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ON VERTICAL ORDER OF ONE-DIMENSIONAL COMPACTA IN E^3

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1. Introduction. Let X be a compactum in E^n of dimension at most $n - 2$. In [**9**, Theorem 4.1] it was shown that there is an arbitrarily small homeomorphism h of E^n, fixed outside any given neighborhood of X, so that $h(X)$ has vertical order $n - 1$ provided $n \neq 3$. If X is a 0-dimensional set or a tame 1-dimensional set in E^3 then the result is still true. However, the examples of tangled continua of Bothe [**2**] and McMillan and Row [**7**] are not amenable to the techniques used in dimensions other than three. This prompted Wright [**9**] to make the following conjecture.

CONJECTURE 1.1. *A* 1*-dimensional compactum* X *in* E^3 *with vertical order* 2 *must be tame.*

We give an affirmative answer to this conjecture. We show much more. Most vertical lines that meet the wild set of X contain a subset of X homeomorphic to a Cantor set.

2. Definitions and notation. We let E^n denote Euclidean n-dimensional space and rB^n denote the solid n-ball of radius r centered at the origin in E^n. We use the usual x, y, z coordinates for E^3. We let P, Q be the projections from E^3 to E^2 and E^1, respectively, defined by

$$P(x, y, z) = (x, y) \quad \text{and} \quad Q(x, y, z) = z.$$

Let $A \subset E^2$ and $X \subset E^3$. For k a non-negative integer, we say that X has *vertical order* k over A if $P^{-1}(a)$ meets X in at most k points for each $a \in A$. We say X has *bounded finite vertical order over* A if X has vertical order k over A for some k. We say X has *finite vertical order over* A if $P^{-1}(a)$ meets X in a finite set for each $a \in A$, and we say X has *countable vertical order over* A if $P^{-1}(a)$ meets X in a countable set for each $a \in A$. For a, $r > 0$ the subset $rB^2 \times [0, a]$ of E^3 is a *right circular cylinder* with *end-disks* $rB^2 \times \{0\}$ and $rB^2 \times \{a\}$. The set $\{(0, 0, t) | 0 \leqq t \leqq a\}$ is the *axis* of the cylinder. Any subset of E^3 which is isometric to $rB^2 \times [0, a]$ is also called a right circular cylinder and the isometry determines the end-disks and axis. By an *arc in a right circular cylinder* we will always mean an arc that runs between the interiors of the end-disks and that meets the boundary of the cylinder precisely in the end-points of the arc.

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We use dim X to denote the dimension of a space X, and if $X \subset E^n$, we let dem X denote the dimension of embedding of X [**3**], [**4**], [**8**], [**5**]. For topological embeddings of 1-dimensional compacta in E^3 there are several notions of tameness [**1**]. We choose to call a 1-dimensional compact subset of E^3 *tame* or *tamely* embedded provided that dem $X = 1$. Let X be a 1-dimensional compactum in E^3 and $p \in X$. We say that X is *locally tame* at p provided that there is a neighborhood N of p in E^3 so that

$$\text{dem}(X \cap N) \leqq 1.$$

The subset of X at which X is not locally tame is called the wild set of X and is denoted by $W(X)$. Elementary facts from the theory of dimension for embeddings [**5**] implies that X is tame if and only if $W(X) = \emptyset$. Furthermore, if X is not tame $W(X)$ is a compact, 1-dimensional set that does not have any points at which it is locally tame. We say X is *wild* if it is not tame and X is *totally wild* if $W(X) = X$.

Let X be a 1-*dimensional compactum in* E^3 and C be a right circular cylinder in E^3 whose end-disks miss X. We call C a *plug for* X if it is impossible to find an unknotted arc in C that misses X. We call C a *vertical plug* for X if the axis of C is parallel to the z-axis.

3. Plugless one-dimensional compacta are tame.

THEOREM 3.1. *Let X be a* 1-*dimensional compactum in E^3. Then X is tame if and only if there are no plugs for X.*

Proof. The forward implication is trivial. Hence, we assume that plugs do not exist for X. Let L be an arbitrary one-dimensional subpolyhedron of E^3, U be a neighborhood of $X \cap L$, and $\epsilon > 0$ be given. We will show that dem $X \leqq 1$ by constructing an ϵ-ambient isotopy of E^3 with support in U that moves L off X.

