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COMPUTING CERTAIN GROMOV-WITTEN INVARIANTS OF THE CREPANT RESOLUTION OF $\mathbb{P}(1,3,4,4)$

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Abstract. We prove a formula computing the Gromov-Witten invariants of genus zero with three marked points of the resolution of the transversal A_3-singularity of the weighted projective space $\mathbb{P}(1,3,4,4)$ using the theory of deformations of surfaces with A_n-singularities. We use this result to check Ruan's conjecture for the stack $\mathbb{P}(1,3,4,4)$.

§1. Introduction

The main results of this paper concern the weighted projective space $\mathbb{P}(1,3,4,4)$. The singular locus of its coarse moduli space $|\mathbb{P}(1,3,4,4)|$ is the disjoint union of an isolated singularity of type $(1/3)(1,1,1)$ (we use Reid's notation in [17]) and a transversal A_3-singularity. Using toric methods, we construct a crepant resolution Z of $|\mathbb{P}(1,3,4,4)|$. In Theorem 3.3.1, we determine a formula for certain Gromov-Witten invariants of Z over the A_3-singularity using the theory of deformations of surfaces with rational double points and the deformation invariance property of Gromov-Witten invariants. We then apply this result in Theorem 5.2.1 to construct a ring isomorphism—predicted in [18] by Ruan's cohomological crepant resolution conjecture—between the quantum corrected cohomology ring of Z and the Chen-Ruan orbifold cohomology of $\mathbb{P}(1,3,4,4)$, after evaluating the quantum parameters related to the transversal A_3-singularity to a fourth root of the unity and putting the last parameter to zero. This last evaluation is quite surprising (in [2], we show that this parameter can be evaluated to 1). To confirm this property, we show in Proposition 5.3.1 that, for all weighted projective spaces $\mathbb{P}(1,\ldots,1,n)$ with n weights equal to 1 which have only

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one isolated singular point $(1/n)(1,\dots,1)$, the predicted ring isomorphism can be obtained simply by putting the quantum parameter to zero.

§2. Weighted projective spaces

Let $n \geq 1$ be an integer, and let $w = (w_0,\dots,w_n)$ be a sequence of integers greater than or equal to 1. The multiplicative group $\mathbb{C}^\star$ acts on $\mathbb{C}^{n+1} \setminus \{0\}$ by

$$\lambda \cdot (x_0,\dots,x_n) := (\lambda^{w_0} x_0,\dots,\lambda^{w_n} x_n).$$

The *weighted projective space* $\mathbb{P}(w)$ is defined as the quotient stack $[\mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^\star]$. It is a smooth Deligne-Mumford stack whose coarse moduli space, denoted $|\mathbb{P}(w)|$, is a projective variety of dimension n.

According to Borisov, Chen, and Smith [4], $\mathbb{P}(w)$ is a toric stack associated to the stacky fan

$$\Big(N := \mathbb{Z}^{n+1}/\sum_{i=0}^{n} w_i v_i, \quad \beta\colon \mathbb{Z}^{n+1} \to N, \quad \Sigma\Big), \tag{1}$$

where $v_0,\dots,v_n$ is the standard basis of $\mathbb{Z}^{n+1}$, β is the canonical projection, and $\Sigma \subset N \otimes_{\mathbb{Z}} \mathbb{Q}$ is the fan whose cones are generated by any proper subset of the set $\{\beta(v_0) \otimes 1,\dots,\beta(v_n) \otimes 1\}$.

The weighted projective space $\mathbb{P}(w)$ comes with a natural invertible sheaf $\mathcal{O}_{\mathbb{P}(w)}(1)$ defined as follows: for any scheme Y and any morphism $Y \to \mathbb{P}(w)$ given by a principal $\mathbb{C}^\star$-bundle $P \to Y$ and a $\mathbb{C}^\star$-equivariant morphism $P \to \mathbb{C}^{n+1} \setminus \{0\}$, $\mathcal{O}_{\mathbb{P}(w)}(1)_Y$ is the sheaf of sections of the associated line bundle of P.

