking a choice of representatives for Cl^+_F, we have $\mathrm{Cl}^+_F(p^\infty) \cong \bigsqcup_{\mathrm{Cl}^+_F} (\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times / \overline{E(1)}$. (Here note that $E(1) = \mathcal{O}^\times_{F,+}$, and we have taken $\overline{E(1)}$ to be the completion of $E(1)$ in $(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$). Indeed, for any $\mathfrak{f}$, we can go further and write $\mathrm{Cl}^+_F(p^\infty) \cong \bigsqcup_{\mathbf{y} \in \mathrm{Cl}^+_F(\mathfrak{f})} G_{\mathbf{y}}$, where

$$G_{\mathbf{y}} := \{z \in \mathrm{Cl}^+_F(p^\infty) : z \mapsto \mathbf{y} \text{ under the map } \mathrm{Cl}^+_F(p^\infty) \to \mathrm{Cl}^+_F(\mathfrak{f})\}. \tag{10.1}$$

Note that multiplication by $a_{\mathbf{y}}^{-1}$ gives an isomorphism

$$G_{\mathbf{y}} \cong G := \{z \in (\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times : z \equiv 1 \pmod{\mathfrak{f}}\} / \overline{E(\mathfrak{f})}.$$

10.2.2 Properties of Locally Analytic Functions

For a choice of idelic representatives $\{a_{\mathbf{y}}\} \subset \mathbb{A}_F^\times$ of $\mathrm{Cl}^+_F(\mathfrak{f})$, we can consider any function $\varphi : \mathrm{Cl}^+_F(p^\infty) \to L$ as a collection $\{\varphi_{a_{\mathbf{y}}} : \mathbf{y} \in \mathrm{Cl}^+_F(\mathfrak{f})\}$, for

$$\varphi_{a_{\mathbf{y}}} : G \longrightarrow L,$$
$$z \longmapsto \varphi(a_{\mathbf{y}}^{-1} z).$$

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Then, in a slight abuse of notation, $\varphi_{a_\mathbf{y}}$ can be thought of as a function

$$\varphi_{a_\mathbf{y}} \colon \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow L$$

with support on a subset of $(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$ and with $\varphi_{a_\mathbf{y}}(ez) = \varphi_{a_\mathbf{y}}(z)$ for all $e \in \overline{E(1)}$. A simple calculation then shows the following.

Proposition 10.3 *Suppose $a'_\mathbf{y} = a_\mathbf{y} \gamma u r$ is a different representative of the class $\mathbf{y} \in \mathrm{Cl}^+_F(\mathfrak{f})$, where $\gamma \in F^\times$, $u \in U(\mathfrak{f})$, and $r \in F^+_\infty$. Then $\varphi_{a_\mathbf{y}}(z) = \varphi_{a'_\mathbf{y}}(\widetilde{u}z)$ as functions on G, where $\widetilde{u}$ is the image of $u \in U(\mathfrak{f})$ in $U(\mathfrak{f})/U(p^\infty) \subset (\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$.*

10.3 Constructing μ_Ψ in $\mathcal{D}(\mathrm{Cl}^+_F(p^\infty), L)$

Notation We write $A_\mathfrak{f} = \{a_\mathbf{y}\}$ to denote our system of class group representatives for $\mathrm{Cl}^+_F(\mathfrak{f})$.

We now construct a distribution $\mu_\Psi^{A_\mathfrak{f}}$ associated with this choice of representatives. Let φ be a locally analytic function on $\mathrm{Cl}^+_F(p^\infty)$. Via the above construction, we obtain functions $\varphi_{a_\mathbf{y}} \colon G_\mathbf{y} \to L$, each of which we can view as a function $\varphi_{a_\mathbf{y}} \colon \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p \to L$ with support on the open subset $\{z \in (\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times : z \equiv 1 \pmod{\mathfrak{f}}\}$ and satisfying $\varphi_{a_\mathbf{y}}(ez) = \varphi_{a_\mathbf{y}}(z)$ for all $e \in E(\mathfrak{f})$. Now, $\mathrm{Ev}^{a_\mathbf{y}}_{\mathfrak{f},\dagger}(\Psi) \in \mathcal{D}^+_\lambda(L)$. This is a distribution that takes as input functions $\psi \colon (\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \to L$ with $\psi(ez) = e^{\mathbf{k}+\mathbf{v}}\psi(z)$. To force $\varphi_{a_\mathbf{y}}$ to satisfy this condition, we twist it.

Definition 10.4 If $\psi \colon \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p \to L$ is a function with support on elements congruent to $1 \pmod{\mathfrak{f}}$ and that satisfies $\psi(ez) = \psi(z)$ for all $e \in E(\mathfrak{f})$, then we define $\psi^* \in \mathcal{A}^+_\lambda(L)$ by

$$\psi^*(z) = \begin{cases} z^{\mathbf{k}+\mathbf{v}}\psi(z^{-1}) & \text{if } z \in (\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times, \\ 0 & \text{otherwise.} \end{cases}$$

Since ψ has support inside the units, this remains continuous. It is simple to see that this now satisfies the condition required. We use z^{-1} rather than z for reasons of compatibility in later calculations.

Now we can evaluate $\mathrm{Ev}^{a_\mathbf{y}}_{\mathfrak{f},\dagger}(\Psi)$ at $\varphi^*_{a_\mathbf{y}}$. This motivates the following.

Definition 10.5 Define $\mu_\Psi^{A_\mathfrak{f}} \in \mathcal{D}(\mathrm{Cl}^+_F(p^\infty), L)$ by

$$\mu_\Psi^{A_\mathfrak{f}}(\varphi) = \sum_{\mathbf{y} \in \mathrm{Cl}^+_F(\mathfrak{f})} \mathrm{Ev}^{a_\mathbf{y}}_{\mathfrak{f},\dagger}(\Psi)(\varphi^*_{a_\mathbf{y}}) \in L.$$

Proposition 10.6 *For fixed $\mathfrak{f}$, this is independent of the choice of class group representatives.*

Proof There are two layers to this. Choosing representatives fixes the collection of maps $\{\mathrm{Ev}^{a_\mathbf{y}}_{\mathfrak{f},\dagger}(\Psi) : a_\mathbf{y} \in A_\mathfrak{f}\}$, and the identification of φ with $(\varphi_{a_\mathbf{y}})_{\mathbf{y} \in \mathrm{Cl}^+_F(\mathfrak{f})}$. We prove that these choices cancel each other out. To do so, we examine the local systems; see Section 5.1 for descriptions of each local system.