Because X is not dense in E^3, there exists an $(\epsilon/2)$-ambient isotopy h_t of E^3 with support in U so that

(1) $h_1(L)$ is a subpolyhedron of E^3,
(2) $h_1(L)$ has a triangulation T of mesh $\epsilon/2$,
(3) the vertices of T miss X,
(4) each one-simplex of T that meets X lies in U.

For each one-simplex σ of T that meets X, construct a small right circular cylinder C_σ so that

(1) $C_\sigma \subset U$ and has diameter $< \epsilon/2$,
(2) the end-disks of C_σ miss X,
(3) $\sigma \cap C_\sigma$ is the axis of C_σ,
(4) $\sigma \cap X \subset C_\sigma$,
(5) $C_\sigma \cap C_\tau = \emptyset$ for $\sigma \neq \tau$ where τ is any other one-simplex of T that meets X.

Since each C is not a plug we find an unknotted arc σ' in C_σ whose end-points are the end-points of the axis of C_σ. It is now an easy matter to get an $(\epsilon/2)$-ambient isotopy g_t of E^3 with support in $\cup\ C_\sigma$ that takes $\sigma \cap C_\sigma$ to σ' and, therefore, moves $h_1(L)$ off X. Putting together the isotopies h_t and g_t we obtain the desired ϵ-isotopy.

THEOREM 3.2. *Let X be a 1-dimensional compactum in E^3. Then X is tame if and only if there are no vertical plugs for X.*

Proof. As in Theorem 3.1, one direction is trivial. We assume that X has no vertical plugs. Let C be an arbitrary right circular cylinder in E^3 whose end-disks miss X. Since X is 1-dimensional it is possible to find a polygonal arc A in C that misses X. If A is unknotted, then C fails to be a plug; otherwise we assume A has a regular projection to the xy-plane; i.e., there are a finite number of singular points each of which is a transverse double point. For each such double point p, we let L_p be the straight line interval connecting the two points in A which give rise to the double point. The vertical straight line interval L_p lies in the interior of C since C is convex. If for each double point p, $L_p \cap X = \emptyset$, we could unknot A in the complement of X by changing overcrossings and undercrossings. If this is not the case, we use the fact that there are no vertical plugs to adjust X by a small homeomorphism h, fixing the boundary of C, so that $h(X)$ misses A and all such L_p. We then change overcrossings and undercrossings as needed to get an unknotted arc A' in the complement of $h(X)$. Then $h^{-1}(A')$ is our unknotted arc in C missing X. We have shown that there are no plugs for X, and Theorem 3.1 implies that X is tame.

4. Straight line intervals in 1-dimensional compacta. Let X be a subset of E^3. For real numbers $a < b$ we set

$$X[a, b] = \{x \in E^2 | [a, b] \subset Q(P^{-1}(x) \cap X)\}.$$

We also set

$$X_1 = \{x \in E^2 | \dim(P^{-1}(x \cap X) = 1\}.$$

LEMMA 4.1. *If X is a 1-dimensional compactum in E^3, then for $a < b$ $X[a, b]$ is compact and* $\dim X[a, b] \leqq 0$.

Proof. One easily checks that $X[a, b]$ is a closed subset of the compact set $P(X)$. Hence, $X[a, b]$ is compact. If $\dim X[a, b] > 0$, then

$$\dim(X[a, b] \times [a, b]) > 1,$$

[**6**, page 34]. However, since $X[a, b] \times [a, b] \subset X$, this would imply that $\dim X > 1$ which is a contradiction.

THEOREM 4.2. *Let X be a 1-dimensional compactum in E^3. Then* $\dim X_1 \leqq 0$ *and X_1 is the countable union of compact sets.*

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Proof. If $P^{-1}(x) \cap X$ is 1-dimensional, then $P^{-1}(x) \cap X$ must contain an open interval. Hence $X_1 = \cup X[p, q]$ where p and q range over all rational numbers with $p < q$. Since there are a countable number of such ordered pairs (p, q) and each $X[p, q]$ is compact of dimension $\leqq 0$, we conclude that $\dim X_1 \leqq 0$ [**6**, page 30].

