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ALGEBRAIC CYCLES ON COMPACT QUATERNIONIC SHIMURA FOURFOLDS AND POLES OF L-FUNCTIONS

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Abstract. In this article we prove Tate conjecture for a large class of compact quaternionic Shimura fourfolds.

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1. Introduction. Let X be a smooth projective variety of dimension n defined over a number field F and let

$$\bar{X} = X \times_F \overline{\mathbb{Q}}.$$

For a prime number l, let $H^i_{\text{et}}(X, \overline{\mathbb{Q}}_l)$ be the étale cohomology of $\bar{X}$. If K is a number field, we denote $\Gamma_K := \text{Gal}(\overline{\mathbb{Q}}/K)$. The Galois group Γ_F acts on $H^i_{\text{et}}(X, \overline{\mathbb{Q}}_l)$ by a representation $\phi_{i,l}$. For any $j \in \mathbb{Z}$, let $H^i_{\text{et}}(X, \overline{\mathbb{Q}}_l)(j)$ denote the representation of Γ_F on $H^i_{\text{et}}(X, \overline{\mathbb{Q}}_l)$ defined by $\phi_{i,l} \otimes \xi_l^j$, where ξ_l is the l-adic cyclotomic character. For any finite extension E/F the elements of $V^i(X, E) := H^{2i}_{\text{et}}(X, \overline{\mathbb{Q}}_l)(i)^{\Gamma_E}$ are called *Tate cycles* on X defined over E. The union

$$V^i(X) := \cup_E V^i(X, E)$$

is the space of all *Tate cycles* on X.

To each algebraic subvariety Y of X of codimension i, one can associate a cohomology class

$$[Y] \in H_{2n-2i}(X(\mathbb{C}), \mathbb{Q}) \cong H^{2i}_B(X(\mathbb{C}), \mathbb{Q})(i),$$

where $H^{2i}_B(X(\mathbb{C}), \mathbb{Q})$ is the Betti cohomology. Then using the isomorphism

$$H^{2i}_B(X(\mathbb{C}), \mathbb{Q})(i) \otimes_{\mathbb{Q}} \mathbb{Q}_l \cong H^{2i}_{\text{et}}(X, \mathbb{Q}_l)(i),$$

we obtain a class $[Y] \in H^{2i}_{\text{et}}(X, \mathbb{Q}_l)(i)$. A cohomology class $[Y]$ obtained in this way is called *algebraic*. If Y is defined over a finite extension E of F, then we obtain a class $[Y] \in H^{2i}_{\text{et}}(X, \mathbb{Q}_l)(i)^{\Gamma_E}$. Let $U^i(X, E)$ be the space of algebraic cycles defined over E. Then $U^i(X, E) \subseteq V^i(X, E)$ and the first part of the Tate conjecture **[16]** states that for any finite extension E/F we have

$$U^i(X, E) = V^i(X, E),$$

i.e. every Tate cycle is algebraic.

The L-function $L^{2i}(s, X_{/F})$ (more exactly the Euler product) attached to the representation $\phi_{2i,l}$ converges for $\mathrm{Re}(s) > i+1$. The second part of the Tate conjecture **[16]** states that for any finite extension E/F the L-function $L^{2i}(s, X_{/E})$ has a meromorphic continuation to the entire complex plane and has a pole at $s = i+1$ of order equal to

$$\dim_{\overline{\mathbb{Q}}_l} V^i(X, E).$$

We consider a quartic totally real number field F containing a quadratic subfield. Let B be a quaternion division algebra over $\mathbb{Q}$ and let $D := B \otimes_{\mathbb{Q}} F$. We assume that D is a quaternion division algebra over F which splits at the real places. Let G be the algebraic group over F defined by the multiplicative group $D^\times$ of D and let $\bar{G} = \mathrm{Res}_{F/\mathbb{Q}}(G)$. We denote by $S_K := S_{\bar{G},K}$ the canonical model of the quaternionic Shimura variety associated with an open compact subgroup K of $\bar{G}(\mathbb{A}_f)$, where $\mathbb{A}_f$ is the finite part of the ring of adeles $\mathbb{A}_{\mathbb{Q}}$ of $\mathbb{Q}$. Then, S_K is a four-dimensional proper smooth variety defined over $\mathbb{Q}$.

