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ON THE REGULAR DIGRAPH OF IDEALS OF COMMUTATIVE RINGS

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Abstract

Let R be a commutative ring. The regular digraph of ideals of R, denoted by $\Gamma(R)$, is a digraph whose vertex set is the set of all nontrivial ideals of R and, for every two distinct vertices I and J, there is an arc from I to J whenever I contains a nonzero divisor on J. In this paper, we study the connectedness of $\Gamma(R)$. We also completely characterise the diameter of this graph and determine the number of edges in $\Gamma(R)$, whenever R is a finite direct product of fields. Among other things, we prove that R has a finite number of ideals if and only if $\mathrm{N}_{\Gamma(R)}(I)$ is finite, for all vertices I in $\Gamma(R)$, where $\mathrm{N}_{\Gamma(R)}(I)$ is the set of all adjacent vertices to I in $\Gamma(R)$.

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1. Introduction

The investigation of graphs related to various algebraic structures is a very large and growing area of research. Several classes of graphs associated with algebraic structures have been actively investigated. For example, Cayley graphs have been studied in [8, 11, 12, 15, 18, 20], power graphs and divisibility graphs have been considered in [9, 10], zero-divisor graphs have been studied in [2–4, 6, 7], and cozero-divisor graphs have been introduced in [1]. Also, comaximal graphs have been studied in [13, 16, 19].

In [14], Nikmehr and Shaveisi defined the regular digraph of ideals of R, denoted by $\overrightarrow{\Gamma_{\mathrm{reg}}}(R)$, as a digraph whose vertex set is the set of all nontrivial ideals of R, and, for every two distinct vertices I and J, there is an arc from I to J, denoted by $I \longrightarrow J$, whenever I contains an element x such that $xy \neq 0$ for all $y \in J$. In other words, I contains a J-regular element. They studied and investigated some properties of this graph. For simplicity of notation, we denote this graph by $\Gamma(R)$.

In commutative algebra, regular elements play an important role in the structure of rings (see, for example, [17, Sections 16 and 17]). Thus in this paper we study some more properties of the graph $\Gamma(R)$. In Section 2 we study the connectedness and diameter of $\Gamma(R)$. Also, we give a very short proof of [14, Theorem 2.1]. Moreover, we generalise [14, Proposition 2.1] and provide necessary and sufficient conditions for connectedness of $\Gamma(R)$, whenever R is an arbitrary commutative ring. Finally, we completely investigate and determine the diameter of $\Gamma(R)$. In Section 3 we determine the isolated vertices in $\Gamma(R)$, and we compute the number of edges in $\Gamma(R)$, whenever R is a finite direct product of fields.

We now recall some definitions and notation on graphs. We use the standard terminology of graphs following [5]. Let $G = (V, E)$ be a simple graph, where V is the set of vertices and E is the set of edges. The graph $H = (V_0, E_0)$ is a subgraph of G if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, H is called a *subgraph induced* by V_0, denoted by $G[V_0]$, if $V_0 \subseteq V$ and $E_0 = \{\{u, v\} \in E \mid u, v \in V_0\}$. The *distance* between two distinct vertices a and b in G, denoted by $\mathrm{d}(a, b)$, is the length of the shortest path connecting a and b, if such a path exists; otherwise, we set $\mathrm{d}(a, b) := \infty$. The *diameter* of a graph G is denoted by $\mathrm{diam}(G)$ and is defined as the supremum of the set of all distances $\mathrm{d}(a, b)$ for all pairs (a, b), where a and b are distinct vertices of G. Also, for two distinct vertices a and b in G, the notation $a - b$ means that a and b are adjacent. A graph G is said to be *connected* if there exists a path between any two distinct vertices, and it is *complete* if it is connected with diameter one. We use K_n to denote the complete graph with n vertices. We say that G is *empty* if no two vertices of G are adjacent. For a vertex x in G, we denote the set of all vertices adjacent to x by $\mathrm{N}_G(x)$, and the size of $\mathrm{N}_G(x)$ is called the *degree* of x in G, denoted by $\deg(x)$. A vertex x is *isolated*, if $\mathrm{N}_G(x) = \emptyset$.

Let Γ be a digraph. An arc from a vertex x to another vertex y of Γ is denoted by $x \to y$. We say that Γ is *weakly connected* if the undirected underlying simple graph obtained by replacing all directed edges of Γ with undirected edges is a connected graph. Also, the *in-degree* (*out-degree*) of a vertex x in a digraph G is the number of arcs to (away from) x which is denoted by $\mathrm{d}^+(x)$ $(\mathrm{d}^-(x))$.

Throughout this paper, all rings are assumed to be commutative with identity. We denote by $\mathrm{Max}(R)$ and $\mathrm{Nil}(R)$ the set of all maximal ideals and the set of all nilpotent elements of R, respectively. Also, the set of all zero-divisors of an R-module M, which is denoted by $Z(M)$, is the set

$$Z(M) = \{r \in R \mid rx = 0, \text{ for some nonzero element } x \text{ in } M\}.$$

An element $r \in R$ is called *M-regular* if $r \notin Z(M)$. An *R-sequence* is a d-tuple $r_1, \ldots, r_d$ in R such that, for every $i \leq d$, r_i is $R/(r_1, r_2, \ldots, r_{i-1})$-regular. We say that $\mathrm{depth}(R) = 0$, whenever every nonunit element of R is a zero-divisor.

2. Connectedness of $\Gamma(R)$

In this section we study the weak connectedness of $\Gamma(R)$. We also completely characterise the diameter of $\Gamma(R)$.

For an arbitrary element $r \in R$, we set

$$\mathrm{Ann}(r) = \mathrm{Ann}(rR) = \{s \in R \mid sr = 0\}.$$

Also, for an ideal I of R we put $\mathrm{Ann}(I) = \bigcap_{s\in I} \mathrm{Ann}(s)$. Moreover, the set of all associated prime ideals of R is defined as follows: $\mathrm{Ass}(R) = \{\mathfrak{p} \mid \mathfrak{p}$ is a prime ideal of R and there exists $r \in R$ such that $\mathfrak{p} = \mathrm{Ann}(r)\}$.

Remark 2.1. If R is Noetherian and $\mathrm{depth}(R) = 0$, then R contains a finite number of maximal ideals and $\mathrm{Ann}(\mathfrak{m}) \neq 0$, for all maximal ideals $\mathfrak{m}$ in R.

The following lemma is needed in the rest of the paper.