5. One-dimensional compacta of finite vertical order. In this section we show that a 1-dimensional compactum X with bounded finite vertical order over a dense subset of E^2 is tame thus answering the question posed in [**9**]. We also show that X need not be tame if "bounded" is omitted from the hypothesis.

LEMMA 5.1. *Let X be a 1-dimensional compactum in E^3 so that for some dense subset D of E^2, X has vertical order* 1 *over D. Then X is tame.*

Proof. If X is wild then there is a vertical plug C for X. Since $\dim X = 1$, there is a polygonal arc A in C that misses X. We assume by general position that the arc A has a regular projection into the xy-plane and that the double points lie in D. For each double point p, let p_1 and p_2 be the points in A so that $P(p_1) = P(p_2) = p$. We assume that the z-coordinate of p_1 is less than that of p_2. We call the double point p *essential* if the straight line interval connecting p_1 and p_2 meets X; otherwise, we call p an *inessential* double point. We will find an unknotted arc in C missing X by induction on the number of essential double points for arcs such as the arc A. If there are no essential double points then we can easily change some of the overcrossings of A to undercrossings to unknot A in C. If p is an essential double point, then the vertical line segment B from p_2 to the top end-disk of C must miss X because $p \in D$. Consider the arc A' in C consisting of B and the subarc of A which runs from p_2 to the bottom end-disk of C. By adjusting A' only in a neighborhood of B we may eliminate the double point p and obtain an arc A'' whose essential double points form a proper subset of the essential double points of A. This shows the existence of an unknotted arc in C in the complement of X and contradicts the fact that C is a vertical plug for X. Hence, there are no vertical plugs for X, and X is tame.

LEMMA 5.2. *Let C be a vertical plug for a totally wild* 1*-dimensional compactum X in E^3. Then the projection of $C \cap X$ in E^2 equals the projection of the end disks of C, and for any open set $U \subset P(C)$ and $\epsilon > 0$ there exist disjoint vertical plugs C_1, C_2 contained in C whose axes have length less than ϵ and whose projections in E^2 are equal and contained in U.*

Proof. That the projection of $C \cap X$ equals the projection of the end disks of C follows easily from the definition of a plug. By Theorem 4.2 the set

$$D = \{x \in E^2 | \dim(P^{-1}(x) \cap X) \leqq 0\}$$

is dense in E^2. If for each $x \in D \cap U$ the set $P^{-1}(x) \cap X$ has at most one point, then the 1-dimensional compactum, P^{-1} (closure of U) $\cap C \cap X$, is tame by Lemma 5.1. But this is impossible since X is totally wild. Hence there is a point $w \in U$ so that

$$\dim(P^{-1}(w) \cap X \cap C) = 0$$

and $P^{-1}(w) \cap X \cap C$ contains at least two points. We now find vertical right circular cylinders $C_1, C_2, \dots, C_n$ ($n \geqq 2$) so that

(1) the axis of each C_i is collinear with $P^{-1}(w)$ and has length less than ϵ,
(2) $P(C_i) = P(C_j) \subset U$ for all i, j,
(3) $C_i \cap C_j = \emptyset$ for $i \neq j$,
(4) the end-disks of each C_i miss X,
(5) the interior of each C_i contains a point of X,
(6) $P^{-1}(w) \cap X \cap C \subset \cup_{i=1}^n C_i$.

If each C_i fails to be a plug for X, it is an easy matter that C also fails to be a plug for X. We may assume, therefore, that C_1 is a plug for X. Since X is totally wild, $X \cap C_2$ is a wild 1-dimensional set. Hence, we can find a vertical plug C_2' for $X \cap C_2$ which we may assume lies in C_2. Clearly C_2' is also a plug for X. Since C_1 is a plug for X,

$$C_1' = C_1 \cap P^{-1}(P(C_2'))$$

is also a plug for X. The plugs C_1' and C_2' show that our lemma is true.