In this paper we prove the first part of the Tate conjecture for S_K for non-CM submotives if we assume that the field F is Galois over $\mathbb{Q}$. We prove the second part of the Tate conjecture for S_K, without assuming that F is Galois over $\mathbb{Q}$, but only for solvable number fields (see Theorem 8.2 for details). We remark that similar results were obtained by Ramakrishnan **[12]** in the case of Hilbert modular fourfolds and by Harder, Langlands, Rapoport **[4]**, Murty, Ramakrishnan **[11]**, Klingenberg **[8]**, Lai **[11]** and Flicker and Hakim **[3]** in the case of Hilbert modular surfaces and compact quaternionic Shimura surfaces.

2. Quaternionic Shimura fourfolds and surfaces. Let F be a totally real field of degree 4 over $\mathbb{Q}$ such that F contains a quadratic number field F_0. We consider a quaternion division algebra B over $\mathbb{Q}$ and let $D := B \otimes_{\mathbb{Q}} F$. Assume that D is a quaternion division algebra over F which splits at the real places (we remark that given a quaternion division algebra D over F which splits at the real places, there exists a quaternion division algebra B over $\mathbb{Q}$ such that $D := B \otimes_{\mathbb{Q}} F$ if and only if for each rational prime p we have $\sum_{v|p} \mathrm{inv}_v D_v = 0$, where v runs over the places of F dividing p, and inv_v denotes the invariant of D at v). Let G be the algebraic group over F defined by the multiplicative group $D^\times$. By restricting the scalars, we obtain the algebraic group $\bar{G} = \mathrm{Res}_{F/\mathbb{Q}}(G)$ over $\mathbb{Q}$ defined by the propriety $\bar{G}(A) = G(A \otimes_{\mathbb{Q}} F)$ for all $\mathbb{Q}$-algebras A.

Then, $\bar{G}(\mathbb{R})$ is isomorphic to $\mathrm{GL}_2(\mathbb{R})^4$. Let $h : \mathbb{C}^* \to \bar{G}(\mathbb{R})$ be defined by $a + bi \mapsto \delta(\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right))$, where δ denotes the diagonal embedding of $\mathrm{GL}(2, \mathbb{R})$ in $\bar{G}(\mathbb{R})$. Let K_∞ be the centralizer of h in $\bar{G}(\mathbb{R})$. For each open compact subgroup $K \subset \bar{G}(\mathbb{A}_f)$ set

$$S_K(\mathbb{C}) = \bar{G}(\mathbb{Q}) \setminus \bar{G}(\mathbb{A}_{\mathbb{Q}})/KK_\infty.$$

For K sufficiently small, $S_K(\mathbb{C})$ is a complex manifold which is the set of the complex points of a proper smooth four-dimensional variety S_K defined over $\mathbb{Q}$, which is called a compact quaternionic Shimura fourfold.

Let D_0 be a quaternion algebra over F_0 which splits at the real places such that $D = D_0 \otimes_{F_0} F$ (we remark that $B \otimes_{\mathbb{Q}} F_0$ is a quaternionic division algebra over F_0 which has this propriety). Let G_0 be the algebraic group over F_0 defined by the multiplicative group $D_0^\times$. As above by restricting the scalars, we obtain the algebraic group

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$\bar{G}_0 = \mathrm{Res}_{F_0/\mathbb{Q}}(G_0)$. Then, $\bar{G}_0(\mathbb{R})$ is isomorphic to $\mathrm{GL}_2(\mathbb{R})^2$. Let $h_0 : \mathbb{C}^* \to \bar{G}_0(\mathbb{R})$ be defined by $a + bi \mapsto \delta_0((\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}))$, where δ_0 denotes the diagonal embedding of $\mathrm{GL}(2, \mathbb{R})$ in $\bar{G}_0(\mathbb{R})$. Let L_∞ be the centralizer of h_0 in $\bar{G}_0(\mathbb{R})$. For each open compact subgroup $L \subset \bar{G}_0(\mathbb{A}_f)$ set

$$S_{0L}(\mathbb{C}) = \bar{G}_0(\mathbb{Q}) \setminus \bar{G}_0(\mathbb{A}_\mathbb{Q})/LL_\infty.$$

For L sufficiently small, $S_{0L}(\mathbb{C})$ is a complex manifold which is the set of the complex points of a proper smooth two-dimensional variety S_{0L} defined over $\mathbb{Q}$, which is called a compact quaternionic Shimura surface.