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Recall that we have $\zeta_*\eta_{\mathfrak{f}}^*\Psi \in \mathrm{H}_c^q(X_{\mathfrak{f}}, \mathcal{L}_{\mathfrak{f},2}(\mathcal{D}_\lambda(L)))$ (canonically), and then that we can pull back to $X_{\mathfrak{y}}$ under the canonical inclusion. At the first stage where our representatives come into play, the map of local systems induced by

$$\tau_{a_{\mathfrak{y}}}\colon E(\mathfrak{f})F_\infty^1\backslash F_\infty^+ \xrightarrow{\sim} X_{\mathfrak{y}}$$

can be described by the map

$$F^\times\backslash(F^\times a_{\mathfrak{y}}U(\mathfrak{f})F_\infty^+ \times \mathcal{D}_\lambda(L))/U(\mathfrak{f})F_\infty^1 \longrightarrow E(\mathfrak{f})F_\infty^1\backslash(F_\infty^+ \times \mathcal{D}_\lambda(L)) \tag{10.2}$$
$$(\gamma a_{\mathfrak{y}} u r, \mu) \longmapsto \left(r, \mu * \begin{pmatrix} \widetilde{u}^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right),$$

recalling that $\tau_{a_{\mathfrak{y}}}$ is given by $z \mapsto a_{\mathfrak{y}} z$ and that $\widetilde{u}$ is the image of u in $(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$. This map is well defined; indeed, consider

$$\begin{aligned}\gamma'[(\gamma a_{\mathfrak{y}} u r, \mu)]vs &= \left[\left(\gamma'\gamma a_{\mathfrak{y}} uvrs, \mu * \begin{pmatrix} \widetilde{v} & 0 \\ 0 & 1 \end{pmatrix}\right)\right] \longmapsto \left[\left(rs, \left(\mu * \begin{pmatrix} \widetilde{v} & 0 \\ 0 & 1 \end{pmatrix}\right) * \begin{pmatrix} (\widetilde{uv})^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)\right] \\ &= \left[\left(r, \mu * \begin{pmatrix} \widetilde{u}^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)\right] = \mathrm{Im}([\gamma a_{\mathfrak{y}} u r, \mu]).\end{aligned}$$

Now suppose we choose a different set of representatives $\{a'_{\mathfrak{y}}\}$, with, as before,

$$a'_{\mathfrak{y}} = a_{\mathfrak{y}}\gamma u r, \quad \gamma \in F^\times, u \in U(\mathfrak{f}), r \in F_\infty^1.$$

Then under the map of equation (10.2), we have

$$[(a'_{\mathfrak{y}}, \mu)] = [(a_{\mathfrak{y}}\gamma u r, \mu)] \longmapsto \left[\left(r, \mu * \begin{pmatrix} \widetilde{u}^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)\right].$$

Thus, when we restrict, we find that $\mathrm{Ev}_{\mathfrak{f},\dagger}^{a'_{\mathfrak{y}}}(\Psi) = \mathrm{Ev}_{\mathfrak{f},\dagger}^{a_{\mathfrak{y}}}(\Psi) * \left(\begin{smallmatrix} \widetilde{u}^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We have already shown that, for $\varphi \in \mathcal{A}(\mathrm{Cl}_F^+(p^\infty), L)$, we have $\varphi_{a'_{\mathfrak{y}}}(\widetilde{u}z) = \varphi_{a_{\mathfrak{y}}}(z)$. Then an easy calculation shows that $\varphi^*_{a'_{\mathfrak{y}}}(z) = \left(\begin{smallmatrix} \widetilde{u} & 0 \\ 0 & 1 \end{smallmatrix}\right) * \varphi^*_{a_{\mathfrak{y}}}(z)$. Accordingly,

$$\mathrm{Ev}_{\mathfrak{f},\dagger}^{a'_{\mathfrak{y}}}(\Psi)(\varphi^*_{a'_{\mathfrak{y}}}) = \mathrm{Ev}_{\mathfrak{f},\dagger}^{a_{\mathfrak{y}}}(\Psi) * \left(\begin{smallmatrix} \widetilde{u}^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\left(\begin{smallmatrix} \widetilde{u} & 0 \\ 0 & 1 \end{smallmatrix}\right) * \varphi^*_{a_{\mathfrak{y}}}\right) = \mathrm{Ev}_{\mathfrak{f},\dagger}^{a_{\mathfrak{y}}}(\Psi)(\varphi^*_{a_{\mathfrak{y}}}).$$

Thus this is independent of the choice of representatives, as desired. ∎

Definition 10.7 For some choice of representatives $A_{\mathfrak{f}} = \{a_{\mathfrak{y}}\}$ of $\mathrm{Cl}_F^+(\mathfrak{f})$, define $\mu_\Psi^{\mathfrak{f}} := \mu_\Psi^{A_{\mathfrak{f}}}$. (Note that, by the proposition, this is well defined for each $\mathfrak{f}$).

10.4 Compatibility Over Choice of $\mathfrak{f}$

We have defined, for each $\mathfrak{f}|p^\infty$, a distribution $\mu_\Psi^{\mathfrak{f}} \in \mathcal{D}(\mathrm{Cl}_F^+(p^\infty), L)$. We now investigate how this distribution varies with the choice of $\mathfrak{f}$. Since we have independence of choice, we now choose class group representatives that are compatible in the following sense.