THEOREM 5.3. *Let X be a 1-dimensional compactum in E^3 that has bounded finite vertical order over a dense subset of E^2. Then X is tame.*

Proof. The proof is by induction on k where X has vertical order k over the dense set D. If $k \leqq 1$, then the theorem is true by Lemma 5.1. So assume $k \geqq 2$ and suppose X is wild. Let $W(X) \neq \emptyset$ be the wild set of X. Observe that $W(X)$ also has vertical order k. Since $W(X)$ is wild there must be a vertical plug C for $W(X)$. Lemma 5.2 with $U =$ interior $[P(C)]$ and ϵ arbitrary we can find disjoint vertical plugs C_1 and C_2 for $W(X)$ so that

$$P(C_1) = P(C_1 \cap W(X)) = P(C_2 \cap W(X)) = P(C_2).$$

Hence, $C_i \cap W(X)$ has vertical order $k - 1$ over D. Since $W(X)$ is totally wild, each $C_i \cap W(X)$ must be wild. But induction implies that each $C_i \cap W(X)$ must be tame. Hence we are forced to conclude that $W(X) = \emptyset$ and X is tame.

Example 5.4. A wild 1-dimensional compactum with finite vertical order over a dense subset of E^2.

Let X be a wild 1-dimensional compactum in E^3 [**2**], [**7**]. Recall from Theorem 4.2 that

$$X_1 = \{x \in E^2 | \dim(P^{-1}(x) \cap X) = 1\}$$

is a set of dimension 0. Hence we can find a countable dense subset $\{a_i\}$ in E^2 in the complement of X_1. For each i such that $P^{-1}(a_i) \cap X \neq \emptyset$, we find a finite collection of disjoint line segments A_{ij}, $1 \leqq j < n_i$, each of length less than $1/i$ so that the A_{ij} cover $P^{-1}(a_i) \cap X$ and for each j, $A_{ij} \cap X \neq \emptyset$. Let G be the decomposition of E^3 into points and the arcs A_{ij}. The decomposition G is easily seen to satisfy the Bing shrinking criterion. In fact the shrinking homeomorphisms do not need to move the x, y coordinates of any point. We let $h_t:E^3 \to E^3$ be a pseudoisotopy, fixing the x, y coordinates so that, h_0 = identity and h_1 realizes the decomposition.

Let $Y = h_1(X)$. Clearly Y is compact and has finite vertical order over $\{a_i\}$. We will show that any map α: [0, 1] $\to E^3$ can be approximated by a map α' whose image misses Y. By techniques in [**7**] this implies dim $(Y) \leqq 1$. Let α: [0, 1] $\to E^3$ be a given map and $\epsilon > 0$ be given. Choose β so that h_t, $\beta \leqq t \leqq 1$ is a pseudoisotopy that moves points less than ϵ. Without loss of generality we may assume that α [0, 1] misses the 1-dimensional set $h_\beta(X)$ and the countable union of lines $\cup P^{-1}\{a_i\}$. The map $\alpha' = h_1\alpha$ is the desired map.

Let C be a vertical plug for X so that the end-disks miss all the nondegenerate elements of G. We may assume that the pseudoisotopy h_t leaves the end-disks of C fixed. If dim $Y = 0$ or if y is tame and 1-dimensional, then there is an unknotted arc A in C that misses Y. For t sufficiently close to 1, $h_t(X)$ misses A. Hence $h_t^{-1}(A)$ is an unknotted arc in C that misses $X = h_t^{-1}h_t(X)$. This contradicts the fact that C is a plug for Y. Therefore, we are forced to conclude that dim $Y = 1$ and Y is wild.

6. Totally wild one-dimensional compacta in E^3.

THEOREM 6.1. *Let X be a totally wild* 1*-dimensional compactum in E^3 and U be the interior of $P(X)$ in E^2. Then $P(X)$ is equal to the closure of U.*

Proof. Let V be an open subset of E^3 so that $V \cap X \neq \emptyset$. Since X is totally wild, dem $(X \cap V) = 2$ and closure $(X \cap V) = X'$ is wild. If $P(X')$ is nowhere dense in E^2, then there are no vertical plugs for X', and, by Theorem 3.2, X' is tame, a contradiction. So $P(X')$ contains an open set. Since V was an arbitrary open subset that meets X, our theorem follows.

For X a totally wild 1-dimensional set in E^3, let

$$X_0 = \{x \in E^2 | P^{-1}(x) \text{ is an uncountable 0-dimensional set}\}.$$

We will show that "most" points in $P(X)$ actually lie in X_0.