3. Cohomologies for quaternionic Shimura fourfolds. Let K be a sufficiently small open compact subgroup of $\bar{G}(\mathbb{A}_f)$.

If l is a prime number, let $\mathbb{H}_K$ be the Hecke algebra generated by the bi-K-invariant $\bar{\mathbb{Q}}_l$-valued compactly supported functions on $\bar{G}(\mathbb{A}_f)$ under the convolution. If $\pi' = \pi'_f \otimes \pi'_\infty$ is an automorphic representation of $\bar{G}(\mathbb{A}_\mathbb{Q})$, we denote by $\pi_f'^K$ the space of K-invariants in π'_f. The Hecke algebra $\mathbb{H}_K$ acts on $\pi_f'^K$.

We have an action of the Hecke algebra $\mathbb{H}_K$ and an action of the Galois group $\Gamma_\mathbb{Q}$ on the étale cohomology $H^4_{\mathrm{et}}(S_K, \bar{\mathbb{Q}}_l)$ and these two actions commute. We say that the representation π' is *cohomological* if $H^*(\mathfrak{g}, \bar{G}_\infty, \pi'_\infty) \neq 0$, where $\mathfrak{g}$ is the Lie algebra of $\bar{G}_\infty$ (the cohomology is taken with respect to the $(\mathfrak{g}, \bar{G}_\infty)$-module associated with π'_∞).

We know the following result (see for example Propositions 1.5 and 1.8 of [**15**]).

PROPOSITION 3.1. *The representation of $\Gamma_\mathbb{Q} \times \mathbb{H}_K$ on the étale cohomology $H^4_{et}(S_K, \bar{\mathbb{Q}}_l)(2)$ is isomorphic to*

$$\oplus_{\pi'} \rho(\pi') \otimes \pi_f'^K,$$

where $\rho(\pi')$ is a representation of the Galois group $\Gamma_\mathbb{Q}$. The above sum is over weight 2 cohomological automorphic representations π' of $\bar{G}(\mathbb{A}_\mathbb{Q})$, such that $\pi_f'^K \neq 0$, and the $\mathbb{H}_K$-representations $\pi_f'^K$ are irreducible and mutually inequivalent, i.e. the decomposition is isotypic with respect to the action of $\mathbb{H}_K$.

The representations π' that appear in Proposition 3.1 are one dimensional or cuspidal and infinite dimensional. If π' is one dimensional, then $\rho(\pi')$ is six dimensional and if π' is cuspidal and infinite dimensional, then $\rho(\pi')$ is 16 dimensional. From now on in this paper we assume that π' is cuspidal and infinite dimensional, because for π' one dimensional the algebraicity of the Tate cycles corresponding to the π'-component of $H^4_{et}(S_K, \bar{\mathbb{Q}}_l)$ (see Proposition 3.1) could be proved in the same way as in Proposition 4.11 of [**12**], and the second part of the Tate conjecture in this case is also trivial (see [**12**]). We denote by π the cuspidal automorphic representation of $\mathrm{GL}(2)_{/F}$ corresponding to π' by Jaquet–Langlands correspondence.

Let $\rho_{\pi'} = \rho_\pi$ be the l-adic two-dimensional semisimple representation of $\Gamma_\mathbb{Q}$ associated with π' or with π (see [**2**, **17**]). Then the representation $\rho(\pi')$ is semisimple (see section 7 of [**13**]) and $\rho(\pi') = \mathrm{As}_{F/\mathbb{Q}}\rho_{\pi'}$, where $\mathrm{As}_{F/\mathbb{Q}}\rho_{\pi'}$ is the Asai (or tensor induction) representation (see section 6 of [**12**]).

We fix an isomorphism $j : \bar{\mathbb{Q}}_l \to \mathbb{C}$ and define the L-function

$$L^4(s, S_K) := \prod_{\pi'} \prod_q \det(1 - q^{-s+2} j(\rho(\pi')(\mathrm{Frob}_q)) | H^4_{\mathrm{et}}(S_K, \bar{\mathbb{Q}}_l)(2)^{I_q})^{-1},$$

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where Frob_q is a geometric Frobenius element at a rational prime q and I_q is a inertia group at q (in order to define the local factor at l one has to use actually the l'-adic cohomology for some $l' \neq l$ and Theorem 3 of [**1**] which gives us the expression of the local factors of the zeta functions of quaternionic Shimura varieties).

