Canad. Math. Bull. Vol. **30** (3), 1987

SUMS OF SQUARES FORMULAE WITH INTEGER COEFFICIENTS

BY

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ABSTRACT. Hidden behind a sums of squares formula

$$(x_1^2 + \ldots + x_r^2)(y_1^2 + \ldots + y_s^2) = f_1^2 + \ldots + f_n^2$$

are other such formulae not obtainable by restriction. This drastically simplifies the combinatorics involved in the existence problem of sums of squares formulae, and leads to a proof that the product of two sums of 16 squares cannot be rewritten as a sum of 28 squares, if only integer coefficients are permitted. We also construct all $[10, 10, 16]_{\mathbb{Z}}$ formulae.

Introduction. For given integers r and s, denote by $r*_{\mathbb{Z}}s$ the least integer n for which there exists an $[r, s, n]_{\mathbb{Z}}$ formula of the type

$$(x_1^2 + \ldots + x_r^2)(y_1^2 + \ldots + y_s^2) = f_1^2 + \ldots + f_n^2$$

where $f_1, \ldots, f_n$ are bilinear forms with integer coefficients in $x_1, \ldots, x_r$ and $y_1, \ldots, y_s$. Historically, the problem of determining $r*_{\mathbb{Z}}s$ arose from the nonexistence of a $[16, 16, 16]_{\mathbb{Z}}$ formula ([2], [3]); even the precise value of $16*_{\mathbb{Z}}16$ remains undecided to date. Adem [1] has constructed a $[16, 16, 32]_{\mathbb{Z}}$ formula, showing that $16*_{\mathbb{Z}}16 \leq 32$. On the other hand, we prove in [13] that $16*_{\mathbb{Z}}16 \geq 25$. The construction of an $[r, s, n]_{\mathbb{Z}}$ formula can be regarded as a combinatoric problem of "appropriately signing an intercalate matrix of type (r, s, n)" (Section 1 below). In this paper, we simplify the formidable combinatorics by the recent geometric and topological results of [6], [7] and [13] to show that $16*_{\mathbb{Z}}16 \geq 29$. We shall also construct all $[10, 10, 16]_{\mathbb{Z}}$ formulae. For background notions and results, we refer to [13].

This paper is a revision of part of the doctoral thesis of the author, written under the supervision of Professor Kee Yuen Lam. The author wishes to express his sincere thanks to Professor Lam for guidance and encouragement.

1. **Intercalate matrices**. Consider an $[r, s, n]_{\mathbb{Z}}$ formula with

$$f_k = \sum_{i,j} a_{i,j}^k x_i y_j, \qquad 1 \leq k \leq n,$$

Received by the editors December 27, 1985, and, in revised form, May 14, 1986.

Key words and phrases: Sums of squares formulae, intercalate matrices, stable homotopy classes of spheres, nonsingular bilinear maps, Hopf-Stiefel condition.

AMS Subject Classification (1980): Primary 11E25; Secondary 55Q45, 05B99.

https://doi.org/10.4153/CMB-1987-045-6 Published online by Cambridge University Press

where each coefficient $a_{i,j}^k$ is an integer. Comparison of coefficients shows that for given $i = 1, \ldots, r$ and $j = 1, \ldots, s$, there is exactly one index $k = k(i,j)$ for which $a_{i,j}^k$ is nonzero, this nonzero coefficient being necessarily ± 1. We can, therefore, represent the $[r,s,n]_{\mathbb{Z}}$ formula as a Hadamard product $A \circ K$ where

(i) A is the $(1,-1)$-matrix $(a_{i,j})$ with $a_{i,j} = a_{i,j}^k$;

(ii) $K = (c_{i,j})$ is an $r \times s$ matrix whose entries are "colors" $c_1, \ldots, c_n$; precisely, $c_{i,j} = c_{k(i,j)}$.

Further comparison of coefficients shows that

(1.1) the colors along each row (respectively column) of K are distinct;

(1.2) if $c_{i,j} = c_{i',j'}$, $i \neq i'$, $j \neq j'$, then $c_{i,j'} = c_{i',j}$;

(1.3) $a_{i,j}a_{i,j'}a_{i',j'}a_{i',j} = -1$ under the same hypothesis of (1.2).