Lemma 2.2. *Let R be a ring and $\mathfrak{m}$ be a maximal ideal in R. If $\mathrm{Ann}(\mathfrak{m}) \neq 0$, then $\mathfrak{m} = \mathrm{Z}(\mathrm{Ann}(\mathfrak{m}))$.*

Proof. Since $\mathfrak{m}\mathrm{Ann}(\mathfrak{m}) = 0$, we have $\mathfrak{m} \subseteq \mathrm{Z}(\mathrm{Ann}(\mathfrak{m}))$. Now, suppose that x is an arbitrary element in $\mathrm{Z}(\mathrm{Ann}(\mathfrak{m}))$. Then there is a nonzero element $y \in \mathrm{Ann}(\mathfrak{m})$ such that $xy = 0$. Assume to the contrary that $x \notin \mathfrak{m}$. So $y \in \mathfrak{m}$ and $Rx + \mathfrak{m} = R$. Therefore $rx + h = 1$, for some $r \in R$ and $h \in \mathfrak{m}$, and hence $rxy + hy = y$. Now, since $xy = 0 = yh$, we have that $y = 0$, which is a contradiction. □

The following corollary immediately follows from Lemma 2.2.

Corollary 2.3. *Suppose that R is Noetherian with $\mathrm{depth}(R) = 0$ and $\mathrm{Max}(R) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$. Then we have the following statements.*

(i) *If $n \geq 2$, then $\mathfrak{m}_i \longrightarrow \mathrm{Ann}(\mathfrak{m}_j)$ in $\Gamma(R)$, for all $i \neq j$.*
(ii) $\mathrm{Z}(\sum_{i=1}^{n} \mathrm{Ann}(\mathfrak{m}_i)) = \mathrm{Z}(R)$.

Lemma 2.4. *Assume that R is a Noetherian ring with $\mathrm{depth}(R) = 0$. Put $n := |\mathrm{Max}(R)|$. Then we have the following statements.*

(i) *If $\mathrm{Ann}(\mathfrak{m}_i) \subseteq \mathfrak{m}_i$, for all $1 \leq i \leq n$, then $\mathrm{Z}(\mathrm{Nil}(R)) = \mathrm{Z}(R)$.*
(ii) *If R is reduced, then R is a finite direct product of fields.*

Proof. (i) Since $\mathfrak{m}_i\mathrm{Ann}(\mathfrak{m}_i) = 0$ and $\mathrm{Ann}(\mathfrak{m}_i) \subseteq \mathfrak{m}_i$, for all $1 \leq i \leq n$, we have $\mathrm{Ann}(\mathfrak{m}_i) \subseteq \bigcap_{\mathfrak{p}\in\mathrm{Min}(R)} \mathfrak{p}$, for $i = 1, \ldots, n$. Hence $\sum_{i=1}^{n} \mathrm{Ann}(\mathfrak{m}_i) \subseteq \mathrm{Nil}(R)$. Now the result follows from Corollary 2.3(ii).

(ii) If $(R, \mathfrak{m})$ is a local ring, then clearly $\mathrm{Ann}(\mathfrak{m}) \subseteq \mathrm{Nil}(R)$. If $\mathfrak{m} \neq 0$, then $\mathrm{Ann}(\mathfrak{m}) \neq 0$, which implies that $\mathrm{Nil}(R) \neq 0$. This violates our assumption. Therefore $\mathfrak{m} = 0$, and so R is a field. Now suppose that R is not local. Then, by (i), there exists a maximal ideal $\mathfrak{m}_i$ such that $\mathrm{Ann}(\mathfrak{m}_i) \nsubseteq \mathfrak{m}_i$. Thus $R \cong R/\mathfrak{m}_i \times R/\mathrm{Ann}(\mathfrak{m}_i)$. Now $R/\mathrm{Ann}(\mathfrak{m}_i)$ is also reduced, $\mathrm{depth}(R/\mathrm{Ann}(\mathfrak{m}_i)) = 0$, and $R/\mathrm{Ann}(\mathfrak{m}_i)$ has $n - 1$ maximal ideals. Now, by using induction on n, the result holds. □

In the following theorem, we provide a very short proof of [14, Theorem 2.1].

Theorem 2.5. *Let R be a Noetherian ring. Then $\Gamma(R)$ is an empty graph if and only if R is either an Artinian local ring or a direct product of two fields.*

Proof. First suppose that $\Gamma(R)$ is an empty graph. If R contains a regular element, then $\Gamma(R)$ is a refinement of a star graph, which is impossible. So we have that $\mathrm{depth}(R) = 0$.

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If R is a local ring with a maximal ideal $\mathfrak{m}$ and $\mathfrak{p}$ is a minimal prime ideal of R, then $\mathfrak{p} \in \mathrm{Ass}(R)$, and so $\mathfrak{p} = \mathrm{Ann}(x)$, for some $x \in R$. Hence $R/\mathfrak{p} \cong Rx$ and $\mathrm{Z}(Rx) = \mathfrak{p}$. Now since $\Gamma(R)$ is an empty graph, we have that $\mathrm{Z}(I) = \mathfrak{m}$, for all nontrivial ideals I in R. In particular we have $\mathfrak{p} = \mathrm{Z}(Rx) = \mathfrak{m}$, and this implies that R is Artinian. Hence, in this situation, R is an Artinian local ring. Now, if R is not local, then there exist two distinct maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Since $\Gamma(R)$ is empty, by Corollary 2.3, we have $\mathrm{Ann}(\mathfrak{m}_1) = \mathfrak{m}_2$. Also we have $\mathfrak{m}_1 + \mathfrak{m}_2 = R$. Therefore we can easily see that R is a direct product of two fields.