We have the canonical isomorphisms:

$$\Phi_K : H^4_B(S_K(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_l \to H^4_{\mathrm{et}}(S_K, \bar{\mathbb{Q}}_l)$$

and

$$\Phi : H^4_B(S(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_l \to H^4_{\mathrm{et}}(S, \bar{\mathbb{Q}}_l),$$

where $H^4_B(S_K(\mathbb{C}), \mathbb{Q})$ is the Betti cohomology, and $S := \varprojlim S_K$. We denote by $V(\pi')$ the π'-component of $H^4_{\mathrm{et}}(S, \bar{\mathbb{Q}}_l)(2)$ in the decomposition of Proposition 3.1 and by $V_B(\pi')$ the corresponding π'-component of $H^4_B(S(\mathbb{C}), \mathbb{Q})(2)$. Thus,

$$V_B(\pi') \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_l \cong V(\pi').$$

Since $\rho(\pi') = \mathrm{As}_{F/\mathbb{Q}}\rho_{\pi'}$, for each finite-order Hecke character ν of F, we have $\rho(\pi') \otimes \nu|_{I_{\mathbb{Q}}} \cong \rho(\pi' \otimes \nu)$, where $I_{\mathbb{Q}}$ is the idele group of $\mathbb{Q}$, i.e. $V(\pi') \otimes \nu|_{I_{\mathbb{Q}}} \cong V(\pi' \otimes \nu)$ as $\Gamma_{\mathbb{Q}}$-modules.

4. Meromorphic continuation. For π' being a cuspidal representation as in Proposition 3.1, we denote by $As_{F/F_0}(\pi')$ the isobaric automorphic representation of $\mathrm{GL}(4, \mathbb{A}_{F_0})$ defined in Theorem D of [**14**]. Let

$$\rho_{As_{F/F_0}(\pi')} : \Gamma_{F_0} \to \mathrm{GL}_4(\bar{\mathbb{Q}}_l)$$

be the l-adic representation associated with $As_{F/F_0}(\pi')$. Then, $\rho_{As_{F/F_0}(\pi')} = As_{F/F_0}\rho_{\pi'}$ and $L(s, \rho_{As_{F/F_0}(\pi')}) = L(s, As_{F/F_0}(\pi'))$. From the proprieties of the Asai representations we know that $\rho(\pi') = As_{F/\mathbb{Q}}\rho_{\pi'} = As_{F_0/\mathbb{Q}}(As_{F/F_0}\rho_{\pi'})$, and because $As_{F/F_0}\rho_{\pi'}$ is automorphic, from Theorem 6.11 of [**12**], and using the solvable base change for GL(2) (see [**10**]) and the main theorem of [**7**], one obtains easily that (see also [**1**])

PROPOSITION 4.1. *If $k/\mathbb{Q}$ is solvable, then the function $L(s, \rho(\pi')|_{\Gamma_k})$ has a meromorphic continuation to the entire complex plane and satisfies a functional equation $s \leftrightarrow 1 - s$.*

5. Some definitions. For k being a number field, define

$$\mathbf{V}(\pi', k) := \{x \in V(\pi') | \rho(\pi')(a)x = x \text{ for all } a \in \Gamma_k\},$$

and

$$\mathbf{V}(\pi', \bar{\mathbb{Q}}) := \cup_k \mathbf{V}(\pi', k),$$

where $V(\pi')$ is the space corresponding to $\rho(\pi')$. The elements of $\mathbf{V}(\pi', k)$ are called *Tate cycles* defined over k, and the elements of $\mathbf{V}(\pi', \bar{\mathbb{Q}})$ are called *Tate cycles*. We denote by $\mathbf{U}(\pi', k) \subseteq \mathbf{V}(\pi', k)$ the subspace of *algebraic cycles* defined over k.

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We denote by $r_{\mathrm{alg}}(\pi', k) := \dim_{\bar{\mathbb{Q}}_l} \mathbf{U}(\pi', k)$, by $r_l(\pi', k) := \dim_{\bar{\mathbb{Q}}_l} \mathbf{V}(\pi', k)$, by $r_l(\pi', \bar{\mathbb{Q}}) := \dim_{\bar{\mathbb{Q}}_l} \mathbf{V}(\pi', \bar{\mathbb{Q}})$ and for k solvable number filed by $r_{an}(\pi', k)$ the order of the pole of $L(s, \rho(\pi')|_{\Gamma_k})$ at $s = 1$. Then, $r_{\mathrm{alg}}(\pi', k) \leq r_l(\pi', k)$.