Notation Throughout this section, take $\mathfrak{f}|p^\infty$ and let $\mathfrak{p}|p$ be a prime. We will make the following important assumption throughout this section: *the ideal $\mathfrak{f}$ is divisible by all of the primes above p.* Let $A_{\mathfrak{f}} = \{a_{\mathfrak{y}}\}$ be a full set of representatives for $\mathrm{Cl}_F^+(\mathfrak{f})$, and let $\{u_r \in U(\mathfrak{f}) : r \in R\}$, for $R = U(\mathfrak{f})/E(\mathfrak{f})U(\mathfrak{f}\mathfrak{p})$, be elements of $U(\mathfrak{f})$ such that the set $A_{\mathfrak{f}\mathfrak{p}} := \{a_{\mathfrak{y}}u_r : \mathfrak{y} \in \mathrm{Cl}_F^+(\mathfrak{f}), r \in R\}$ is a full set of representatives for $\mathrm{Cl}_F^+(\mathfrak{f}\mathfrak{p})$.

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Lemma 10.8 (i) *There is a commutative diagram*

$$\begin{array}{ccc} \mathrm{H}^q_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L))) & \xrightarrow{U_p} & \mathrm{H}^q_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L))) \\ \downarrow{\scriptstyle \zeta_* \eta^*_{\mathfrak{f}\mathfrak{p}}} & & \downarrow{\scriptstyle \zeta_* \eta^*_{\mathfrak{f}}} \\ \mathrm{H}^q_{\mathrm{c}}(X_{\mathfrak{f}\mathfrak{p}}, \mathcal{L}_{\mathfrak{f}\mathfrak{p},2}(\mathcal{D}_\lambda(L))) & \xrightarrow{\mathrm{Tr}} & \mathrm{H}^q_{\mathrm{c}}(X_{\mathfrak{f}}, \mathcal{L}_{\mathfrak{f},2}(\mathcal{D}_\lambda(L))) \end{array}$$

where the bottom map is the natural trace map on cohomology [Hid93, §7].

(ii) *We have, for* $\Psi \in \mathrm{H}^q_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L)))$, *the relation*

$$\mathrm{Ev}^{a_\mathfrak{y} u_r}_{\mathfrak{f}\mathfrak{p},\dagger}(\Psi) * \left(\begin{smallmatrix} \widetilde{u_r} & 0 \\ 0 & 1 \end{smallmatrix}\right) = \mathrm{Ev}^{a_\mathfrak{y}}_{\mathfrak{f},\dagger}(\Psi|U_\mathfrak{p})|_{G_r},$$

where

$$G_r := \{z \in \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p : \text{ there exists } e \in E(\mathfrak{f}) \text{ such that } ez \equiv u_r \pmod{\mathfrak{f}\mathfrak{p}}\}.$$

Proof For part (i), see [BS13, Lemme 5.2]; the proof generalises immediately to the general number field setting. For part (ii), we bring in our explicit dependence on class group representatives. In particular, note that there is a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^q_{\mathrm{c}}(X_{\mathfrak{f}\mathfrak{p}}, \mathcal{L}_{\mathfrak{f}\mathfrak{p},2}(\mathcal{D}_\lambda(L))) & \xrightarrow{\mathrm{Tr}} & \mathrm{H}^q_{\mathrm{c}}(X_{\mathfrak{f}}, \mathcal{L}_{\mathfrak{f},2}(\mathcal{D}_\lambda(L))) \\ \downarrow{\scriptstyle \nu_*(\tau^{\mathfrak{f}\mathfrak{p}}_{a_\mathfrak{y} u_r})^*} & & \downarrow{\scriptstyle \nu_*(\tau^{\mathfrak{f}}_{a_\mathfrak{y} u_r})^*} \\ \mathcal{D}^{\mathfrak{f}\mathfrak{p},+}_\lambda(L) & \xrightarrow{\text{restriction to } G_r} & \mathcal{D}^{\mathfrak{f},+}_\lambda(L) \end{array}$$

where we have written $(\tau^{\mathfrak{f}}_{a_\mathfrak{y} u_r})^*$ to emphasise the dependence of this map on the ideal. Hence $\mathrm{Ev}^{a_\mathfrak{y} u_r}_{\mathfrak{f}\mathfrak{p},\dagger}(\Psi) = \mathrm{Ev}^{a_\mathfrak{y} u_r}_{\mathfrak{f},\dagger}(\Psi|U_\mathfrak{p})|_{G_r}$. Using the results of the previous section, we have the equality $\mathrm{Ev}^{a_\mathfrak{y}}_{\mathfrak{f},\dagger}(\Psi|U_\mathfrak{p}) = \mathrm{Ev}^{a_\mathfrak{y} u_r}_{\mathfrak{f},\dagger}(\Psi|U_\mathfrak{p}) * \left(\begin{smallmatrix} \widetilde{u_r} & 0 \\ 0 & 1 \end{smallmatrix}\right)$, hence the result. ∎

Proposition 10.9 *Let* $\mathfrak{f}|p^\infty$ *be divisible by all of the primes above p, and let* $\mathfrak{p}$ *be a prime above p. Let* $\Psi \in \mathrm{H}^q_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L)))$ *be an eigensymbol for all the Hecke operators at p, with* $U_\mathfrak{p}$*-eigenvalue* $\lambda_\mathfrak{p}$. *Then* $\mu^{\mathfrak{f}\mathfrak{p}}_\Psi = \lambda_\mathfrak{p} \mu^{\mathfrak{f}}_\Psi$.

Proof Let $\varphi \in \mathcal{A}(\mathrm{Cl}^+_F(p^\infty))$. We evaluate $\mu^{\mathfrak{f}\mathfrak{p}}_\Psi$ at φ by using the class group representatives $A_{\mathfrak{f}\mathfrak{p}}$, and then evaluate $\mu^{\mathfrak{f}}_{\Psi|U_\mathfrak{p}}$ at φ using the representatives $A_\mathfrak{f}$, and use the previous lemma to show that they are equal.