Conversely, assume that R is either an Artinian local ring or a direct product of two fields. Firstly suppose that R is an Artinian local ring with the maximal ideal $\mathfrak{m}$. Since $\mathrm{Ass}(R) = \{\mathfrak{m}\}$, we have that $\mathrm{Z}(I) = \mathrm{Z}(J) = \mathfrak{m}$, for all ideals I, J of R. This means that $\Gamma(R)$ is empty. Also if R is a direct product of two fields, then $\Gamma(R) \cong 2\overline{K_2}$. □

Lemma 2.6. *Let R be a nonreduced ring. Then $\mathrm{Nil}(R) \subseteq \mathrm{Z}(I)$, for all nontrivial ideals I of R.*

Proof. Assume to the contrary that $\mathrm{Nil}(R) \nsubseteq \mathrm{Z}(I)$, for some nontrivial ideal I of R. Thus there is a nonzero element x in $\mathrm{Nil}(R)$ such that $xy \neq 0$, for all nonzero elements $y \in I$. Hence $x^2y = x(xy) \neq 0$, and so we can easily see that $x^ny \neq 0$, for all positive integers n. But since $x^k = 0$, for some $k \geq 2$, we have that $x^ky = 0$, which is a contradiction. □

Lemma 2.7. *Let R be a Noetherian ring. Then R is an Artinian local ring if and only if the graph $\Gamma(F \times R)$ is disconnected, where F is a field.*

Proof. First suppose that R is an Artinian local ring. If R is a field, then, by Theorem 2.5, $\Gamma(F \times R)$ is an empty graph. So we may assume that R is not a field. Then, for any nontrivial ideal I of R, the element $(1, 0) \in F \times I$ is $(F \times 0)$-regular, and so we have that $F \times I \longrightarrow F \times 0$ in $\Gamma(R)$. Hence the induced subgraph of $\Gamma(R)$ with vertex set $A = \{F \times 0, F \times I \mid I \text{ is a nontrivial ideal of } R\}$ is a star graph. Also $(0, 1) \in 0 \times R$ is a $(0 \times J)$-regular element, for all nontrivial ideals J of R, and hence $0 \times R \longrightarrow 0 \times J$ in $\Gamma(R)$. This implies that the induced subgraph of $\Gamma(R)$ with vertex set $B = \{0 \times R, 0 \times J \mid J \text{ is a nontrivial ideal of R}\}$ is a star graph. Now, by Theorem 2.5, it is easy to see that $\Gamma(F \times R)$ is disconnected with connected components A and B.

Conversely, suppose that $\Gamma(F \times R)$ is disconnected. Assume to the contrary that R is not an Artinian local ring. If R is a direct product of two fields, then $\Gamma(F \times R) \cong C_6$ (see [14, Proposition. 3.6]), which is impossible. So R is not a direct product of two fields. Now, by Theorem 2.5, we have an edge $I \longrightarrow J$ in $\Gamma(R)$. Let A and B be the sets as defined in the first paragraph in this proof. Then the edge $F \times I \longrightarrow 0 \times J$ connects the sets A and B. Thus $\Gamma(R)$ is connected, which is a contradiction. □

In [14, Proposition 2.1], the authors establish a result on the connectedness of $\Gamma(R)$ for Noetherian local rings. In the following theorem, we generalise [14, Proposition 2.1] and provide necessary and sufficient conditions for connectedness of $\Gamma(R)$, where R is an arbitrary commutative ring.

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Theorem 2.8. *Let R be a Noetherian ring. The graph $\Gamma(R)$ is connected if and only if one of the following statements holds.*

(i) $\mathrm{depth}(R) \neq 0$.

(ii) $\mathrm{depth}(R) = 0$ *and* $R = F \times R'$, *where* F *is a field and* R' *is not an Artinian local ring.*

Proof. Suppose that $\Gamma(R)$ is connected and $\mathrm{depth}(R) = 0$. If $\mathrm{Ann}(\mathfrak{m}) \subseteq \mathfrak{m}$, for all maximal ideals $\mathfrak{m}$, then, by Lemma 2.4, we have $\mathrm{Z}(\mathrm{Nil}(R)) = \mathrm{Z}(R)$. Also, in view of Lemma 2.6, $\mathrm{Nil}(R) \neq 0$ is an isolated vertex which implies that $\Gamma(R)$ is disconnected. So there exists a maximal ideal $\mathfrak{m}$ such that $\mathrm{Ann}(\mathfrak{m}) \nsubseteq \mathfrak{m}$. Hence $\mathrm{Ann}(\mathfrak{m}) + \mathfrak{m} = R$. Also, since $\mathfrak{m}\mathrm{Ann}(\mathfrak{m}) = 0$, we have that $R = R/\mathfrak{m} \times R/\mathrm{Ann}(\mathfrak{m})$. Moreover, by Lemma 2.7, R is not an Artinian local ring.

Conversely, assume that one of the conditions (i) or (ii) is satisfied. Condition (i) implies that $\Gamma(R)$ is a refinement of a star graph, and so it is connected. If (ii) is satisfied, then, by Lemma 2.7, the result holds. □

According to Theorem 2.8, $\Gamma(R)$ has isolated vertices if $\Gamma(R)$ is disconnected and R is indecomposable. We denote the number of nonsingular connected components of $\Gamma(R)$ by $\pi(R)$. In the next theorem we compute $\pi(R)$.

Theorem 2.9. *Let R be a Noetherian ring. Suppose that $\Gamma(R)$ is disconnected and* $\mathrm{depth}(R) = 0$. *Then the following statements hold.*

(i) *If* $|\mathrm{Max}(R)| \geq 3$, *or* $|\mathrm{Max}(R)| = 2$ *and R is not Artinian, then* $\pi(R) = 1$.

(ii) *If* $R = R_1 \times R_2$, *where* R_1, R_2 *are two Artinian local rings which are not both fields, then* $\pi(R) = 2$.

(iii) *If* $R = R_1 \times R_2$, *where* R_1, R_2 *are two fields, or R is a local Artinian ring, then* $\pi(R) = 0$.

Proof. (i) Suppose that $|\mathrm{Max}(R)| \geq 3$. Clearly, for each arc $I \longrightarrow J$ in $\Gamma(R)$, we have $\mathfrak{m} \longrightarrow J$, for all maximal ideals $\mathfrak{m}$ containing I. Hence, if C is a nonsingular connected component of $\Gamma(R)$, then it contains a maximal ideal. Thus it is enough to show that there is a path joining any two maximal ideals of R. To this end, suppose that $\mathfrak{m}_1, \mathfrak{m}_2$ are arbitrary maximal ideals and $\mathfrak{m}_3$ is a maximal ideal distinct from $\mathfrak{m}_1, \mathfrak{m}_2$ in R. By Corollary 2.3, we have the path $\mathfrak{m}_1 \longrightarrow \mathrm{Ann}(\mathfrak{m}_3) \longleftarrow \mathfrak{m}_2$. Thus all maximal ideals lie in the same component, which implies that $\pi(R) = 1$.