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THEOREM 6.2. *Let X be a totally wild 1-dimensional compactum in E^3. Then X_0 contains a dense G_δ subset of $P(X)$.*

Before we can prove Theorem 6.2 we will need some lemmas.

LEMMA 6.3. *Let X be a totally wild 1-dimensional compactum in E^3 and $C_1, C_2, \ldots, C_n$ be disjoint vertical plugs for X with a common projection D in E^2. Let U be an open subset of D, and let $\epsilon > 0$. For each i, $1 \leqq i \leqq n$, there exists a pair of disjoint vertical plugs $C_i(1)$, $C_i(2)$ in C_i such that the axes of $C_i(j)$, $j = 1, 2$, have length less than ϵ. Furthermore, the plugs $C_i(j)$ have a common projection in E^2 that lies in U*

Proof. For $n = 1$ this is just Lemma 5.2. By induction we assume $C_i(j)$ exist satisfying the conclusion for $1 \leqq i \leqq k$ and $j = 1, 2$ with common projection D' in E^2. We apply Lemma 5.2 to $C = C_{k+1} \cap P^{-1}(D')$ to obtain $C'_{k+1}(1)$ and $C'_{k+1}(2)$ in C with common projection D'' in E^2. For $1 \leqq i \leqq k$ and $j = 1, 2$, let

$$C'_i = C_i(j) \cap P^{-1}(D'').$$

The collection $C'_i(j)$ satisfies the conclusion of the lemma.

LEMMA 6.4. *Let X be a totally wild compactum in E^3 and $C_1, C_2, \ldots, C_n$ be vertical plugs for X (not necessarily disjoint) such that*

$$U = \bigcap_{i=1}^{n} \text{interior}\,(P(C_i)) \neq \emptyset.$$

Let D be any round disk in U. Then there is a collection $E_1, E_2, \ldots, E_m$ of disjoint vertical plugs for X with common projection D so that for each C_i, $1 \leqq i \leqq n$, there is an E_j, $1 \leqq j \leqq m$, with $E_j \subset C_i$.

Proof. If $n = 1$, let $E_1 = C_1 \cap (D \times E^1)$. Consider the case $n = k + 1$. By induction we assume the existence of vertical plugs $E_1, \ldots, E_r$ for X with common projection D in E^2 and such that for each C_i, $1 \leqq i \leqq k$, there is an E_j, $1 \leqq j \leqq r$, with $E_j \subset C_i$. Now consider

$$C = C_{k+1} \cap (D \times E^1) = D \times [a, b].$$

The set C is a plug for X that lies in C_{k+1}. By shrinking the vertical axis slightly, if necessary, we may assume that the end-disks of C are at different levels than the end disks of the E_j. If $C \cap E_j = \emptyset$ for $1 \leqq j \leqq r$, set $E_{r+1} = C$. If $E_j \subset C$ for some j, then $E_j \subset C_{k+1}$ and the collection $E_1, E_2, \ldots, E_r$ suffices. There are a few remaining possibilities. We will consider the case where C meets a single $E_j = D \times [a', b']$ and $a < a' < b < b'$. The other cases are similar. Since $D \times [a', b']$ is a plug for X, either $D \times [a', b]$ or $D \times [b, b']$ is a plug for X. If $D \times [a', b]$ is a plug for X, replace E_j by this plug and we are done. If $D \times [b, b']$ is a plug for X, replace E_j by this plug and construct E_{r+1} from C by slightly shrinking the vertical axis. The family $E_1, E_2, \ldots, E_r, E_{r+1}$ forms the desired collection.

LEMMA 6.5 *Let X be a totally 1-dimensional compactum in E^3 and C_1, $C_2, \ldots, C_n$ be vertical plugs for X (not necessarily disjoint) such that*

$$U = \bigcap_{i=1}^{n} \text{interior}\,(P(C_i)) \neq \emptyset.$$

Then for each $\epsilon > 0$ and each open subset V of U there exists a subset $\mathcal{M}$ of E^3 so that:

1) *$\mathcal{M}$ has a finite number of components each of which is a vertical plug for X.*

2) *The components of $\mathcal{M}$ have the identical projection in V.*

3) *Each component of $\mathcal{M}$ has diameter at most ϵ.*

4) *Each C_i contains at least two components of $\mathcal{M}$.*

Proof. First apply Lemma 6.4, and then apply Lemma 6.3.