Fix $\mathfrak{y} \in \mathrm{Cl}^+_F(\mathfrak{f})$ and $r \in R$. Then we see that $\varphi_{a_\mathfrak{y} u_r}(z) = \varphi_{a_\mathfrak{y}}(u_r^{-1} z)$ for $z \in G_r$. In particular, we have $\varphi^*_{a_\mathfrak{y}}|_{G_r} = \left(\begin{smallmatrix} \widetilde{u_r}^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) * \varphi^*_{a_\mathfrak{y} u_r}(z)$. Observe now that by the previous lemma, we have

$$\mathrm{Ev}^{a_\mathfrak{y}}_{\mathfrak{f},\dagger}(\Psi|U_\mathfrak{p})(\varphi^*_{a_\mathfrak{y}}) = \sum_{r \in R} \mathrm{Ev}^{a_\mathfrak{y} u_r}_{\mathfrak{f}\mathfrak{p}}(\Psi) \,\Big|\, \left(\begin{smallmatrix} \widetilde{u_r} & 0 \\ 0 & 1 \end{smallmatrix}\right)(\varphi^*_{a_\mathfrak{y}}|_{G_r}) = \sum_{r \in R} \mathrm{Ev}^{a_\mathfrak{y} u_r}_{\mathfrak{f}\mathfrak{p}}(\Psi)(\varphi^*_{a_\mathfrak{y} u_r}).$$

Summing over $\mathfrak{y} \in \mathrm{Cl}^+_F(\mathfrak{f})$ on both sides and replacing $\Psi|U_\mathfrak{p}$ with $\lambda_\mathfrak{p}\Psi$ on the left-hand side now shows the result. ∎

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We have now proved the following.

Theorem 10.10 *Let $\Psi \in \mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L)))$ be an eigenclass for the $U_\mathfrak{p}$ operators for all $\mathfrak{p}|p$, and let $\mathfrak{f}|(p^\infty)$ be some choice of ideal with $\mathfrak{f}$ divisible by all the primes above p. Define $U_\mathfrak{f} := \prod_{\mathfrak{p}^r||\mathfrak{f}} U_\mathfrak{p}^r$, write $\lambda_\mathfrak{f}$ for the eigenvalue of $U_\mathfrak{f}$, and define $\mu_\Psi := \lambda_\mathfrak{f}^{-1}\mu_\Psi^\mathfrak{f}$. This is well defined and independent of choices up to a fixed choice of uniformisers at primes above p. Thus, for such Ψ, there is a way of attaching an element μ_Ψ of $\mathcal{D}(\mathrm{Cl}_F^+(p^\infty), L)$ to Ψ that is independent of choices.*

Definition 10.11 In the set-up above, we call μ_Ψ the *p-adic L-function of* Φ.

10.5 Evaluating at Hecke Characters

Let φ be a Hecke character of infinity type $\mathbf{r} \in \mathbb{Z}[\Sigma]$ and conductor $\mathfrak{f}|(p^\infty)$, where $\mathfrak{f}$ is divisible by every prime above p. In this section we describe the evaluation of the distribution μ_Ψ at $\varphi_{p\text{-fin}}$ (as defined in Section 2.2.2).

Choosing representatives $\{a_\mathbf{y}\}$ for $\mathrm{Cl}_F^+(\mathfrak{f})$, we see that $(\varphi_{p-\mathrm{fin}})_{a_\mathbf{y}} = \mathbf{1}_{G_\mathbf{y}}\varepsilon_\varphi\varphi_f(a_\mathbf{y})\mathbf{z}^\mathbf{r}$, where $\mathbf{1}_{G_\mathbf{y}}$ is the indicator function of the open subset of $\mathrm{Cl}_F^+(p^\infty)$ corresponding to $\mathbf{y} \in \mathrm{Cl}_F^+(\mathfrak{f})$ (see equation (10.1)), and $\mathbf{z}$ is a variable on $\mathcal{O}_F \otimes_\mathbb{Z} \mathbb{Z}_p$. We see that, for Ψ as above,

$$\mu_\Psi(\varphi_{p-\mathrm{fin}}) = \lambda_\mathfrak{f}^{-1} \sum_\mathbf{y} \varepsilon_\varphi\varphi_f(a_\mathbf{y})\,\mathrm{Ev}^{a_\mathbf{y}}_{\mathfrak{f},\dagger}(\Psi)(\mathbf{z}^{\mathbf{k}+\mathbf{v}-\mathbf{r}}). \tag{10.3}$$

11 Interpolation of L-values

In previous sections, we have defined the maps denoted by solid arrows in the following diagram.

$$\begin{array}{ccccc}
\mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(L)^*)) & & \xrightarrow{\quad\mathrm{Ev}^{a_\mathbf{y}}_{\mathfrak{f},\mathbf{j},1}\quad} & & L \\
\| & & & & \downarrow \beta \\
\mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_2(V_\lambda(L)^*)) & & \xrightarrow{\quad\mathrm{Ev}^{a_\mathbf{y}}_{\mathfrak{f},\mathbf{j},2}\quad} & & L \\
\uparrow \rho & & & & \uparrow \delta \\
\mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L))) & & \xrightarrow{\mathrm{Ev}^{a_\mathbf{y}}_{\mathfrak{f},\dagger}} & \mathcal{D}^+_\lambda(L) \xrightarrow{\text{ev. at } z^{\mathbf{k}-\mathbf{j}}} & L
\end{array} \tag{11.1}$$

In particular, the isomorphism is induced by the isomorphism of local systems given in Remark 4.3, the top (classical) evaluation map was defined in Section 5.2, the map ρ is induced from the specialisation $\mathcal{D}_\lambda(L) \to V_\lambda(L)^*$, and the bottom (overconvergent) evaluation map was defined in Section 10.1. In this section, we define the maps above denoted by dotted arrows in a manner such that the diagram commutes. By doing so, we will be able to use our previous results to relate the evaluation of the distribution μ_Φ at Hecke characters with critical L-values of Φ.