For ν being a finite-order character of $\Gamma_{\mathbb{Q}}$, define

$$\mathbf{V}(\pi'; \nu) := \{x \in V(\pi') | \rho(\pi')(a)x = \nu^{-1}(a)x \text{ for all } a \in \Gamma_{\mathbb{Q}}\},$$

and

$$\mathbf{V}(\pi', \mathbb{Q}^{ab}) := \cup_\nu \mathbf{V}(\pi'; \nu).$$

Let $\mathbf{U}(\pi'; \nu) \subseteq \mathbf{V}(\pi'; \nu)$ and $\mathbf{U}(\pi', \mathbb{Q}^{ab}) \subseteq \mathbf{V}(\pi', \mathbb{Q}^{ab})$ be the subspaces of *algebraic cycles*. We remark that when π' is non-CM, for k sufficiently large we have $\mathbf{V}(\pi', k) = \mathbf{V}(\pi', \mathbb{Q}^{ab})$, i.e. all the Tate cycles are defined over abelian extensions of $\mathbb{Q}$. For π' of CM type, it is possible to have for all k that $\mathbf{V}(\pi', k) \neq \mathbf{V}(\pi', \mathbb{Q}^{ab})$.

We denote by $r_{\mathrm{alg}}(\pi'; \nu) := \dim_{\bar{\mathbb{Q}}_l} \mathbf{U}(\pi', \nu)$, by $r_l(\pi'; \nu) := \dim_{\bar{\mathbb{Q}}_l} \mathbf{V}(\pi', \nu)$ and by $r_{an}(\pi', \nu)$ the order of the pole of $L(s, \rho(\pi') \otimes \nu)$ at $s = 1$. Then, $r_{\mathrm{alg}}(\pi'; \nu) \leq r_l(\pi'; \nu)$, $r_l(\pi', \mathbb{Q}^{ab}) \leq r_l(\pi', \bar{\mathbb{Q}})$.

6. Matching Tate cycles and poles. We say that an automorphic representation π of $\mathrm{GL}(2)/F$ for some number field F is of CM *type* if there exists some quadratic Galois character $\eta : I_F/F^\times \to \bar{\mathbb{Q}}_l^\times$, with $\eta \neq 1$ such that $\pi \cong \pi \otimes \eta$. We say that a representation ρ of a group G is *dihedral* if there exists a normal subgroup N of index 2 in G and a character $\chi : N \to \mathbb{C}^\times$ such that $\rho = \mathrm{Ind}_N^G \chi$. If π is an automorphic representation of $\mathrm{GL}(2)/F$ as in Proposition 3.1, then π is of CM type if and only if ρ_π is a dihedral representation. If π corresponds to an automorphic representation π' of $\bar{G}(\mathbb{A}_f)$ by Jacquet–Langlands correspondence, then we say that π' is CM if π is CM.

Using in particular the decomposition (in some cases) of $\rho(\pi')$ as a sum of automorphic representations, more exactly as a direct sum of l-adic representations associated Hecke characters and of twists by Hecke characters of $\mathrm{Sym}^2\pi''$ and $\mathrm{Sym}^4\pi''$ for non-CM representations π'' of GL(2) (which from **[5]** and **[8]** we know that are cuspidal and irreducible), in **[12]** (Propositions 8.6 and 8.8) are proved the following two lemmas (for the definition of π' and π see Proposition 3.1 and the comments after it):

LEMMA 6.1. *For π' non-CM, all the Tate classes in $V(\pi')$ are rational over an abelian number field k, with*

$$r_l(\pi', k) \leq 2;$$

hence,

$$r_l(\pi', \mathbb{Q}^{ab}) = r_l(\pi', \bar{\mathbb{Q}}) \leq 2.$$

LEMMA 6.2. *Let $F/\mathbb{Q}$ be Galois, and π' non-CM. Then*

(*a*) $r_l(\pi', \mathbb{Q}^{ab}) \neq 0$ *iff a twist of π is a base change from a quadratic subextension of F.*

(*b*) $r_l(\pi', \mathbb{Q}^{ab}) = 2$ *iff a twist of π is a base change from $\mathbb{Q}$.*