Recall that an *orbifold* is by definition a smooth Deligne-Mumford stack over $\mathbb{C}$ with generically trivial stabilizers. A *Gorenstein orbifold* is an orbifold such that, at each point, the stabilizer acts with determinant 1 on the tangent space. This implies that the coarse moduli space is a Gorenstein variety, but that it is not equivalent. For instance, the variety $|\mathbb{P}(1,3)|$ is Gorenstein (in fact, smooth, isomorphic to $\mathbb{P}^1$) but $\mathbb{P}(1,3)$ is not a Gorenstein orbifold, as is easily seen using the following classical result.

PROPOSITION 2.0.1. *We have the following.*

(1) *The Deligne-Mumford stack $\mathbb{P}(w)$ is an orbifold if and only if the greatest common divisor of $w_0,\dots,w_n$ is 1.*
(2) *An orbifold $\mathbb{P}(w)$ is Gorenstein if and only if w_i divides $\sum_{j=0}^{n} w_j$ for any i.*

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In dimension n, the problem of determining all Gorenstein orbifolds $\mathbb{P}(w)$ is equivalent to the problem of *Egyptian fractions*, that is, the number of solutions of $1 = 1/x_0 + \cdots + 1/x_n$ with $1 \leq x_0 \leq \cdots \leq x_n$ (see [19]). Hence, there is a finite number of such $\mathbb{P}(w)$ (only $\mathbb{P}^1$ in dimension 1, three in dimension 2, and 14 in dimension 3). All weighted projective spaces that we will be considering satisfy these two conditions.

§3. Gromov-Witten invariants of the resolution of $|\mathbb{P}(1,3,4,4)|$

3.1. The Mori cone

Let $\mathcal{X}$ be a Gorenstein orbifold with coarse moduli space X. Recall that a resolution of singularities $\rho\colon Z \to X$ is called *crepant* if $\rho^* K_X \cong K_Z$. Assume furthermore that X and Z are projective. Let $\mathrm{N}^+(Z) \subset \mathrm{A}_1(Z;\mathbb{Z})$ be the cone of effective 1-cycles in Z, and set $\mathrm{M}_\rho(Z) := \mathrm{Ker}(\rho_\star) \cap N^+(Z)$, where $\rho_\star\colon \mathrm{A}_\star(Z;\mathbb{Z}) \to \mathrm{A}_\star(X;\mathbb{Z})$ is the morphism of Chow groups induced by ρ. The set $\mathrm{M}_\rho(Z)$ is called the *Mori cone of contracted effective curves.*

Lemma 3.1.1. *Let $\mathbb{P}(w)$ be a Gorenstein orbifold, and let $\rho\colon Z \to |\mathbb{P}(w)|$ be a toric crepant resolution associated to a subdivision Σ' of Σ and the identity morphism of N. Then the cone $\mathrm{M}_\rho(Z)$ is polyhedral.*

Proof. Let $\Sigma'(n-1)$ be the set of $(n-1)$-dimensional cones of Σ'. Then

$$\mathrm{M}_\rho(Z) = \Big\{ \sum_{\nu \in \Sigma'(n-1)} \gamma_\nu [\mathrm{V}(\nu)] \;\Big|\; \gamma_\nu \in \mathbb{N}, \rho_\star\Big(\sum_{\nu \in \Sigma'(n-1)} \gamma_\nu [\mathrm{V}(\nu)]\Big) = 0 \Big\},$$

where, for any $\nu \in \Sigma'(n-1)$, $\mathrm{V}(\nu)$ denotes the rational curve in Z stable under the torus action which is associated to ν, and where $[\mathrm{V}(\nu)]$ is the induced Chow class (see Fulton [11]). Now let $L \in \mathrm{Pic}(|\mathbb{P}(w)|)$ be an ample line bundle. From standard intersection theory, we have (see, e.g., [12])

$$\rho_\star\Big(\sum_{\nu \in \Sigma'(n-1)} \gamma_\nu [\mathrm{V}(\nu)]\Big) = 0 \quad \text{if and only if}$$

$$c_1(\rho^\star L) \cap \Big(\sum_{\nu \in \Sigma'(n-1)} \gamma_\nu [\mathrm{V}(\nu)]\Big) = 0.$$