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11.1 Classical Evaluations, II

We start by defining the missing evaluation map. We have already touched on all of the key points of this construction; it is essentially a blend of our previous two evaluation maps. Taking notation from Section 5, we pullback along $\eta_{\mathfrak{f}}$, giving a local system $\eta_{\mathfrak{f}}^*\mathcal{L}_2(V_\lambda(L)^*)$ on $X_{\mathfrak{f}}$ that can be described by sections of the projection

$$F^\times\backslash(\mathbb{A}_F^\times \times V_\lambda(L)^*)/U(\mathfrak{f})F_\infty^1,$$

with action

$$f(x,P)ur = \left(fxur, P * \begin{pmatrix} u & ((u-1)\pi_{\mathfrak{f}}^{-1})_{v|\mathfrak{f}} \\ 0 & 1 \end{pmatrix}\right).$$

This bears relation with the overconvergent case in that we have an action of units that is not particularly nice. As in that case, we untwist this action using the map $(\zeta_{\mathfrak{f}})_*$ from Section 10.1, so that units act via the matrix $\left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We can then pull back under the injection $\tau_{a_\gamma}\colon E(\mathfrak{f})F_\infty^1\backslash F_\infty^+ \hookrightarrow X_{\mathfrak{f}}$ of previous sections. Finally, as in the classical case, we push forward under evaluation at the polynomial $\mathbf{X}^{\mathbf{k}-\mathbf{j}}\mathbf{Y}^{\mathbf{j}}$, which lands us in a cohomology group with coefficients in a constant sheaf (see Section 5.2). Combining all of these maps, we get a map

$$\mathrm{Ev}_{\mathfrak{f},\mathbf{j},2}^{a_\gamma}\colon \mathrm{H}_{\mathrm{c}}^q(Y_1(\mathfrak{n}), \mathcal{L}_2(V_\lambda(L)^*)) \longrightarrow L,$$

which gives the definition of the dotted horizontal arrow in the diagram.

The following lemma determines the definition of the map β in the diagram. For ease of notation, write Ev_k for the map $\mathrm{Ev}_{\mathfrak{f},\mathbf{j},k}^{a_\gamma}$.

Lemma 11.1 *Let α denote the isomorphism*

$$\alpha\colon \mathrm{H}_{\mathrm{c}}^q(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(L)^*)) \xrightarrow{\sim} \mathrm{H}_{\mathrm{c}}^q(Y_1(\mathfrak{n}), \mathcal{L}_2(V_\lambda(L)^*))$$

induced by the isomorphism $\mathcal{L}_1(V_\lambda(L)^*) \xrightarrow{\sim} \mathcal{L}_2(V_\lambda(L)^*)$ of local systems given by $(g,P) \longmapsto (g, P|g_p)$ (see Remark 4.3). Then $\mathrm{Ev}_2(\alpha(\phi)) = \pi_{\mathfrak{f}}^{\mathbf{j}+\mathbf{v}}\,\mathrm{Ev}_1(\phi)$.

Remark Here, in an abuse of notation, we write $\pi_{\mathfrak{f}}$ for the natural element of L corresponding to $(\pi_{\mathfrak{f}})_{v|p} \in \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$ under our fixed choice of uniformisers at primes above p. Note that under this map, a uniformiser $\pi_{\mathfrak{p}}$ is mapped to $N_{K_{\mathfrak{p}}/\mathbb{Q}_p}(\pi_{\mathfrak{p}})$, so that as elements of L, we have $\pi_{\mathfrak{f}}^{\mathbf{j}+\mathbf{v}} = N(\mathfrak{f})^{[\mathbf{j}+\mathbf{v}]}$ up to multiplication by a p-adic unit. In particular, *multiplication by* $\pi_{\mathfrak{f}}^{\mathbf{j}+\mathbf{v}}$ is a well-defined concept.

Proof We look at the local systems in each case. A simple check shows that there is a commutative diagram

$$\begin{array}{ccc} \mathrm{H}_{\mathrm{c}}^q(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(L)^*)) & \xrightarrow{\eta_{\mathfrak{f}}^*} & \mathrm{H}_{\mathrm{c}}^q(X_{\mathfrak{f}}, \mathcal{L}_{\mathfrak{f},1}(V_\lambda(L)^*)) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} \\ \mathrm{H}_{\mathrm{c}}^q(Y_1(\mathfrak{n}), \mathcal{L}_2(V_\lambda(L)^*)) & \xrightarrow{(\zeta_{\mathfrak{f}})_*\eta_{\mathfrak{f}}^*} & \mathrm{H}_{\mathrm{c}}^q(X_{\mathfrak{f}}, \mathcal{L}_{\mathfrak{f},2}(V_\lambda(L)^*)) \end{array},$$

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where α' is the map induced by the map

$$(x, P) \longmapsto \left(x, P\middle|\begin{pmatrix} x_p & 0 \\ 0 & (\pi_{\mathfrak{f}})_{v|\mathfrak{f}} \end{pmatrix}\right)$$

of local systems. Then continuing, we see that there is a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^q_{\mathrm{c}}(X_{\mathfrak{f}}, \mathcal{L}_{\mathfrak{f},1}(V_\lambda(L)^*)) & \xrightarrow{\tau^*_{a_{\mathbf{y}}}} & \mathrm{H}^q_{\mathrm{c}}(E(\mathfrak{f})F^1_\infty \backslash F^+_\infty, \mathcal{L}_{\mathfrak{f},\mathbf{y},1}(V_\lambda(L)^*)) \\ \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha''} \\ \mathrm{H}^q_{\mathrm{c}}(X_{\mathfrak{f}}, \mathcal{L}_{\mathfrak{f},2}(V_\lambda(L)^*)) & \xrightarrow{\tau^*_{a_{\mathbf{y}}}} & \mathrm{H}^q_{\mathrm{c}}(E(\mathfrak{f})F^1_\infty \backslash F^+_\infty, \mathcal{L}_{\mathfrak{f},\mathbf{y},2}(V_\lambda(L)^*)) \end{array}$$

where α'' is the map induced by the map

$$(r, P) \longmapsto \left(r, \middle|\begin{pmatrix} 1 & 0 \\ 0 & (\pi_{\mathfrak{f}})_{v|\mathfrak{f}} \end{pmatrix}\right)$$