Now suppose that $\mathrm{Max}(R) = \{\mathfrak{m}, \mathfrak{n}\}$ and R is not Artinian. We claim that $\mathrm{Ann}(\mathfrak{m}) \subseteq \mathfrak{m}$ and $\mathrm{Ann}(\mathfrak{n}) \subseteq \mathfrak{n}$. Assume to the contrary that $\mathrm{Ann}(\mathfrak{m}) \nsubseteq \mathfrak{m}$. Then we have $R = R/\mathfrak{m} \times R/\mathrm{Ann}(\mathfrak{m})$. Since $|\mathrm{Max}(R)| = 2$, the ring $R/\mathrm{Ann}(\mathfrak{m})$ is local. Now, since $\Gamma(R)$ is disconnected, by Theorem 2.8, we have that $R/\mathrm{Ann}(\mathfrak{m})$ is Artinian, and so R is Artinian, which is the required contradiction. Now, by Lemma 2.4, it is easy to see that $\mathrm{Z}(\mathfrak{m} \cap \mathfrak{n}) = \mathrm{Z}(R)$. This means that $\mathrm{d}^+(\mathfrak{m} \cap \mathfrak{n}) = 0$. Hence if $\mathfrak{m} \cap \mathfrak{n}$ is an isolated vertex, then $\mathfrak{m} \cap \mathfrak{n} \subseteq Z(I)$, for all nontrivial ideals I of R. Hence we deduce that $\mathfrak{m} \cap \mathfrak{n} = \mathrm{Nil}(R)$, and so R is an Artinian ring, which is impossible. Hence $\mathfrak{m} \cap \mathfrak{n} \longrightarrow J$, for some nontrivial ideal J of R. Now consider the path $\mathfrak{m} \longrightarrow J \longleftarrow \mathfrak{n}$ to deduce that $\mathfrak{m}$ and $\mathfrak{n}$ lie in the same component, and hence $\pi(R) = 1$.

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(ii) Suppose that $R = R_1 \times R_2$, where R_1, R_2 are two Artinian local rings at least one of which is not a field. Without loss of generality, we may assume that R_1 is not a field. Consider two sets of edges

$$A = \{R_1 \times J \longrightarrow I \times 0\}_{J \neq R_2, I \neq 0}$$

and

$$B = \{0 \times J \longleftarrow I \times R_2\}_{J \neq 0, I \neq R_1}.$$

It is easy to see that $\Gamma(R)$ has only two nonsingular connected components A and B. Note that all other parts of $\Gamma(R)$ are isolated vertices. Thus $\pi(R) = 2$.

(iii) By Theorem 2.5, the result holds. □

For any two subsets A and B of vertices in $\Gamma(R)$, we use the notation $\mathrm{d}(A, B)$ to denote the maximum distance between vertices in A and vertices in B.

Theorem 2.10. *Let R be a Noetherian ring such that $\Gamma(R)$ is connected. Then the following statements hold.*

(i) $\mathrm{diam}(\Gamma(R)) = 1$ *if and only if R is an integral domain.*
(ii) $\mathrm{diam}(\Gamma(R)) = 2$ *if and only if R is not an integral domain and* $\mathrm{depth}(R) \neq 0$.
(iii) *If* $\mathrm{depth}(R) = 0$, *then we have the following statements.*
    (a) $\mathrm{diam}(\Gamma(R)) = 3$ *if and only if $R = F_1 \times F_2 \times R_2$, where F_1 and F_2 are fields and* $\mathrm{Z}(\mathrm{Nil}(R_2)) \neq \mathrm{Z}(R_2)$.
    (b) $\mathrm{diam}(\Gamma(R)) = 4$ *if and only if $R = F_1 \times F_2 \times R_2$, where F_1 and F_2 are fields and* $\mathrm{Z}(\mathrm{Nil}(R_2)) = \mathrm{Z}(R_2)$.
    (c) $\mathrm{diam}(\Gamma(R)) = 5$ *if and only if R is non of the rings as in (a) and (b).*

Proof. (i) The result follows by [14, Thoerem 3.1].

(ii) Firstly, suppose that R is not an integral domain and $\mathrm{depth}(R) \neq 0$. Thus clearly $\Gamma(R)$ is a refinement of a star graph, and so $\mathrm{diam}(\Gamma(R)) = 2$.

Conversely, suppose that $\mathrm{diam}(\Gamma(R)) = 2$. By part (i), R is not an integral domain. Now assume to the contrary that $\mathrm{depth}(R) = 0$. Since $\Gamma(R)$ is connected, by Theorem 2.8, we have $R = F_1 \times R_1$, where F_1 is a field and R_1 is a commutative ring which is not an Artinian local ring. Then it is easy to see that $\mathrm{d}(F_1 \times 0, 0 \times R_1) = 3$, which is a contradiction.

(iii) Since $\Gamma(R)$ is connected, by Theorem 2.8, we have that $R = F_1 \times R_1$, where F_1 is a field and F_1 is not an Artinian local ring. We have two cases to consider.

Case 1. If there exists a field F_2 such that $R_1 = F_2 \times R_2$ for some nonzero commutative ring R_2, then we claim that $\mathrm{diam}(\Gamma(R)) = 3$ or 4. Whenever R_2 is a field, then as we mentioned in the proof of Lemma 2.7, the graph $\Gamma(F_1 \times F_2 \times R_2)$ is isomorphic to C_6. So we may assume that R_2 is not a field. Now, we consider the following partition for vertices of $\Gamma(R)$:

$$C_1 := \{F_1 \times F_2 \times 0,\ F_1 \times 0 \times 0,\ 0 \times F_2 \times 0,\ F_1 \times F_2 \times I \mid I \text{ is a nontrivial ideal of } R_2\},$$
$$C_2 := \{0 \times 0 \times R_2,\ F_1 \times 0 \times R_2,\ 0 \times F_2 \times R_2,\ 0 \times 0 \times I \mid I \text{ is a nontrivial ideal of } R_2\},$$

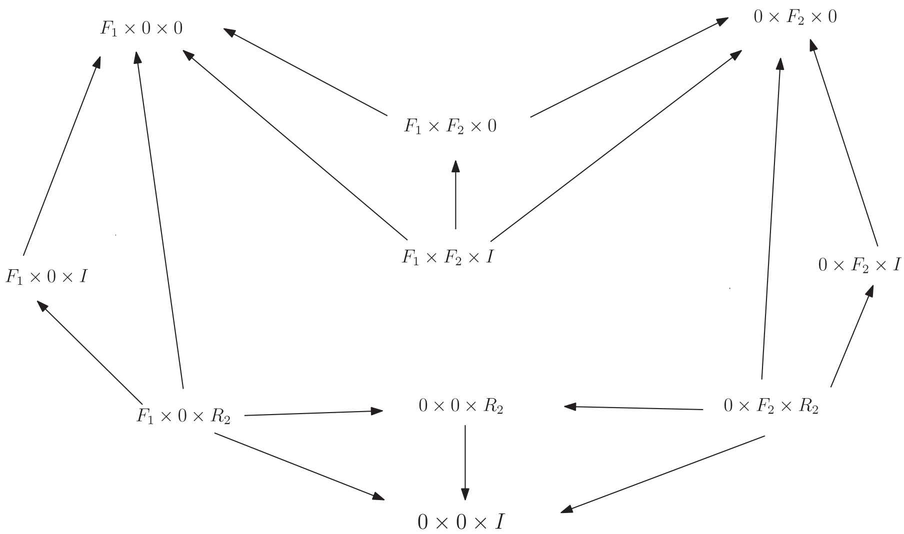

FIGURE 1. A part of $\Gamma(F_1 \times F_2 \times R_2)$.