Since $c_1(\rho^\star L) \cap [\mathrm{V}(\nu)] \geq 0$ for any ν, it follows that

$$\mathrm{M}_\rho(Z) = \Big\{ \sum_{\nu \in \Sigma'(n-1)} \gamma_\nu [\mathrm{V}(\nu)] \;\Big|\; \gamma_\nu \in \mathbb{N}, \rho_\star([\mathrm{V}(\nu)]) = 0 \Big\},$$

hence the claim. ∎

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3.2. Crepant resolution of $|\mathbb{P}(1,3,4,4)|$

The coarse moduli space of $\mathbb{P}(1,3,4,4)$ has a transversal A_3-singularity on the line $\{[0:0:x_2:x_3]\} \cong \mathbb{P}^1$ and an isolated singularity of type $(1/3)(1,1,1)$ at the point $[0:1:0:0]$. We identify the stacky fan (N,β,Σ) with $\big(\mathbb{Z}^3, \{\Lambda(\beta(v_i))\}_{i\in\{0,1,2,3\}}, \Sigma\big)$, where the v_i are defined in (1) and where $\Lambda\colon N \to \mathbb{Z}^3$ is the isomorphism defined by sending $v_0 \mapsto (-3,-4,-4)$, $v_1 \mapsto (1,0,0)$, $v_2 \mapsto (0,1,0)$, and $v_3 \mapsto (0,0,1)$. A crepant resolution of $|\mathbb{P}(1,3,4,4)|$ can be constructed using standard methods in toric geometry. Consider the integral points

$$\begin{aligned}
\mathrm{P}_1 &:= (0,-1,-1) = \frac{3}{4}\Lambda\big(\beta(v_1)\big) + \frac{1}{4}\Lambda\big(\beta(v_0)\big),\\
\mathrm{P}_2 &:= (-1,-2,-2) = \frac{1}{2}\Lambda\big(\beta(v_1)\big) + \frac{1}{2}\Lambda\big(\beta(v_0)\big),\\
\mathrm{P}_3 &:= (-2,-3,-3) = \frac{1}{4}\Lambda\big(\beta(v_1)\big) + \frac{3}{4}\Lambda\big(\beta(v_0)\big),\\
\mathrm{P}_4 &:= (-1,-1,-1) = \frac{1}{3}\Lambda\big(\beta(v_0)\big) + \frac{1}{3}\Lambda\big(\beta(v_2)\big) + \frac{1}{3}\Lambda\big(\beta(v_3)\big),
\end{aligned}$$

and subdivide Σ by inserting the rays generated by P_1, P_2, P_3, and P_4 as shown in Figure 1. Let Σ' be the fan obtained after this subdivision, let Z be

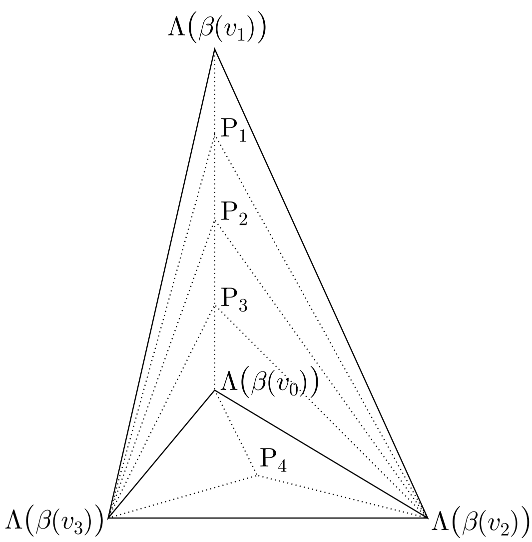

Figure 1: Polytope of $\mathbb{P}(1,3,4,4)$ and of the crepant resolution Z

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the associated toric variety, and let $\rho\colon Z \to |\mathbb{P}(1,3,4,4)|$ be the birational morphism associated to the identity on $\mathbb{Z}^3$. One checks easily that Z is smooth and ρ is crepant. This follows from the existence of a continuous piecewise linear function $|\Sigma| \to \mathbb{R}$, which is linear when restricted to each cone of Σ and associates the value -1 to the minimal lattice points of the rays of Σ' (see [11, Section 3.4]).