of local systems. Finally, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^q_{\mathrm{c}}(E(\mathfrak{f})F^1_\infty \backslash F^+_\infty, \mathcal{L}_{\mathfrak{f},\mathbf{y},1}(V_\lambda(L)^*)) & \xrightarrow{(\text{ev. at } \mathbf{X}^{\mathbf{k}-\mathbf{j}}\mathbf{Y}^{\mathbf{j}})_*} & L \\ \downarrow{\scriptstyle \alpha''} & & \downarrow{\scriptstyle \times \pi_{\mathfrak{f}}^{\mathbf{j}+\mathbf{v}}} \\ \mathrm{H}^q_{\mathrm{c}}(E(\mathfrak{f})F^1_\infty \backslash F^+_\infty, \mathcal{L}_{\mathfrak{f},\mathbf{y},2}(V_\lambda(L)^*)) & \xrightarrow{\text{ev. at } \mathbf{X}^{\mathbf{k}-\mathbf{j}}\mathbf{Y}^{\mathbf{j}})_*} & L \end{array}$$

Putting these diagrams together gives the required result. ∎

Recall the definition of Ev_φ in Definition 6.3, and relabel $\mathrm{Ev}_{\varphi,1} := \mathrm{Ev}_\varphi$. Similarly define

$$\mathrm{Ev}_{\varphi,2} := \sum_{\mathbf{y} \in \mathrm{Cl}^+_F(\mathfrak{f})} \varepsilon_\varphi \varphi_f(a_{\mathbf{y}}) \, \mathrm{Ev}^{a_{\mathbf{y}}}_{\mathfrak{f},\mathbf{j},2},$$

where this makes sense, and note that by an identical argument to the previous one, this is independent of class group representatives. Using the results above with the results in Section 6, we obtain the following.

Corollary 11.2 *Recall the definition of $\theta_K \in \mathrm{H}^q_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathcal{L}_1(V_\lambda(K)^*))$ from Definition 4.8, and recall that we set θ_L to be its image in $\mathrm{H}^q_{\mathrm{c}}(Y_1(\mathfrak{n}), \mathcal{L}_2(V_\lambda(L)^*))$ under the inclusions of equation (4.2) and (4.3). Then*

$$\mathrm{Ev}_{\varphi,2}(\theta_L) = \pi_{\mathfrak{f}}^{\mathbf{j}+\mathbf{v}} \, \mathrm{Ev}_{\varphi,1}(\theta_K) = (-1)^{R(\mathbf{j},\mathbf{k})} \left[\frac{|D|\tau(\varphi)\pi_{\mathfrak{f}}^{\mathbf{j}+\mathbf{v}}}{2^{r_2}\Omega_\Phi^{\varepsilon_\varphi}} \right] \cdot \Lambda(\Phi, \varphi),$$

where $R(\mathbf{j}, \mathbf{k}) = \sum_{v \in \Sigma(\mathbb{R})} j_v + k_v + \sum_{v \in \Sigma(\mathbb{C})} k_v$.

Note here that this holds for any conductor $\mathfrak{f}|(p^\infty)$, with no condition on ramification.

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11.2 Relating Classical and Overcovergent Evaluations

Returning to the commutative diagram in equation (11.1), we now show that the map δ is actually nothing but the identity map. For a suitable automorphic form Φ, this will then allow us to prove the required interpolation property for the distribution μ_Φ.

Proposition 11.3 *There is a commutative diagram*

$$\begin{array}{ccc} \mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_2(\mathcal{D}_\lambda(L))) & \xrightarrow{\mathrm{Ev}^{a_\mathbf{y}}_{\mathfrak{f},\dagger}} & \mathcal{D}^+_\lambda(L) \\ \downarrow \rho & & \downarrow \textit{ev. at } z^{\mathbf{k}-\mathbf{j}} \\ \mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_2(V_\lambda(L)^*)) & \xrightarrow{\mathrm{Ev}^{a_\mathbf{y}}_{\mathfrak{f},\mathbf{j},2}} & L \end{array}$$

where the left vertical arrow is the specialisation map and the right vertical arrow is evaluation at the polynomial $z^{\mathbf{k}-\mathbf{j}}$.

Proof This is easily shown by looking at each step of the construction of the maps $\mathrm{Ev}^{a_\mathbf{y}}_{\mathfrak{f},\dagger}$ and $\mathrm{Ev}^{a_\mathbf{y}}_{\mathfrak{f},\mathbf{j},2}$ in the previous sections. At each of steps 1, 2 and 3 we can write down a specialisation map by restricting the coefficients, and by looking at the level of local systems, we can clearly see that these specialisations commute with the maps $\eta_\mathfrak{f}$, $\zeta_\mathfrak{f}$, and $\tau_{a_\mathbf{y}}$. It remains to show compatibility over step 4, where the construction is slightly different. This amounts to showing that the diagram

$$\begin{array}{ccc} \mathrm{H}^q_c(E(\mathfrak{f})F^1_\infty \backslash F^+_\infty, \mathcal{L}_{\mathfrak{f},\mathbf{y},2}(\mathcal{D}_\lambda(L))) & \xrightarrow{\text{res}} & \mathcal{D}^+_\lambda(L) \\ \downarrow & & \downarrow \textit{ev. at } z^{\mathbf{k}-\mathbf{j}} \\ \mathrm{H}^q_c(E(\mathfrak{f})F^1_\infty \backslash F^+_\infty, \mathcal{L}_{\mathfrak{f},\mathbf{y},2}(V_\lambda(L)^*)) & \xrightarrow{\text{ev. at } \mathbf{X}^{\mathbf{k}-\mathbf{j}}\mathbf{Y}^\mathbf{j}} & L \end{array}$$

commutes, where the left-hand map is restriction of the coefficients, the map res is the restriction of coefficients to $\mathcal{D}^+_\lambda(L)$ followed by integration over a fixed cycle, and the bottom map is the composition of $(\rho_\mathbf{j})_*$ with integration over the same cycle. Since $V_\lambda(L)^* \hookrightarrow \mathcal{A}_\lambda(L)$ via

$$P(\mathbf{X}, \mathbf{Y}) \mapsto P(z, 1),$$

we see that when we look at the corresponding local systems, we are evaluating at the same element in each case; thus the diagram commutes. ∎

By combining this with equation (10.3) for $\mu_\Psi(\varphi_{p-\mathrm{fin}})$, we get the following.