$C_3 := \{F_1 \times 0 \times I \mid I$ is a nontrivial ideal of $R_2\}$,
$C_4 := \{0 \times F_2 \times I \mid I$ is a nontrivial ideal of $R_2\}$.

We need only determine $\mathrm{d}(C_i, C_j)$ for $1 \le i, j \le 4$. For an arbitrary nontrivial ideal I of R_2, consider Figure 1, which shows some parts of $\Gamma(R)$. Now, it is routine to check that

$$\mathrm{d}(C_i, C_j) = \begin{cases} 1 \text{ or } 2 & \text{for } i = j = 1, 2, 3, 4 \\ 3 \text{ or } 4 & \text{for } i = 3 \text{ and } j = 4 \\ 1 \text{ or } 2 \text{ or } 3 & \text{otherwise.} \end{cases}$$

This implies that whenever $\mathrm{d}(C_3, C_4) = 4$, then $\mathrm{diam}(\Gamma(R)) = 4$. Otherwise $\mathrm{diam}(\Gamma(R)) = 3$. So we need only consider the situations in which $\mathrm{diam}(\Gamma(R)) = 4$. We claim that $\mathrm{diam}(\Gamma(R)) = 4$ if and only if $\Gamma(R_2)$ has an isolated vertex and R_2 is not reduced. To prove the claim we consider the following situations.

(α) $\Gamma(R_2)$ has no isolated vertex. Let $F_1 \times 0 \times I$ and $0 \times F_2 \times J$ be arbitrary vertices in C_3 and C_4, respectively. Since I is not an isolated vertex in $\Gamma(R_2)$, there exists an ideal I' of R_2 such that $I \longrightarrow I'$ or $I' \longrightarrow I$ in $\Gamma(R_2)$. Hence we have the paths

$$F_1 \times 0 \times I \longrightarrow 0 \times 0 \times I' \longleftarrow 0 \times F_2 \times R_2 \longrightarrow 0 \times F_2 \times J$$

or

$$F_1 \times 0 \times I \longleftarrow F_1 \times F_2 \times I' \longrightarrow 0 \times F_2 \times 0 \longleftarrow 0 \times F_2 \times J.$$

These imply that $\mathrm{d}(C_3, C_4) = 3$.

(β) R_2 is a reduced ring. Since $\mathrm{depth}(R) = 0$, we have that $\mathrm{depth}(R_2) = 0$. Now, in view of Lemma 2.4(ii), R_2 is a finite direct product of fields $F'_1, \ldots, F'_n$. Since R_2 is not a field, $n \ge 2$. If $n = 2$, then $\mathrm{V}(\Gamma(R_2)) = \{F'_1 \times 0, 0 \times F'_2\}$. Now if I and J are vertices

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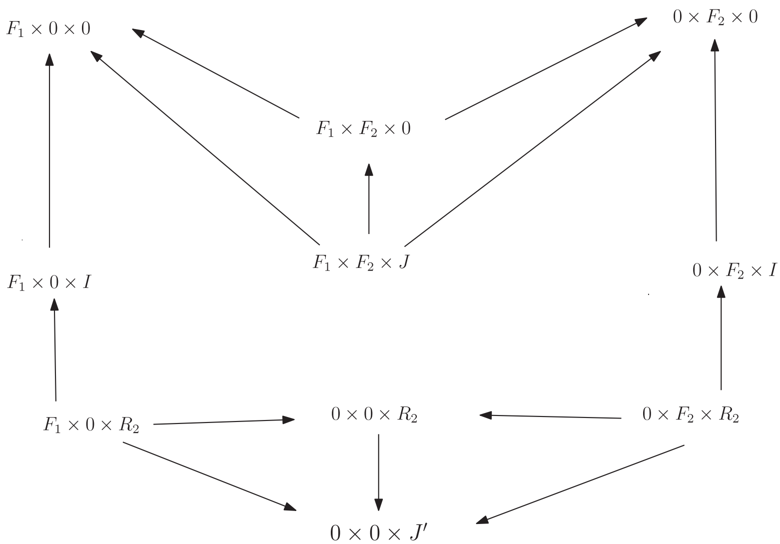

FIGURE 2. A part of $\Gamma(F_1 \times F_2 \times R_1)$ where $J' \neq 0$ and $J \neq R_2$.

in $\Gamma(R_2)$, then we have the following path in $\Gamma(R)$:

$$F_1 \times 0 \times I \longleftarrow F_1 \times F_2 \times I \longrightarrow 0 \times F_2 \times 0 \longleftarrow 0 \times F_2 \times J.$$

This means that $d(C_3, C_4) = 3$. Also, whenever $n \geq 3$, by Theorem 2.8(ii), $\Gamma(R_2)$ is connected, and so it has no isolated vertices. Thus, by (α), we have that $d(C_3, C_4) = 3$.
(γ) R_2 is not reduced and there exists an isolated vertex I in $\Gamma(R_2)$. Now consider the vertices $F_1 \times 0 \times I$ and $0 \times F_2 \times I$ in $\Gamma(R)$. It follows from Figure 2 that $d(F_1 \times 0 \times I, 0 \times F_2 \times I) = 4$. This implies that $d(C_3, C_4) = 4$.