For convenience, we shall call K an intercalate matrix of type (r,s,n) if it satisfies (1.1) and (1.2) above. Such an intercalate matrix can be signed to give an $[r,s,n]_{\mathbb{Z}}$ formula if there is a $(1,-1)$-matrix A satisfying (1.3). Compare [14]. Without loss of generality, such a signing matrix A can be taken to be standard: each entry in the first row and the first column being 1.

There is an obvious notion of equivalence of intercalate matrices of type (r,s,n): two such matrices are equivalent if one can be brought to the other by permutations of rows and columns, and relabelling of colors.

There is also an obvious notion of tensor product of intercalate matrices. Let $K' = (c'_{i,j})$ and $K'' = (c''_{i,j})$ be intercalate matrices of types (r_1, s_1, n_1) and (r_2, s_2, n_2) respectively. There is an intercalate matrix $K' \otimes K'' = (c_{i,j})$ of type (r_1r_2, s_1s_2, n_1n_2), where $c_{i,j} = (c'_{i',j'}, c''_{i'',j''})$ if $i = i'r_2 + i''$, $0 \leq i'' < r_2$, and $j = j's_2 + j''$, $0 \leq j'' < s_2$.

Up to equivalence, $D_1 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ is the only intercalate matrix of type $(2,2,2)$. The k-fold tensor product $D_k = D_1 \otimes \ldots \otimes D_1$ (k copies) is an intercalate matrix of type $(2^k, 2^k, 2^k)$. For convenience, we relabel the color $(a_1, \ldots, a_k)$ of D_k by the integer $1 + \Sigma_{i=1}^k a_i 2^{i-1}$.

In a square intercalate matrix of type (r,r,n), call a color ubiquitous if it appears in every row and every column. It is clear from (1.2) that, up to equivalence, a square intercalate matrix with at least one ubiquitous color is symmetric.

LEMMA 1.1. *Suppose an intercalate matrix K of type (r,r,n) has two ubiquitous colors. Then r and n are even, and K is equivalent to a tensor product $K_1 \otimes K_2$, where K_1 and K_2 are intercalate matrices of types (r',r',n') and $(2,2,2)$ respectively, $r = 2r'$, $n = 2n'$.*

PROOF. Assume one ubiquitous color is along the principal diagonal so that K is symmetric. Permute the rows and columns if necessary to arrange a second ubiquitous color along the principal 2×2 blocks. From this, it is clear that r is even, say $r = 2r'$. Furthermore, we have a partition of K into 2×2 blocks, each of which is an intercalate matrix of type $(2,2,2)$. "Contraction" of these 2×2 matrices leads to an intercalate matrix K_1 of type (r',r',n'), $n = 2n'$. ∎

PROPOSITION 1.2. *An intercalate matrix of type* (n, n, n) *exists if and only if* $n = 2^k$ *for some integer* k. *Furthermore, every intercalate matrix of type* $(2^k, 2^k, 2^k)$ *is equivalent to* D_k. ■

For $r, s \leq 2^k$, denote by $D_{r,s}$ the submatrix of D_k consisting of the first r rows and first s columns.

PROPOSITION 1.3. *The intercalate matrix* $D_{10,10}$ *can be signed to give a* $[10, 10, 16]_{\mathbb{Z}}$ *formula, but this is not possible with* $D_{10,11}$.

PROOF. Every standard signing matrix of $D_{10,10}$ is of the form

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & a_1 & -a_1 & a_2 & -a_2 & -a_3 & a_3 & a_8 & -a_8 \\ 1 & -a_1 & -1 & a_1 & a_4 & a_5 & -a_4 & -a_5 & 1 & 1 \\ 1 & a_1 & -a_1 & -1 & a_6 & -a_7 & a_7 & -a_6 & a_9 & -a_9 \\ 1 & -a_2 & -a_4 & -a_6 & -1 & a_2 & a_4 & a_6 & 1 & 1 \\ 1 & a_2 & -a_5 & a_7 & -a_2 & -1 & -a_7 & a_5 & a_{10} & -a_{10} \\ 1 & a_3 & a_4 & -a_7 & -a_4 & a_7 & -1 & -a_3 & 1 & 1 \\ 1 & -a_3 & a_5 & a_6 & -a_6 & -a_5 & a_3 & -1 & -a_{11} & a_{11} \\ 1 & -a_8 & -1 & -a_9 & -1 & -a_{10} & -1 & a_{11} & -1 & a_8 \\ 1 & a_8 & -1 & a_9 & -1 & a_{10} & -1 & -a_{11} & -a_8 & -1 \end{bmatrix}$$