REMARK 3.2.1. In [2, Proposition 2.2], we showed that the coarse moduli space $|\mathbb{P}(1,3,4,4)|$ admits a unique crepant resolution, up to isomorphism. So the toric crepant resolution Z constructed above is unique.

3.3. Statement of the main result

By Lemma 3.1.1, the cone $\mathrm{M}_\rho(Z)$ can be directly determined from the combinatorial data Σ and Σ'. In our case, $\mathrm{M}_\rho(Z)$ is generated by four curves: an A_3-chain Γ_1, Γ_2, Γ_3 over the transversal A_3-singularity and one curve Γ_4 over the isolated singularity (see Section 5.2 for the toric equations of these curves). We compute the Gromov-Witten invariants of the crepant resolution Z of genus zero, homology class $\Gamma = d_1\Gamma_1 + d_2\Gamma_2 + d_3\Gamma_3$, and without marked points. We denote by $\overline{\mathcal{M}}_{0,0}(Z,\Gamma)$ the corresponding moduli space. Note that the expected dimension is zero. Our result confirms Perroni's conjecture [16, Conjecture 5.1].

THEOREM 3.3.1. *Let $\rho : Z \to |\mathbb{P}(1,3,4,4)|$ be the crepant resolution of $|\mathbb{P}(1,3,4,4)|$ defined in Section 3.2, and let $\Gamma = d_1\Gamma_1 + d_2\Gamma_2 + d_3\Gamma_3$. Then*

$$\deg[\overline{\mathcal{M}}_{0,0}(Z,\Gamma)]^{\mathrm{vir}} = \begin{cases} 1/d^3 & \text{if } \Gamma = d\sum_{i=\mu}^{\nu}\Gamma_i, \text{ with } 1 \le \mu \le \nu \le 3 \text{ and } d \in \mathbb{N}^*, \\ 0 & \text{otherwise.} \end{cases}$$

§4. Proof of Theorem 3.3.1

To prove Theorem 3.3.1, we use the deformation invariance property of the Gromov-Witten invariants. We define an open neighborhood V of the singular locus

$$\big\{[0:0:x_2:x_3] \in |\mathbb{P}(1,3,4,4)|\big\} \cong \mathbb{P}^1$$

of $|\mathbb{P}(1,3,4,4)|$, we construct an explicit deformation of V, and then we construct a simultaneous resolution. This gives a deformation of $\rho^{-1}(V)$, a neighborhood of the component of the exceptional divisor which lies over $|\mathbb{P}(4,4)|$. We will denote this deformation by $\overline{\mathrm{Graph}(\mu)}_t$, $t \in \Delta$. Then we

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relate the Gromov-Witten invariants of Z we are interested in with some Gromov-Witten invariants of $\overline{\mathrm{Graph}(\mu)}_t$ that we can explicitly compute.

4.1. The neighborhood

By abuse of notation, for any $a \in \mathbb{Z}$, we denote by the same symbol $\mathcal{O}(a)$ the sheaf $\mathcal{O}_{\mathbb{P}^1}(a)$ and the corresponding vector bundle, and we identify $\mathcal{O}(a) \otimes \mathcal{O}(b)$ with $\mathcal{O}(a+b)$ using the canonical isomorphism. For any vector bundle E, we denote by $\underline{0}_E$ its zero section.