Case 2. R_1 is indecomposable or R_1 has no decomposition $R_1 = F_2 \times R_2$, for some field F_2. Now let $\mathfrak{m}$ be an arbitrary maximal ideal of R_1. If $\mathrm{Ann}(\mathfrak{m}) \nsubseteq \mathfrak{m}$, then $\mathrm{Ann}(\mathfrak{m})$ and $\mathfrak{m}$ are comaximal, and so $R_1 = R_1/\mathfrak{m} \times R_1/\mathrm{Ann}(\mathfrak{m})$ which is impossible. Hence $\mathrm{Ann}(\mathfrak{m}) \subseteq \mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of R_1. Now, by Lemmas 2.4 and 2.6, $\mathrm{Nil}(R_1)$ is an isolated vertex in $\Gamma(R_1)$. Thus

$$d^-(F_1 \times \mathrm{Nil}(R_1)) = d^+(0 \times \mathrm{Nil}(R_1)) = 1$$

and

$$d^+(F_1 \times \mathrm{Nil}(R_1)) = d^-(0 \times \mathrm{Nil}(R_1)) = 0.$$

On the other hand, since the graph $\Gamma(F_1 \times R_1)$ is connected, there exists an arc $F_1 \times I \longrightarrow 0 \times J$ for some ideals I and J of R_1. Hence there exists a path (with minimum length)

$$F_1 \times \mathrm{Nil}(R_1) \longrightarrow F_1 \times 0 \longleftarrow F_1 \times I \longrightarrow 0 \times J \longleftarrow 0 \times R_1 \longrightarrow 0 \times \mathrm{Nil}(R_1)$$

in $\Gamma(R)$. Therefore in this situation we have $\mathrm{diam}(\Gamma(R)) = 5$. □

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3. Degrees of the vertices and counting the edges

We begin this section with the following proposition which determines the isolated vertices in $\Gamma(R)$.

PROPOSITION 3.1. *Let R be a nonreduced Noetherian ring with* $\operatorname{depth}(R) = 0$. *Then I is an isolated vertex in $\Gamma(R)$ if and only if I is a nilpotent ideal and $Z(I) = Z(R)$.*

PROOF. Let I be an isolated vertex in $\Gamma(R)$. We claim that $I \subseteq Z(I)$. Assume to the contrary that $I \nsubseteq Z(I)$. We consider the following cases.

Case 1. There exists an ideal J of R such that $I \subset J$. Thus $J \nsubseteq Z(I)$. Hence there exists an arc $J \longrightarrow I$ in $\Gamma(R)$. Since I is an isolated vertex, this is impossible.

Case 2. There exists an ideal K of R such that $K \subset I$. By using a method similar to that used in Case 1, $K \longrightarrow I$, which again is impossible.

Now it follows from the above cases that I is both a minimal and maximal ideal of R. Hence $\operatorname{Ann}(I)$ is a maximal ideal of R, and so $R \cong R/I \times R/\operatorname{Ann}(I)$. This implies that R is reduced, which is the required contradiction. Thus $I \subseteq Z(I)$. Now it follows from our claim that $I \subseteq Z(J)$ for all ideal J of R. Hence $I \subseteq \bigcap_{J \trianglelefteq R} Z(J)$. Moreover, for any minimal prime ideal $\mathfrak{p}$ of R, there exists an element x in R such that $\mathfrak{p} = \operatorname{Ann}(x)$. Thus $\mathfrak{p} = Z(Rx)$. Hence $I \subseteq \mathfrak{p}$, and so $I \subseteq \operatorname{Nil}(R)$. Also it follows from our claim that $\mathfrak{m} \subseteq Z(I)$ for all maximal ideals $\mathfrak{m}$ of R. Hence $\bigcup_{\mathfrak{m} \in \operatorname{Max}(R)} \mathfrak{m} \subseteq Z(I)$, which implies that $Z(R) = Z(I)$.

The converse implication follows from Lemma 2.6 □

COROLLARY 3.2. *If R is a nonreduced ring such that $\Gamma(R)$ contains an isolated vertex, then* $\operatorname{Nil}(R)$ *is an isolated vertex in $\Gamma(R)$.*

PROOF. Suppose that I is an isolated vertex in $\Gamma(R)$. Then, in view of Proposition 3.1, $I \subseteq \operatorname{Nil}(R)$ and $Z(I) = Z(R)$. This implies that $Z(\operatorname{Nil}(R)) = Z(R)$. Again, by Proposition 3.1, $\operatorname{Nil}(R)$ is an isolated vertex in $\Gamma(R)$. □

THEOREM 3.3. *Suppose that R is a Noetherian ring such that* $\operatorname{depth}(R) = 0$ *and that the graph $\Gamma(R)$ is not empty. Then the following conditions are equivalent.*

(i) *R has a finite number of ideals.*
(ii) *$N_{\Gamma(R)}(I)$ is finite, for all vertices I in $\Gamma(R)$.*
(iii) *$V(\Gamma(R)) - N_{\Gamma(R)}(I)$ is finite, for all vertices I in $\Gamma(R)$.*

PROOF. The implications (i) $\Longrightarrow$ (ii) and (i) $\Longrightarrow$ (iii) are trivial.

(ii) $\Longrightarrow$ (i) First note that if R is reduced, then, by Lemma 2.4(ii), R is a finite direct product of fields, and so it has a finite number of ideals. So we may assume that R is not reduced.

Now we claim that R has a decomposition as $R \cong R_1 \times R_2$ for some nonzero rings R_1 and R_2. We consider two cases.

Case 1. There exists a maximal ideal $\mathfrak{m}$ of R such that $\operatorname{Ann}(\mathfrak{m}) \nsubseteq \mathfrak{m}$. This implies that R has the decomposition $R \cong R/\operatorname{Ann}(\mathfrak{m}) \times R/\mathfrak{m}$.

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Case 2. For all maximal ideals $\mathfrak{m}$ of R, $\mathrm{Ann}(\mathfrak{m}) \subseteq \mathfrak{m}$. Hence, by Lemma 2.4, $\mathrm{Z}(\mathrm{Nil}(R)) = \mathrm{Z}(R)$. Also, since $\mathrm{Nil}(R) \subseteq \mathrm{J}(R)$, we have that $\mathrm{Z}(\mathrm{J}(R)) = \mathrm{Z}(R)$, and so, for any ideal I of R, we have no arc $I \longrightarrow \mathrm{J}(R)$ in $\Gamma(R)$. Now we show that $\mathrm{J}(R)$ is an isolated vertex in $\Gamma(R)$. To achieve this, assume to the contrary that $\mathrm{J}(R)$ is not an isolated vertex. Hence there exists an arc $\mathrm{J}(R) \to I$ in $\Gamma(R)$, for some ideal I of R. Clearly $I\mathrm{J}(R)^i \neq 0$ for all $i \geq 0$. Moreover, by Nakayama's lemma, $I\mathrm{J}(R)^i \neq I\mathrm{J}(R)^j$ for all $i \neq j$. But $\mathrm{J}(R) \longrightarrow I\mathrm{J}(R)^i$ for all $i > 0$, which contradicts (ii). Thus $\mathrm{J}(R)$ is an isolated vertex, and so, by Proposition 3.1, $\mathrm{J}(R)$ is nilpotent. This implies that the ring R is Artinian, and so $R \cong R_1 \times R_2$ for some nonzero rings R_1 and R_2. Now, clearly, for any ideal I of R_1, there exists an arc $R_1 \times 0 \longrightarrow I \times 0$ in $\Gamma(R)$. So, by using assumption (ii), R_1 has a finite number of ideals. Similarly R_2 has a finite number of ideals. This implies that the ring R has finite number of ideals.