where $a_1, a_2, a_3, a_4, a_8 = \pm 1$, and $a_5 = a_2a_3a_4$, $a_6 = a_1a_3a_4$, $a_7 = a_1a_2a_4$, $a_9 = a_1a_8$, $a_{10} = a_2a_8$, $a_{11} = a_3a_8$.

However, extension to a standard signing matrix $A' = (a_{i,j})$ for $D_{10,11}$ is impossible: putting $a_{1,11} = 1$, we find $a_{9,11} = 1$ and $a_{10,11} = -a_9$. Condition (1.3) gives $a_{5,11} = a_4$ with $i = 5$, $i' = 9$, $j = 7$, $j' = 11$. On the other hand, with $i = 5$, $i' = 10$, $j = 8$, $j' = 11$, we have $a_{5,11} = -a_6a_9a_{11} = -a_4$, a contradiction. ■

2. **Hidden sums of squares formulae**. Suppose an intercalate matrix K of type (r, s, n) is signed by a $(1, -1)$-matrix A to give an $[r, s, n]_{\mathbb{Z}}$ formula. This is equivalent to a normed bilinear map $f: R^r \times R^s \to R^n$ with Hopf construction $F: S^{r+s-1} \to S^n$. Let c be a color of K corresponding to an equatorial lattice point $q = (0, c) \in S^n$. In [13], we showed that hidden at q there is a sums of squares formula. This formula also has integer coefficients, and can be easily retrieved as follows. For details, we refer to [8].

THEOREM 2.1 (Theorem 8 of [8]). *Suppose the color* c *appears* k *times in the signed intercalate matrix* $A \circ K$ *representing an* $[r, s, n]_{\mathbb{Z}}$ *formula. Permute the rows and columns, and change the signs of all colors along certain rows of* $A \circ K$ *if necessary to bring* $A \circ K$ *into the form*

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix},$$

in which each diagonal entry of the $k \times k$ *submatrix* K_{11} *is the color* c *with a positive* *sign. Then the signed intercalate matrix*

$$[K_{11} \quad K_{12} \quad K_{21}^t]$$

gives the $[k, r+s-k, n]_{\mathbb{Z}}$ *formula hidden at* $q = (0, c) \in S^n$. ■

REMARK. The original $[r, s, n]_{\mathbb{Z}}$ formula can be regarded as hidden at the north and south poles $(\pm 1, 0) \in S^n$.

3. **Structure of $[10, 10, 16]_{\mathbb{Z}}$ formulae.**

LEMMA 3.1. *Every* $[10, 10, 16]$ *formula, not necessarily with integer coefficients, represents a nontrivial homotopy class, which is an even multiple of the generator* ν *of the stable 3-stem* π_3^s.

PROOF. Let f be a normed bilinear map of type $[10, 10, 16]$. Then the adjoint $h: S^9 \to V_{16,10}$ must be nontrivial, for otherwise it admits an extension to a skew map $S^{10} \to V_{16,10}$. This gives 10 sections of the bundle $16\xi_{10}$ over the real projective space RP^{10}, which is impossible according to the tables of [4].

Observe that the homomorphism $J: \pi_9(V_{16,10}) \to \pi_{19}(S^{16})$ maps $\mathbb{Z}_{12}$ injectively into $\mathbb{Z}_{24}$: Lam [5] has exhibited a $[10, 10, 16]_{\mathbb{Z}}$ formula representing $\pm 2\nu$, showing that the image of J is a group of order 12. It follows that J is injective and f represents a nonzero homotopy class.

It further follows that the Hopf construction of f is surjective, and there is a regular value by Sard's theorem. The sums of squares formula hidden at a regular value is of type $[4, 16, 16]$, and so represents $\pm\nu\pm\nu\pm\nu\pm\nu$, an even multiple of ν. ■

For later reference, we record a useful lemma which follows by an argument similar to Step 4 of the proof of Theorem 8.2 of [13].