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(*c*) *The following are equivalent:*
(*i*) $r_l(\pi';\nu) = 2$ *for some* ν.
(*ii*) *A twist of* π *is a base change from* $\mathbb{Q}$, *and* F *is biquadratic.*

From Lemma 6.1 above and section 8 of [**12**] we know that

PROPOSITION 6.3. *For* π' *non-CM we have*

$$r_l(\pi';\nu) = r_{an}(\pi';\nu) \leq 2,$$

and thus because $r_{alg}(\pi';\nu) \leq r_l(\pi';\nu)$, *we have*

$$r_{alg}(\pi';\nu) \leq r_l(\pi';\nu) = r_{an}(\pi';\nu) \leq 2.$$

We also know that (see Proposition 8.5 of [**12**])

PROPOSITION 6.4. *If* π' *is of CM type, we have for any* k,

$$r_l(\pi',k) = r_{an}(\pi',k).$$

7. Twisted Hirzebruch–Zagier cycles. We use the same notations as in section 2, i.e. we consider a quaternion division algebra D_0 over some quadratic subextension F_0 of F such that $D = D_0 \otimes_{F_0} F$. Then, the map h factors through the map h_0 of $R_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^*)$ into $\bar{G}_{0\mathbb{R}}$. The natural diagonal embedding of $\bar{G}_0$ into $\bar{G}$ defines a morphism

$$\delta_{L,K} : S_{0L} \hookrightarrow S_K$$

over $\mathbb{Q}$, if L is contained in K.

For any $g \in \bar{G}(\mathbb{A}_f)$, and any open compact subgroup K of $\bar{G}(\mathbb{A}_f)$, define the corresponding *Hirzebruch–Zagier cycle* (or *H-Z cycle*) (relative to $\bar{G}_0$) to be the algebraic cycle of codimension 2 of S_K given by

$$^{D_0}Z_{g,K} = R(g)(\delta_{\bar{G}_0(\mathbb{A}_f)\cap gKg^{-1},gKg^{-1}}(S_{0\bar{G}_0(\mathbb{A}_f)\cap gKg^{-1}})),$$

where $R(g) : S_{gKg^{-1}} \to S_K$ is the right translation action on Shimura varieties.

Now for each character of finite order μ of F, we have the usual *twisted correspondence* $R(\mu) \subset S_K \times S_{K[\mu]}$, where $K[\mu]$ is some level which depends on K and μ (see for example section 5 of [**12**] for details). This twisting correspondence is algebraic and acts on any cohomology group, Betti or étale, of the fourfold $S = \varprojlim S_K$. The induced operator sends the π'-component to the $\pi' \otimes \mu$-component. The twisting correspondence $R(\mu)$ is rational over $\mathbb{Q}(\mu_1)$, where $\mu_1 = \mu|_{I_\mathbb{Q}}$, and $I_\mathbb{Q}$ is the idele group of $\mathbb{Q}$.

For each character of finite order μ of F and each H-Z cycle Z on S, let $Z(\mu)$ be the *μ-twisted H-Z cycle* obtained by pushing forward Z under $R(\mu)$. Then, $Z(\mu)$ is algebraic and rational over $\mathbb{Q}(\mu_1)$.

8. Matching algebraic cycles and poles. We prove

PROPOSITION 8.1. *Let* F *be a quartic, Galois, totally real number field, and* π' *be a non-CM cuspidal automorphic representation of* $\bar{G}(\mathbb{A}_\mathbb{Q})$ *of weight 2 that appears in*

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Proposition 3.1. Then for any character of finite order ν of $\Gamma_{\mathbb{Q}}$, we have

$$r_{alg}(\pi'; \nu) = r_{an}(\pi'; \nu).$$

Proof. From Proposition 6.3 we know that $r_{\mathrm{alg}}(\pi'; \nu) \leq r_{an}(\pi'; \nu) \leq 2$.

We distinguish three cases:

(A) $r_{an}(\pi'; \nu) = 0$. Then $r_{\mathrm{alg}}(\pi'; \nu) = 0$ and we are done.