Proof of Theorem 6.2. We inductively define $\mathfrak{M}_i$, $\mathcal{G}_i$, $\mathcal{H}_i$, Φ_i. Let $\mathfrak{M}_0$ be the collection of all plugs for X and

$$\mathcal{G}_0 = \{\text{interior } P(\mathcal{M}) \mid \mathcal{M} \in \mathfrak{M}_0\}.$$

Let $\mathcal{G}_0^*$ be the union of the elements of $\mathcal{G}_0$. Then $\mathcal{G}_0^*$ is a dense open subset of $P(X)$. Let $\mathcal{H}_0$ be a locally finite refinement of $\mathcal{G}_0$. We define a function $\Phi_0:\mathcal{H}_0 \to \mathfrak{M}_0$ by assigning to each $h \in \mathcal{H}_0$ exactly one $\mathcal{M} \in \mathfrak{M}_0$ with $h \subset$ interior $(P(M))$.

Assume that $\mathfrak{M}_i$, $\mathcal{G}_i$, $\mathcal{H}_i$, Φ_i have been defined for $i \leqq k$. Let $\mathfrak{M}_{k+1}$ be the collection of all subsets of E^3 satisfying

1) $\mathcal{M}$ is the disjoint union of finitely many plugs for X, each plug has the identical projection in E^2 and each plug has diameter at most $1/(k + 1)$.

2) $P(\mathcal{M})$ is contained in some element of $\mathcal{H}_k$.

3) For each $h \in \mathcal{H}_k$ either $P(\mathcal{M}) \cap h = \emptyset$ or $P(\mathcal{M}) \subset h$.

4) $\mathcal{M}$ has at least two components in each component of $\Phi_k(h)$ where $P(\mathcal{M}) \subset h$.

Let

$$\mathcal{G}_{k+1} = \{\text{interior } P(\mathcal{M}) \mid \mathcal{M} \in \mathfrak{M}_{k+1}\}.$$

By Lemma 6.5 and the local finiteness of $\mathcal{H}_k$, $\mathcal{G}^*_{k+1}$, the union of the elements of $\mathcal{G}_{k+1}$, is dense in $\mathcal{G}^*_k$, and hence dense in $P(X)$. Let $\mathcal{H}_{k+1}$ be a locally finite refinement of $\mathcal{G}_{k+1}$. Define a function

$$\Phi_{k+1}:\mathcal{H}_{k+1} \to \mathfrak{M}_{k+1}$$

by assigning to each $h \in \mathcal{H}_{k+1}$ exactly one $\mathcal{M} \in \mathfrak{M}_{k+1}$ with $h \subset$ interior $(P(\mathcal{M}))$.

Recall from Theorem 4.2 that the set

$$X_1 = \{x \in E^2 \mid \dim(P^{-1}(x) \cap X) = 1\}$$

is a 0-dimensional set that is the countable union of compact sets F_i. Let $V_i = \mathcal{G}_i^* - F_i$. The sets V_i are easily seen to also be dense in $P(X)$. Let z be a point in the intersection of the V_i. Since $z \notin X_1$,

$$\dim(P^{-1}(z) \cap X) \leqq 0.$$

Since z is in the intersection of the $\mathcal{G}_i^*$, there is an h_i in each $\mathcal{H}_i$ so that $z \in h_i$. Let

$$\mathcal{M}_i = \Phi_i(h_i) \in \mathfrak{M}_i.$$

Since $\mathcal{M}_{i+1}$ has at least two components in each component of $\mathcal{M}_i$, $\cap \mathcal{M}_i$ is a Cantor set that is contained in $P^{-1}(z) \cap X$ and the theorem is proved.

For X a 1-dimensional compactum in E^3 define

$$X_2 = \{x \in P(X) | P^{-1}(x) \cap X \text{ is at most countable}\}.$$

COROLLARY 6.6. *Let X be a totally wild* 1-*dimensional compactum in E^3. Then each of X_1 and X_2 is the countable union of nowhere dense subsets of $P(X)$.*

Finally, we have the following taming theorem.

COROLLARY 6.7. *Let X be a* 1-*dimensional compactum in E^3 with countable vertical order. Then X is tame.*

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