The transversal A_3-singularity of $|\mathbb{P}(1,3,4,4)|$ is identified with $\mathbb{P}^1$ by the morphism $[z_0 : z_1] \mapsto [0:0:z_0:z_1]$. We set

$$V_i := \left\{[x_0 : x_1 : x_2 : x_3] \in |\mathbb{P}(1,3,4,4)| \text{ such that } x_i \neq 0\right\}$$

for any $i \in \{0,1,2,3\}$, and we set $V := V_2 \cup V_3 \subset |\mathbb{P}(1,3,4,4)|$.

Consider the bundle morphism

$$\begin{aligned} \psi \colon \mathcal{O}(1) \oplus \mathcal{O}(3) \oplus \mathcal{O}(1) &\longrightarrow \mathcal{O}(4) \\ (\xi, \eta, \zeta) &\longmapsto \xi \otimes \eta - \zeta^{\otimes 4}, \end{aligned}$$

and consider the inverse image under ψ of the zero section of $\mathcal{O}(4)$

$$\psi^{-1}\big(\mathrm{Im}(\underline{0}_{\mathcal{O}(4)})\big) \subset \mathcal{O}(1) \oplus \mathcal{O}(3) \oplus \mathcal{O}(1).$$

LEMMA 4.1.1. *The variety V is isomorphic to $\psi^{-1}(\mathrm{Im}(\underline{0}_{\mathcal{O}(4)}))$.*

Proof. Let $\mathcal{U}_4 \subset \mathbb{C}^\star$ be the group of fourth roots of the unity acting linearly on $\mathbb{C}^3$ with weights $(1,3,0)$. We have the identification

$$\begin{aligned} \mathbb{C}^3/\mathcal{U}_4 &\longrightarrow V_2 \\ [(x_0, x_1, x_3)] &\longmapsto [x_0 : x_1 : 1 : x_3], \end{aligned} \tag{2}$$

where (x_0, x_1, x_3) are coordinates on $\mathbb{C}^3$ and where $[(x_0, x_1, x_3)]$ denotes the equivalence class of the corresponding point. On the other hand, we have the isomorphism

$$\mathbb{C}^3/\mathcal{U}_4 \longrightarrow \mathrm{Spec}\big(\mathbb{C}[s,u,v,w]/(uv - w^4)\big) \tag{3}$$

given by setting $s := x_3, u := x_0^4, v := x_1^4$, and $w := x_0 x_1$. The composition of the inverse of (2) with (3) gives the isomorphism $V_2 \simeq \mathrm{Spec}(\mathbb{C}[s,u,v,w]/(uv - w^4))$. In the same way, by setting $t := x_2, x := x_0^4, y := x_1^4$, and $z :=$

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x_0x_1, we have the isomorphism $V_3 \simeq \mathrm{Spec}(\mathbb{C}[t,x,y,z]/(xy-z^4))$. The affine open subvarieties $V_2, V_3 \subset V$ glue together by means of the ring isomorphism

$$\begin{aligned}\frac{\mathbb{C}[s,\frac{1}{s},u,v,w]}{(uv-w^4)} &\longrightarrow \frac{\mathbb{C}[t,\frac{1}{t},x,y,z]}{(xy-z^4)}\\ s &\longmapsto \frac{1}{t}\\ u &\longmapsto \frac{1}{t}x\\ v &\longmapsto \frac{1}{t^3}y\\ w &\longmapsto \frac{1}{t}z.\end{aligned}$$

On the other hand, consider a trivialization of the bundle $\mathcal{O}(1)\oplus\mathcal{O}(3)\oplus\mathcal{O}(1)$ on $W_0 = \{[z_0,z_1] \in \mathbb{P}^1 \mid z_0 \neq 0\}$. On such a trivialization, the morphism ψ is given by

$$\begin{aligned}W_0 \times \mathbb{C}^3 &\longrightarrow W_0 \times \mathbb{C}\\ (s,v_1,v_2,v_3) &\longmapsto (s, v_1v_2 - v_3^4).\end{aligned}$$