(iii) $\Longrightarrow$ (i) Again we may assume that R is not reduced. We have two cases.

Case 1′. There exists a maximal ideal $\mathfrak{m}$ of R such that $\mathrm{Ann}(\mathfrak{m}) \nsubseteq \mathfrak{m}$. Then $R \cong R_1 \times R_2$ for some nonzero rings R_1 and R_2. Now, for any ideal K of R_2, there is no adjacency between two vertices $R_1 \times 0$ and $0 \times K$. Hence (iii) implies that R_2 has a finite number of ideals. Similarly, R_1 also has a finite number of ideals, and so (i) is proved.

Case 2′. For all maximal ideals $\mathfrak{m}$ of R, $\mathrm{Ann}(\mathfrak{m}) \subseteq \mathfrak{m}$. Again, by Lemma 2.4, $\mathrm{Z}(\mathrm{Nil}(R)) = \mathrm{Z}(R)$. Hence, in view of Proposition 3.1, $\mathrm{Nil}(R)$ is an isolated vertex in $\Gamma(R)$. Thus, by (iii), R has only a finite number of ideals. □

COROLLARY 3.4. *The graph $\Gamma(R)$ is finite if and only if R has a finite number of ideals, and so R is an Artinian ring.*

In the rest of the paper, we determine the number of edges in $\Gamma(R)$, denoted by $|\mathrm{E}(R)|$, in the case where R is a finite direct product of fields. To this end, we first prove the following lemmas.

We denote by $\mathbb{I}(R)$ the number of nontrivial ideals of R, and by $r(R)$ the number of nontrivial ideals I such that $I \nsubseteq \mathrm{Z}(I)$. Let $R = R_1 \times \cdots \times R_n$ and I be an ideal of R. We use $\pi_i(I)$ to denote the image of ideal I of R by the natural ring epimorphism $\pi_i : R \longrightarrow R_i$. Also, we use $\mathrm{Supp}(I)$ to denote the set of indices i such that $\pi_i(I) = R_i$.

In the following lemma, we compute $r(R)$, where R is an Artinian ring.

LEMMA 3.5. *Suppose that $R = R_1 \times \cdots \times R_n$, where each R_i is an Artinian local ring, for $i = 1, \ldots, n$. Then $r(R) = 2^n - 2$.*

PROOF. Set

$$\Sigma := \{I \mid I \nsubseteq \mathrm{Z}(I), \text{ for } I \notin 0, R\} \quad \text{and}$$
$$\Sigma' := \{I \mid \pi_i(I) = 0 \text{ or } R, \text{ for } i = 1, \ldots, n\}.$$

Clearly $|\Sigma| = r(R)$ and $|\Sigma'| = 2^n - 2$. So it is enough to show that $\Sigma = \Sigma'$. To do so,

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suppose that $I \in \Sigma'$. Consider the element $\mathbf{x} := (x_i) \in R$, where

$$x_i = \begin{cases} 1 & \text{if } i \in \mathrm{Supp}(I) \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\mathbf{x}$ is I-regular. Hence $I \in \Sigma$ and $\Sigma' \subseteq \Sigma$.

Now, if $\Sigma \nsubseteq \Sigma'$, then there exists $I \in \Sigma$ such that $I \notin \Sigma'$. Without loss of generality, we may assume that $\pi_1(I)$ is a nontrivial ideal of R_1. Since $I \in \Sigma$, there exists an I-regular element $\mathbf{y} = (y_i)$ with $y_1 \notin \mathrm{Z}(I_1)$. Since R_1 is an Artinian local ring, any nonunit element in R_1 is a zero-divisor. Also, by Proposition 3.1, $\mathrm{Z}(I_1) = \mathrm{Z}(R_1)$, because I_1 is an isolated vertex in $\Gamma(R_1)$. Hence $y_1 \notin \Gamma\mathrm{Z}(R_1)$ which is impossible, and so for a maximal ideal $\mathfrak{m}$, we have $\mathrm{Ass}(R) = \{\mathfrak{m}\}$. Thus $\Sigma \subseteq \Sigma'$ as desired. □

Lemma 3.6. *Assume that F is a field and R is an arbitrary ring. Then*

$$|\mathrm{E}(F \times R)| = 3|\mathrm{E}(R)| + 2\mathbb{I}(R) + r(R).$$

Proof. For a subset C in $\Gamma(F \times R)$, we denote the induced subgraph of $\Gamma(F \times R)$ with vertex set C by $\Gamma(F \times R)[C]$. Now consider the collections $A := \{F \times I_i\}_{i=1}^{\mathbb{I}(R)}$ and $B := \{0 \times I_i\}_{i=1}^{\mathbb{I}(R)}$ of vertices in $\Gamma(F \times R)$. It is not hard to see that

$$|\mathrm{E}(\Gamma(F \times R)[B])| = |\mathrm{E}(\Gamma(F \times R)[A])| = |\mathrm{E}(R)|.$$

Also, clearly, for some ideals I and J of R, $F \times I \longrightarrow 0 \times J$ is an arc in $\Gamma(F \times R)[A \cup B]$ if and only if we have one of the following conditions:

(i) $I = J$ and $I \nsubseteq \mathrm{Z}(J)$;
(ii) $I \neq J$ and $I \longrightarrow J$ in $\Gamma(R)$.