(B) $r_{an}(\pi'; \nu) = 1$. Then as in the proof of Theorem 9.1 of **[12]**, one can find a quadratic subfield F_1 of F and a finite-order character μ of F such that $r_{\mathrm{alg}}(\pi'; \nu) = r_{\mathrm{alg}}(\pi' \otimes \mu; 1)$ and such that $L(s, As_{F/F_1}(\pi' \otimes \mu)) = L(s, As_{F/F_1}(\pi \otimes \mu))$ has a simple pole at $s = 1$, which by the residue formula of **[4]** implies that there exists some function ϕ in the space of π such that

$$\int_{\mathrm{GL}(2,F_1)Z(\mathbb{A}_{F_1})\backslash \mathrm{GL}(2,\mathbb{A}_{F_1})} \phi(g)\mu(\det(g))dg \neq 0,$$

where Z denotes the centre of GL(2). From **[6]** (the main theorem) and **[3]** (the appendix), we deduce that that there exists some function ϕ' in the space of π' such that

$$\int_{\bar{G}_1(\mathbb{Q})\bar{Z}_1(\mathbb{A}_{\mathbb{Q}})\backslash \bar{G}_1(\mathbb{A}_{\mathbb{Q}})} \phi'(g)\mu(\det(g))dg \neq 0,$$

where $\bar{Z}_1$ denotes the centre of $\bar{G}_1 = \mathrm{Res}_{F_1/\mathbb{Q}}(G_1)$, and G_1 is the algebraic group over F_1 defined by the multiplicative group $D_1^\times$ of a suitable quaternion division algebra D_1 over F_1 which satisfies that $D = D_1 \otimes_{F_1} F$ (more exactly let $\mathcal{S}$ be the set of places v of F_1 which split into two different places w and $\bar{w}$ of F such that D_w and $D_{\bar{w}}$ are ramified (we remark that because $D = B \otimes_{\mathbb{Q}} F$, we get that $B \otimes_{\mathbb{Q}} F_1$ is a quaternion division algebra over F_1, and thus we have that for each two different places w and $\bar{w}$ of F dividing a place v of F_1, D_w and $D_{\bar{w}}$ have the same invariant). If $|\mathcal{S}|$ is even, then there exists a quaternion division algebra D_1 over F_1 which ramifies at exactly the places v in $\mathcal{S}$ such that $D = D_1 \otimes_{F_1} F$. Then by the main theorem of **[6]**, D_1 satisfies the above propriety. If $|\mathcal{S}|$ is odd, then from **[3]** (appendix) we know that there exists a place (actually infinitely many) v_1 of F_1 outside S which does not split into F and a quaternion division algebra D_1 over F_1 which is ramified at exactly the places v in $\mathcal{S} \cup v_1$, such that $D = D_1 \otimes_{F_1} F$ and has the above propriety). Hence, the integral of a (2, 2)-form $\eta_{\phi'}$ on the compact quaternionic Shimura fourfold S_K defined by ϕ' has a non-zero μ-twisted period over a Hecke translate of the embedded compact quaternionic Shimura surface attached to D_1. Thus, the corresponding twisting self-correspondence of the fourfold defines for some $g \in \bar{G}(\mathbb{A}_f)$ a μ-twisted H-Z cycle $Z(\mu) =^{D_1} Z_{g,K}(\mu)$ of codimension 2 such that

$$\int_{Z(\mu)} \eta_{\phi'} \neq 0,$$

and hence $Z(\mu)$ is *homologically non-trivial*. Thus, $r_{\mathrm{alg}}(\pi'; \nu) \geq 1$, and we obtain that $r_{\mathrm{alg}}(\pi'; \nu) = 1$, and we are done.

(C) $r_{an}(\pi'; \nu) = 2$. From part (c) of Lemma 6.2 we deduce that $F/\mathbb{Q}$ is biquadratic, and then as in **[12]**, one can find a finite-order character μ of F such that $r_{\mathrm{alg}}(\pi'; \nu) = r_{\mathrm{alg}}(\pi' \otimes \mu; 1)$ and such that $\pi \otimes \mu$ is a base change from two quadratic subfields F_1 and