Corollary 11.4 *Let $\phi \in \mathrm{H}^q_c(Y_1(\mathfrak{n}), \mathcal{L}_2(V_\lambda(L)^*))$ be a small slope Hecke eigensymbol with $U_\mathfrak{f}$-eigenvalue $\lambda_\mathfrak{f}$ and with (unique) overconvergent eigenlift Ψ, and let μ_Ψ be the corresponding ray class distribution. Then for a Hecke character φ of infinity type $\mathbf{j} + \mathbf{v}$ and conductor $\mathfrak{f}|(p^\infty)$, where $0 \le \mathbf{j} \le \mathbf{k}$ and $\mathfrak{f}$ is divisible by every prime above p, we have $\mu_\Psi(\varphi_{p-fin}) = \lambda_\mathfrak{f}^{-1} \mathrm{Ev}_{\varphi,2}(\phi)$.*

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In the case that ϕ is the modular symbol attached to an automorphic form, this then gives the desired interpolation property at Hecke characters that ramify at all primes above p as an immediate corollary (see Theorem 12.1 below).

11.3 Interpolating at Unramified Characters

We now consider interpolation of L-values at Hecke characters that are not necessarily ramified at all primes above p. For this, we use Corollary 5.7. Whilst heretofore the results of this section have been for arbitrary modular symbols, to use this corollary we need to restrict to the case where the cohomology classes we consider are those attached to automorphic forms via the Eichler–Shimura isomorphism. Let Φ be such an automorphic form of weight λ and level $\Omega_1(\mathfrak{n})$, and suppose that Φ is a Hecke eigenform that has small slope at the primes above p. Let ϕ_L be the (p-adic) modular symbol attached to Φ, and let Ψ be the associated (unique) overconvergent modular symbol corresponding to ϕ_L under the control theorem. Then we have the following.

Lemma 11.5 *Let φ be a Hecke character of conductor $\mathfrak{f}|(p^\infty)$ (with no additional conditions on $\mathfrak{f}$) and infinity type $\mathbf{j}+\mathbf{v}$, where $0 \le \mathbf{j} \le \mathbf{k}$. Let B be the set of primes above p that do not divide $\mathfrak{f}$, and define $\mathfrak{f}' := \mathfrak{f}\prod_{\mathfrak{p}\in B}\mathfrak{p}$, so that $\mathfrak{f}'$ is divisible by all the primes above p. Then we have*

$$\begin{aligned}\mu_\Psi(\varphi_{p\text{-}fin}) &= \lambda_{\mathfrak{f}'}^{-1}\pi_{\mathfrak{f}'}^{\mathbf{j}+\mathbf{v}}\Big[\prod_{\mathfrak{p}\in B}(\varphi(\mathfrak{p})\lambda_\mathfrak{p}-1)\Big]\,\mathrm{Ev}_{\varphi,1}(\phi_L)\\ &= \lambda_{\mathfrak{f}}^{-1}\pi_{\mathfrak{f}}^{\mathbf{j}+\mathbf{v}}\Big[\prod_{\mathfrak{p}\in B}\varphi_{p\text{-}fin}(\pi_\mathfrak{p})(1-\lambda_\mathfrak{p}^{-1}\varphi(\mathfrak{p})^{-1})\Big]\,\mathrm{Ev}_{\varphi,1}(\phi_L).\end{aligned}$$

Proof By definition, $\mu_\Psi := \lambda_{\mathfrak{f}'}^{-1}\mu_\Psi^{\mathfrak{f}'}$. Hence we see that

$$\mu_\Psi(\varphi_{p-\mathrm{fin}}) = \lambda_{\mathfrak{f}'}^{-1}\sum_{\mathbf{y}\in\mathrm{Cl}_F^+(\mathfrak{f}')}\varepsilon_\varphi\varphi_f(a_\mathbf{y})\,\mathrm{Ev}_{\mathfrak{f}',\dagger}^{a_\mathbf{y}}(\Psi)(\mathbf{z}^{\mathbf{k}-\mathbf{j}}).$$

Using the results of Section 11.2, we can replace the overconvergent evaluations with classical ones, and then using the results of Section 11.1, we get

$$\mu_\Psi(\varphi_{p-\mathrm{fin}}) = \lambda_{\mathfrak{f}'}^{-1}\pi_{\mathfrak{f}'}^{\mathbf{j}+\mathbf{v}}\sum_{\mathbf{y}\in\mathrm{Cl}_F^+(\mathfrak{f}')}\varepsilon_\varphi\varphi_f(a_\mathbf{y})\,\mathrm{Ev}_{\mathfrak{f}',\mathbf{j},1}^{a_\mathbf{y}}(\phi_L).$$

We now use Corollary 5.7, which directly gives the first equality. The second equality follows since for $\mathfrak{p}$ not dividing $\mathfrak{f}$, we have $\pi_\mathfrak{p}^{\mathbf{j}+\mathbf{v}} = \varphi_{p-\mathrm{fin}}(\pi_\mathfrak{p})\varphi(\mathfrak{p})^{-1}$, an identity which follows from the definition of $\varphi_{p-\mathrm{fin}}$. ∎

12 Main Results

The following is a summary of the main results of this paper. Recall the setting: Φ is a small slope cuspidal eigenform for GL_2 over a number field F of weight $\lambda = (\mathbf{k},\mathbf{v}) \in \mathbb{Z}[\Sigma]^2$, where $\mathbf{k}+2\mathbf{v}$ is parallel, and with level $\Omega_1(\mathfrak{n})$, where $(p)|\mathfrak{n}$. Let $\Lambda(\Phi,\cdot)$ be the normalised L-function attached to Φ in Definition 3.2. To Φ, one can attach a unique overconvergent modular symbol Ψ using Theorem 9.7. Using Theorem 10.10 we may construct a distribution $\mu_\Psi \in \mathcal{D}(\mathrm{Cl}_F^+(p^\infty), L)$ attached to Ψ, which we defined to be the p-adic L-function of Φ.