Hence we have that, over W_0, $\psi^{-1}(\mathrm{Im}(\underline{0}_{\mathcal{O}(4)}))$ is $\mathrm{Spec}(\mathbb{C}[s,v_1,v_2,v_3]/(v_1v_2 - v_3^4))$. If we do the same over $W_1 = \{[z_0,z_1] \in \mathbb{P}^1 \mid z_1 \neq 0\}$, we deduce that V and $\psi^{-1}(\mathrm{Im}(\underline{0}_{\mathcal{O}(4)}))$ are a union of the same affine varieties with the same gluing. This proves that they are isomorphic. $\square$

4.2. The deformation

We now construct a deformation of V. The construction is inspired by the theory of deformations of surfaces with A_n-singularities (see Tyurina [20]). Consider the fibration $f\colon V \to \mathbb{P}^1$ defined as the composition of the isomorphism $V \xrightarrow{\sim} \psi^{-1}(\mathrm{Im}(\underline{0}_{\mathcal{O}(4)}))$ in Lemma 4.1.1, followed by the inclusion $\psi^{-1}(\mathrm{Im}(\underline{0}_{\mathcal{O}(4)})) \subset \mathcal{O}(1)\oplus\mathcal{O}(3)\oplus\mathcal{O}(1)$, and then the bundle map. The morphism $f\colon V \to \mathbb{P}^1$ exhibits V as a 3-fold fibered over $\mathbb{P}^1$ with fibers isomorphic to a surface A_3-singularity. Furthermore, the fibration is locally trivial.

The aim is to extend some of the results of Tyurina [20] to V, when viewed as a family of such surfaces with respect to $f\colon V \to \mathbb{P}^1$. Consider the

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bundle morphism

$$\begin{aligned} \chi\colon \mathcal{O}(1)^{\oplus 4} &\longrightarrow \mathcal{O}(1) \\ (\delta_1,\dots,\delta_4) &\longmapsto \delta_1+\cdots+\delta_4, \end{aligned}$$

and set $\mathcal{F} := \chi^{-1}(\mathrm{Im}(\underline{0}_{\mathcal{O}(1)}))$. Then consider the bundle morphism

$$\begin{aligned} \pi\colon \mathcal{O}(1)\oplus\mathcal{O}(3)\oplus\mathcal{O}(1)\oplus\mathcal{F} &\longrightarrow \mathcal{O}(4) \\ (\xi,\eta,\zeta,\delta_1,\dots,\delta_4) &\longmapsto \xi\otimes\eta - \bigotimes_{i=1}^{4}(\zeta+\delta_i), \end{aligned}$$

and set $\mathcal{V}_{\mathcal{F}} := \pi^{-1}(\mathrm{Im}(\underline{0}_{\mathcal{O}(4)}))$. We obtain a Cartesian diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d, "f"] & \mathcal{V}_{\mathcal{F}} \arrow[d, "F"] \\ \mathbb{P}^1 \arrow[r, "\underline{0}_{\mathcal{F}}"] & \mathcal{F} \end{tikzcd}$$

where the arrow F is the composition of the inclusion $\mathcal{V}_{\mathcal{F}} \to \mathcal{O}(1)\oplus\mathcal{O}(3)\oplus\mathcal{O}(1)\oplus\mathcal{F}$ followed by the projection $\mathcal{O}(1)\oplus\mathcal{O}(3)\oplus\mathcal{O}(1)\oplus\mathcal{F}\to\mathcal{F}$. Note that $F\colon \mathcal{V}_{\mathcal{F}}\to\mathcal{F}$ is a family of surfaces. We now construct a simultaneous resolution. Consider the rational map

$$\begin{aligned} \mu\colon \mathcal{V}_{\mathcal{F}} &\dashrightarrow \mathbb{P}\big(\mathcal{O}(1)\oplus\mathcal{O}(1)\big)\times\mathbb{P}\big(\mathcal{O}(1)\oplus\mathcal{O}(2)\big)\times\mathbb{P}\big(\mathcal{O}(1)\oplus\mathcal{O}(3)\big) \\ (\xi,\eta,\zeta,\delta_1,\dots,\delta_4) &\longmapsto (\xi,\zeta+\delta_1)\times\big(\xi,(\zeta+\delta_1)\otimes(\zeta+\delta_2)\big) \\ &\qquad \times\Big(\xi,\bigotimes_{i=1}^{3}(\zeta+\delta_i)\Big), \end{aligned}$$