F_2 of F. Then as in (B) because the functions $L(s, As_{F/F_1}(\pi' \otimes \mu))$ and $L(s, As_{F/F_2}(\pi' \otimes \mu))$ have simple poles at $s = 1$, we get, for some quaternion algebras D_1 and D_2 over F_1 and F_2 and some $g_1, g_2 \in \bar{G}(\mathbb{A}_f)$, two twisted codimension 2 algebraic cycles $Z_1 :=^{D_1} Z_{g_1,K}(\mu)$ and $Z_2 :=^{D_2} Z_{g_2,K}(\mu)$ on S which are *homologically non-trivial* because the period integrals of some $(2, 2)$-forms over these cycles are non-zero. But these two cycles could be proportional in the π'-component of the cohomology, and thus one may have to replace one of them by a twisted version. Then in [**12**], Lemma 9.3, a finite-order character ξ of F is defined such that it has some special signature at the infinite places, and such that $\xi|_{I_{F_1}} = 1$, and thus $\xi|_{I_{\mathbb{Q}}} = 1$ and hence

$$r_{an}(\pi' \otimes \xi; \nu) = r_{an}(\pi'; \nu).$$

Also $L(s, As_{F/F_1}(\pi' \otimes \xi\mu))$ has a simple pole at $s = 1$, and thus, if we define

$$Z_3 :=^{D_1} Z^*_{g_3,K}(\mu\xi),$$

we have for some $g_3 \in \bar{G}(\mathbb{A}_f)$, and some ϕ' in the space of π' that

$$\int_{Z_3} \eta_{\phi'} \neq 0.$$

Then, from Lemma 9.4 of [**12**], we know by looking at the signatures at infinite places of the classes of Z_1, Z_2 and Z_3 in $V_B(\pi')$ that the space spanned by the classes of Z_1, Z_2 and Z_3 in $V_B(\pi')$ has dimension 2. Thus, $r_{\mathrm{alg}}(\pi'; \nu) \geq 2$ and we obtain that $r_{\mathrm{alg}}(\pi'; \nu) = 2$, and we are done. □

We can deduce now the following result.

THEOREM 8.2. *Let F be a quartic totally real number field containing a quadratic subfield. Let π' be an automorphic representation as in Proposition 3.1. Then,*

(a) For any solvable number field k, the function $L(s, \rho(\pi')|_{\Gamma_k})$ has a meromorphic continuation to the complex plane and satisfies a functional equation $s \leftrightarrow 1 - s$.

(b) If $F/\mathbb{Q}$ is Galois, then for any solvable number field k we have that $\dim_{\bar{\mathbb{Q}}_l} \mathbf{V}(\pi', k)$ *is equal to the order of the pole of the function $L(s, \rho(\pi')|_{\Gamma_k})$ at $s = 1$. If π' is CM, this result is true for any number field k.*

(c) If $F/\mathbb{Q}$ is Galois and π' is non-CM, then for any number field k we have

$$\dim_{\bar{\mathbb{Q}}_l} \mathbf{U}(\pi', k) = \dim_{\bar{\mathbb{Q}}_l} \mathbf{V}(\pi', k).$$

Proof. Part (a) is the statement of Proposition 4.1. Now assume that $F/\mathbb{Q}$ is Galois and π' is non-CM. Then from Propositions 6.3 and 8.1 we get that $r_{\mathrm{alg}}(\pi'; \nu) = r_l(\pi'; \nu) = r_{an}(\pi'; \nu) \leq 2$, and from Lemma 6.1 we deduce part (c). Now if π' is CM, part (b) is the statement of Proposition 6.4. For π' non-CM and k solvable we know from (a) that $L(s, \rho(\pi')|_{\Gamma_k})$ has a meromorphic continuation to the complex plane, and we have to match the order of the pole of $L(s, \rho(\pi')|_{\Gamma_k})$ at $s = 1$ with the dimension of the space of the Tate cycles defined over k. But because k is solvable and π' is non-CM we get that $\rho_{\pi'}|_{\Gamma_{Fk}}$ is cuspidal irreducible. Now because K contains a quadratic subextension F_0 we get that $\rho(\pi')|_{\Gamma_k}$ is a tensor product of Asai representations of degree 4, 2 or 1 associated with cuspidal representations of GL(2). When we have degree 4, i.e $\rho(\pi')|_{\Gamma_k} = As_{Fk/k}\rho_{\pi''}$, for some cuspidal non-CM automorphic representation π'' of $\mathrm{GL}(2)/Fk$ (associated with $\rho_{\pi'}|_{\Gamma_{Fk}}$) and Fk has a quadratic subextension k'/k,

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we obtain part (b) exactly as in **[12]**, section 8 (it is proved in the same way as Proposition 6.3). The rest of the cases are similar. □

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