This implies that

$$|\mathrm{E}(\Gamma(F \times R)[A \cup B])| = |\mathrm{E}(\Gamma(F \times R)[B])| + |\mathrm{E}(\Gamma(F \times R)[A])| + |\mathrm{E}(R)| + r(R).$$

Thus

$$|\mathrm{E}(\Gamma(F \times R)[A \cup B])| = 3|\mathrm{E}(R)| + r(R).$$

Now, since all vertices in B are adjacent to $0 \times R$, the number of arcs from the vertex $0 \times R$ to the vertices in B is equal to $\mathbb{I}(R)$. Similarly, the number of arcs from the vertices in A to the vertex $F \times 0$ is equal to $\mathbb{I}(R)$. Also, there exists no edge between the vertices in A and the vertices in $0 \times R$. Therefore, we can easily check that $|\mathrm{E}(F \times R)| = 3|\mathrm{E}(R)| + 2\mathbb{I}(R) + r(R)$. □

Theorem 3.7. *Let $R = F_1 \times \cdots \times F_n$, where F_i is a field, for $i = 1, \ldots, n$, where $n \geq 3$. Then $|\mathrm{E}(R)| = 3^n - 3(2^n - 1)$.*

Proof. Since $\mathbb{I}(R) = 2^n - 2$, by Lemma 3.5, we have $\mathbb{I}(R) = r(R)$. We use induction on n. Clearly, for $n = 3$, $\Gamma(R)$ is isomorphic to C_6, and so the result holds. Now, suppose that $n = k + 1$ and the result holds for smaller values of n. Assume that

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$R = F_1 \times \cdots \times F_k \times F_{k+1}$. By the induction hypothesis, $|\mathrm{E}(F_1 \times \cdots \times F_k)| = 3^k - 3(2^k - 1)$. Hence, by Lemma 3.6,

$$\begin{aligned}|\mathrm{E}(F_1 \times \cdots \times F_k \times F_{k+1})| &= 3|\mathrm{E}(F_1 \times \cdots \times F_k)| + 2(2^k - 2) + (2^k - 2) \\ &= 3(3^k - 3(2^k - 1)) + 3(2^k - 2) \\ &= 3^{k+1} - 3(2^{k+1} - 1).\end{aligned}$$

This concludes the proof. □

REMARK 3.8. We now provide another proof of Theorem 3.7. For this purpose, suppose that $I(k)$ denotes the product

$$F_1 \times \cdots F_k \times \underbrace{0 \times \cdots \times 0}_{n-k \text{ times}},$$

for $k = 1, \ldots, n-1$. First note that, for ideals J_i of F_i, $I(k) \longrightarrow J_1 \times \cdots \times J_n$ is an arc in $\Gamma(R)$, if and only if $J_i = 0$ for all $i = k+1, \ldots, n$. Hence $\mathrm{d}^+(I(k)) = 2^k - 2$. Similarly, $\mathrm{d}^-(I(k)) = 2^{n-k} - 2$, and so $\deg(I(k)) = \mathrm{d}^+ + \mathrm{d}^- = 2^{n-k} + 2^k - 4$.

Now, if J is an ideal of R with $|\mathrm{Supp}(J)| = k$, then, by using a suitable permutation on $\{1, 2, \ldots, n\}$, it is easy to see that the degrees of vertex J and $I(k)$ are equal in $\Gamma(R)$. On the other hand, there are exactly $\binom{n}{k}$ ideals J with $|\mathrm{Supp}(J)| = k$. Thus we have the equality

$$2|\mathrm{E}(R)| = \sum_{k=1}^{n-1} \binom{n}{k} (2^{n-k} + 2^k - 4).$$

Now, by using the expansion of the function $f(x) = (1+x)^n - x^n - 1$, we can easily see that $2|\mathrm{E}(R)| = 2f(2) - 4f(1)$, and so $|\mathrm{E}(R)| = 3^n - 3(2^n - 1)$.

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References

[1] M. Afkhami and K. Khashyarmanesh, 'The cozero-divisor graph of a commutative ring', *Southeast Asian Bull. Math.* **35** (2011), 753–762.

[2] D. F. Anderson, M. C. Axtell and J. A. Stickles, 'Zero-divisor graphs in commutative rings', in: *Commutative Algebra, Noetherian and Non-Noetherian Perspectives*, (eds. M. Fontana, S. E. Kabbaj, B. Olberding and I. Swanson) (Springer, New York, 2011), pp. 23–45.

[3] D. F. Anderson and P. S. Livingston, 'The zero-divisor graph of a commutative ring', *J. Algebra* **217** (1999), 434–447.

[4] I. Beck, 'Coloring of commutative rings', *J. Algebra* **116** (1998), 208–226.

[5] J. A Bondy and U. S. R. Murty, *Graph Theory with Applications* (American Elsevier, New York, 1976).

[6] F. R. DeMeyer and L. DeMeyer, 'Zero-divisor graphs of semigroups', *J. Algebra* **283** (2005), 190–198.

[7] E. Estaji and K. Khashyarmanesh, 'The zero-divisor graph of a lattice', *Results Math.* **61** (2012), 1–11.
[8] A. V. Kelarev, *Graph Algebras and Automata* (Marcel Dekker, New York, 2003).
[9] A. V. Kelarev and S. J. Quinn, 'A combinatorial property and power graphs of groups', *Contrib. General Algebra* **12** (2000), 229–235.
[10] A. V. Kelarev and S. J. Quinn, 'Directed graphs and combinatorial properties of semigroups', *J. Algebra* **251** (2002), 16–26.
[11] A. V. Kelarev, J. Ryan and J. Yearwood, 'Cayley graphs as classifiers for data mining: the influence of asymmetries', *Discrete Math.* **309** (2009), 5360–5369.
[12] C. H. Li and C. E. Praeger, 'On the isomorphism problem for finite Cayley graphs of bounded valency', *European J. Combin.* **20** (1999), 279–292.
[13] H. R. Maimani, M. Salimi, A. Sattari and S. Yassemi, 'Comaximal graph of commutative rings', *J. Algebra* **319** (2008), 1801–1808.
[14] M. J. Nikmehr and F. Shaveisi, 'The regular digraph of ideals of a commutative ring', *Acta Math. Hunger.* **134** (2012), 516–528.
[15] C. E. Praeger, 'Finite transitive permutation groups and finite vertex-transitive graphs', in: *Graph Symmetry: Algebraic Methods and Applications* (Kluwer, Dordrecht, 1997), pp. 277–318.
[16] P. K. Sharma and S. M. Bhatwadekar, 'A note on graphical representation of rings', *J. Algebra* **176** (1995), 124–127.
[17] R. Y. Sharp, *Steps in Commutative Algebra*, 2nd edn, London Mathematical Society Student Texts, 51 (Cambridge University Press, Cambridge, 2000).
[18] A. Thomson and S. Zhou, 'Gossiping and routing in undirected triple-loop networks', *Networks* **55** (2010), 341–349.
[19] H. J. Wang, 'Graphs associated to co-maximal ideals of commutative rings', *J. Algebra* **320** (2008), 2917–2933.
[20] S. Zhou, 'A class of arc-transitive Cayley graphs as models for interconnection networks', *SIAM J. Discrete Math.* **23** (2009), 694–714.

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