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Theorem 12.1 *Let φ be a Hecke character of conductor $\mathfrak{f}|(p^\infty)$ and infinity type $\mathbf{j}+\mathbf{v}$, where $0 \le \mathbf{j} \le \mathbf{k}$, and let ε_φ be the character of $\{\pm 1\}^{\Sigma(\mathbb{R})}$ attached to φ in Section 2.2.1. Let $\varphi_{p-fin} \in \mathcal{A}(\mathrm{Cl}_F^+(p^\infty), L)$ be the p-adic avatar of φ. Let B be the set of primes above p that do not divide $\mathfrak{f}$. Then*

$$\mu_\Psi(\varphi_{p-fin}) = (-1)^{R(\mathbf{j},\mathbf{k})}\left[\frac{|D|\tau(\varphi)\pi_\mathfrak{f}^{\mathbf{j}+\mathbf{v}}}{2^{r_2}\lambda_\mathfrak{f}\Omega_\Phi^{\varepsilon_\varphi}}\right]\Big(\prod_{\mathfrak{p}\in B} Z_\mathfrak{p}\Big)\Lambda(\Phi,\varphi),$$

where $Z_\mathfrak{p} := \varphi_{p-fin}(\pi_\mathfrak{p})(1-\lambda_\mathfrak{p}^{-1}\varphi(\mathfrak{p})^{-1})$ (noting here that $\varphi(\mathfrak{p})$ is well defined since φ is unramified at $\mathfrak{p}$).

Here $R(\mathbf{j},\mathbf{k}) = \sum_{v\in\Sigma(\mathbb{R})} j_v + k_v + \sum_{v\in\Sigma(\mathbb{C})} k_v$, D is the discriminant of F, $\tau(\varphi)$ is the Gauss sum of Definition 2.6, r_2 is the number of pairs of complex embeddings of F, $\lambda_\mathfrak{f}$ is the $U_\mathfrak{f}$-eigenvalue of Φ, $\Omega_\Phi^{\varepsilon_\varphi}$ is the fixed period attached to Φ and ε_φ in Theorem 3.3, and $\Lambda(\Phi,\cdot)$ is the normalised L-function of Φ as defined in Definition 3.2.

13 Remarks on Uniqueness

When F is a totally real or imaginary quadratic field, we can prove a *uniqueness* property of this distribution. In particular, we prove that the distribution constructed above is *admissible* in a certain sense, and any admissible distribution is uniquely determined by its values at functions coming from critical Hecke characters [Col10, Loe14]. For further details of admissibility conditions in these cases, see [BS13, Wil17] for the totally real and imaginary quadratic situations, respectively. In the general case, things are more subtle. There is a good notion of admissibility for distributions on $\mathcal{O}_F \otimes_\mathbb{Z} \mathbb{Z}_p$, but it is not at all clear how this descends to a useful admissibility condition on $\mathrm{Cl}_F^+(p^\infty)$.

In particular, recall that $\mathrm{Cl}_F^+(p^\infty) = \bigsqcup_{\mathrm{Cl}_F^+}(\mathcal{O}_F\otimes\mathbb{Z}_p)^\times/\overline{E(1)}$. When F is imaginary quadratic, the unit group is finite, and in particular, in passing to the quotient, we do not change the rank. In this case, growth properties pass down almost unchanged. When F is totally real, the unit group is in a sense maximal if we assume Leopoldt's conjecture. In particular, provided this, the quotient is just one-dimensional, and we have a canonical direction with which to check growth properties.

Let us illustrate the difficulties of the general case with a conceptual example, for which the authors would like to thank David Loeffler. Let $F = \mathbb{Q}(\sqrt[3]{2})$, and note that F is a cubic field of mixed signature. We see that $(\mathcal{O}_F \otimes_\mathbb{Z} \mathbb{Z}_p)^\times$ is a p-adic Lie group of rank 3, and that the quotient by $\overline{E(1)}$ has rank 2 (since the unit group has rank 1 by Dirichlet's unit theorem). In particular, a distribution on $\mathrm{Cl}_F^+(p^\infty)$ can grow in two independent directions.

As the maximal CM subfield of F is nothing but $\mathbb{Q}$, it follows that the only possible infinity types of Hecke characters of F are parallel. In particular, there is only *one* dimension of Hecke characters. In this sense, even though we have constructed a distribution that interpolates all critical Hecke characters, there are simply not enough Hecke characters to hope that we can uniquely determine a ray class distribution by this interpolation property.

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One might be able to obtain nice growth properties using the extra structure that we obtain from our overconvergent modular symbol; in particular, one might expect the overconvergent cohomology classes we construct to take values in the smaller space of admissible distributions on $\mathcal{O}_F \otimes \mathbb{Z}_p$, which makes sense before we quotient to obtain distributions on $\mathrm{Cl}_F^+(p^\infty)$. Without the theory of admissibility at hand in the latter situation, however, we cannot show that the distribution constructed in this paper is (in general) unique. We have tried to rectify this by proving that the distribution we obtain is independent of choices. As seen in the previous sections, we were able to do this up to a (fixed) choice of uniformisers at the primes above p. Hence, in the spirit of Pollack and Stevens [PS12], we simply define the p-adic L-function to be this distribution.

It remains to comment on the dependence on choices of uniformisers. Whilst this dependence seems intrinsic to our more explicit approach, since submission, Bergdall and Hansen have given a similar, but less hands-on, construction in the Hilbert case that removes this dependency on uniformisers [BH17].

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Universitat Politécnica de Catalunya, Campus Nord, Calle Jordi Girona, 1-3, 08034 Barcelona, Spain
e-mail: daniel.barrera.salazar@upc.edu

Mathematics Department, Imperial College London, South Kensington Campus, London, SW7 2AZ, UK
e-mail: christopher.williams@imperial.ac.uk