LEMMA 3.2. *Suppose an* $[r, s, n]$ *formula contains a hidden* $[k, r+s-k, n]$ *formula. There is a diagram*

$$\begin{tikzcd} \pi_{k-1}(V_{n,r+s-k}) \arrow[r, "\partial"] \arrow[d, "J"'] & \pi_{k-2}(S^{n+k-r-s-1}) \arrow[r, "j_*"] \arrow[dl, "E^{r+s-k+1}"] & \pi_{k-2}(P^{\infty}_{n+k-r-s-1}) \\ \pi_{r+s-1}(S^n) & & \end{tikzcd}$$

commutative up to sign in which the top row is exact. ■

PROPOSITION 3.3. *The sums of squares formulae hidden behind a* $[10, 10, 16]$ *formula are of types* $[k, 20-k, 16]$, *where* $k = 4, 8, 10$.

PROOF. For $k = 5, 6$, $\pi_{k-1}(V_{16,20-k}) = 0$ from the tables of [11]. No hidden $[k, 20-k, 16]$ formula with $k = 5, 6$ can represent the nonzero homotopy class of the $[10, 10, 16]$ formula. From this $k \neq 5, 6$. For $k = 9, 13$, we obtain a contradiction to Lemma 3.2 by noting that in each case, the 2-primary component of Ker j_* is zero according to the tables of [10]. ■

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THEOREM 3.4 *Every* $[10, 10, 16]_{\mathbb{Z}}$ *formula is obtained by signing* $D_{10,10}$.

PROOF. Let K be an intercalate matrix of type $(10, 10, 16)$ that can be signed to give a $[10, 10, 16]_{\mathbb{Z}}$ formula. By Theorem 2.1 and Proposition 3.3, each color of K appears with frequency 4, 8, or 10. A simple enumeration gives the following possible distributions of colors.

	Frequency	4	8	10
	(i)	7	9	0
Number of	(ii)	8	6	2
of colors	(iii)	9	3	4
	(iv)	10	0	6

The last two cases can be easily eliminated by observing that it is impossible to arrange 3 or more ubiquitous in an intercalate 10×10 matrix. We omit the lengthy elimination of case (i) and refer to [12] for details. Granted this, we know, by Lemma 1.1, that K is a tensor product $K_1 \otimes K_2$, where K_1 and K_2 are respectively intercalate matrices of types $(5,5,8)$ and $(2,2,2)$ respectively. Furthermore, K_1 has four colors with frequency 2, three colors with frequency 4, and one ubiquitous color (with frequency 5). It is easy to check that K_1 is equivalent to $D_{5,5}$. From this the result follows. ■

REMARK. With a hidden $[4, 16, 16]_{\mathbb{Z}}$ formula explicitly written down according to Theorem 2.1, it can be shown that every $[10, 10, 16]_{\mathbb{Z}}$ formula represents $\pm 2\nu$.

4. **Nonexistence of $[16, 16, 28]_{\mathbb{Z}}$ formulae**. For given r and s, denote by $r\#s$ the least integer n for which there exists a nonsingular bilinear map $R^r \times R^s \to R^n$.

LEMMA 4.1. $12\#20 = 28$.

PROOF. According to [9], there is a nonsingular bilinear map $R^{12} \times R^{20} \to R^{28}$. On the other hand, a nonsingular bilinear map of type $R^{12} \times R^{20} \to R^{27}$ would give 20 sections of the vector bundle $27\xi_{11}$ over the real projective space RP^{11}, which is impossible according to the tables of [4]. ■

THEOREM 4.2. *There is no* $[16, 16, 28]_{\mathbb{Z}}$ *formula. Consequently*, $16 *_{\mathbb{Z}} 16 \geq 29$.

PROOF. We assume that there is one such formula f obtained by signing an intercalate matrix K of type $(16, 16, 28)$, and derive a contradiction in the following steps.

Step 1. By the Hopf-Stiefel condition, the only hidden sums of squares formulae are of types $[k, 32\text{-}k, 28]$, $k = 4, 8, 12, 16$.

Step 2. Treating f as a hidden $[16, 16, 28]_{\mathbb{Z}}$ formula (Remark following Theorem 2.1), and applying Lemma 3.2 with the 2-primary component of $\operatorname{Ker} j_* = \mathbb{Z}_4$ from the tables of [10], we note that f represents an even multiple of $\nu \in \pi_3^s$.

https://doi.org/10.4153/CMB-1987-045-6 Published online by Cambridge University Press

Step 3. Consequently, there is no hidden [4, 28, 28] formula, for such a hidden formula would represent an odd multiple of ν. It follows from Sard's theorem that f cannot be surjective. Hence, f in fact represents the zero homotopy class.

Step 4. We further exclude the possiblity $k = 12$. Note that every (hidden) normed bilinear map g of type [12, 20, 28], if it exists, must be surjective, for otherwise, we obtain a nonsingular bilinear map $R^{12} \times R^{20} \to R^{27}$ by projecting onto the orthogonal complement of any vector not in the image of g, contradicting Lemma 4.1.

Step 5. It follows, by Theorem 2.1, that in the intercalate matrix K of type (16, 16, 28), each color appears either 8 or 16 times. A simple enumeration shows that there are 24 colors with frequency 8 and 4 ubiquitous colors (with frequency 16). By Lemma 1.1, K is equivalent to a tensor product $K_1 \otimes K_2$, where K_1 and K_2 are intercalate matrices of types (4, 4, 7) and (4, 4, 4) respectively, and K_1 has one ubiquitous color (with frequency 4).

Step 6. It is easy to see that every intercalate matrix of type (4, 4, 7) with a ubiquitous color is equivalent to

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 5 & 6 \\ 3 & 5 & 1 & 7 \\ 4 & 6 & 7 & 1 \end{bmatrix}.$$

Thus, K_1 contains an intercalate submatrix of type (3, 3, 4). It follows that K contains an intercalate submatrix of type (12, 12, 16), which can be signed by the corresponding restriction of the signing matrix of K to give a $[12, 12, 16]_{\mathbb{Z}}$ formula. We arrive at the desired contradiction by observing that not even a [12, 12, 20] formula can exist ([13]). ■

Acknowledgement. This research was partially supported by the NSERC of Canada.

References

1. J. Adem, *Construction of some normed maps*, Bol. Soc. Mat. Mexicana **20** (1975), pp. 59–75.

2. A. Cayley, *On the theorem of the 2, 4, 8 and 16 squares*. Quart. J. Math. **17** (1881), pp. 258–276; reprinted in Collected Papers 763.

3. L. E. Dickson, *On quaternions and their generalization and the history of the eight square theorem*, Annals of Math. **20** (1919), pp. 155–171.

4. K. Y. Lam, *Sectioning vector bundles over real projective spaces*, Quart. J. Math. Oxford (ser. 2) **23** (1972), pp. 97–106.

5. K. Y. Lam, *Some interesting examples of nonsingular bilinear maps*, Topology **16** (1977), pp. 185–188.

6. K. Y. Lam, *Topological methods for studying the composition of quadratic forms*, Canadian Math. Soc. Conf. Proc. **4** (1984), pp. 173–192.

7. K. Y. Lam, *Some new results in composition of quadratic forms*, Invent. Math. **79** (1985), pp. 467–474.

8. K. Y. Lam and P. Yiu, *Retrieving hidden sums of squares formulae*; preprint.

9. R. J. Milgram, *Immersing projective spaces*, Annals of Math. **85** (1967), pp. 473–482.

10. R. J. Milgram, J. Strutt and P. Zvengrowski, *Projective stable stems of spheres*, Bol. Soc. Mat. Mexicana **22** (1977), pp. 48–57.

11. G. F. Paechter, *The groups* $\pi_r(V_{n,m})$, Quart. J. Math. Oxford **7** (1956), pp. 249–268.

12. P. Yiu, Thesis, University of British Columbia, 1985.

13. P. Yiu, *Quadratic forms between spheres and the nonexistence of sums of squares formulae*, Math. Proc. Cambridge Philos. Soc. **100** (1986), pp. 493–504

14. S. Yuzvinsky, *Orthogonal pairings of euclidean spaces*, Michigan J. Math. **28** (1981), pp. 131–145.

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https://doi.org/10.4153/CMB-1987-045-6 Published online by Cambridge University Press