and let $\mathrm{Graph}(\mu)$ be the graph of μ. We denote by $\overline{\mathrm{Graph}(\mu)}$ the closure of $\mathrm{Graph}(\mu)$ in $\mathcal{V}_{\mathcal{F}}\times\big(\bigtimes_{i=1}^{3}\mathbb{P}(\mathcal{O}(1)\oplus\mathcal{O}(i))\big)$. Let $\mathcal{R}\colon \overline{\mathrm{Graph}(\mu)}\to\mathcal{V}_{\mathcal{F}}$ be the composition of the inclusion $\overline{\mathrm{Graph}(\mu)}\to\mathcal{V}_{\mathcal{F}}\times\big(\bigtimes_{i=1}^{3}\mathbb{P}(\mathcal{O}(1)\oplus\mathcal{O}(i))\big)$ followed by the projection on the first factor $\mathcal{V}_{\mathcal{F}}\times\big(\bigtimes_{i=1}^{3}\mathbb{P}(\mathcal{O}(1)\oplus\mathcal{O}(i))\big)\to\mathcal{V}_{\mathcal{F}}$.

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Lemma 4.2.1. *The diagram*

$$\begin{tikzcd} \overline{\mathrm{Graph}(\mu)} \arrow[r, "\mathcal{R}"] \arrow[d, "F\circ\mathcal{R}"] & \mathcal{V}_{\mathcal{F}} \arrow[d, "F"] \\ \mathcal{F} \arrow[r, "\mathrm{id}"] & \mathcal{F} \end{tikzcd} \tag{4}$$

is a simultaneous resolution of $F\colon \mathcal{V}_{\mathcal{F}} \to \mathcal{F}$.

Proof. The property of being a simultaneous resolution is local in $\mathcal{F}$. Diagram (4) is fibered over $\mathbb{P}^1$. If we restrict it to an open subset of $\mathbb{P}^1$ where $\mathcal{O}(1)$ is trivial, then the assertion is exactly the result obtained by Brieskorn [5]. ∎

Set $\Delta := \mathbb{C}$. For any section $\theta \in H^0(\mathbb{P}^1, \mathcal{F})$, we get a deformation of V parameterized by Δ

$$\begin{tikzcd} V \arrow[r] \arrow[d, "f"] & \mathcal{V}_\theta \arrow[r] \arrow[d, "f_\theta"] & \mathcal{V}_{\mathcal{F}} \arrow[d, "F"] \\ \mathbb{P}^1 \arrow[r] & \mathbb{P}^1\times\Delta \arrow[r, "\Theta"] & \mathcal{F} \end{tikzcd}$$

where $\Theta\colon \mathbb{P}^1\times\Delta \to \mathcal{F}$ sends $([z_0:z_1],t)$ to $t\cdot\theta([z_0:z_1])$, where $\mathcal{V}_\theta$ is defined by the requirement that the diagram is Cartesian, and where the map $\mathbb{P}^1 \to \mathbb{P}^1\times\Delta$ is the inclusion $[z_0:z_1] \mapsto ([z_0:z_1],0)$. The pullback of diagram (4) with respect to Θ gives the diagram

$$\begin{tikzcd} \overline{\mathrm{Graph}(\mu)}_\theta \arrow[r, "\rho_\theta"] \arrow[d] & \mathcal{V}_\theta \arrow[d, "f_\theta"] \\ \mathbb{P}^1\times\Delta \arrow[r, "\mathrm{id}"] & \mathbb{P}^1\times\Delta \end{tikzcd} \tag{5}$$

where ρ_θ is the pullback of $\mathcal{R}$ in (4). Observe that (5) is a simultaneous resolution of $\mathcal{V}_\theta$ over $\mathbb{P}^1\times\Delta$.

4.3. Computation of the invariants

We specialize the previous construction in the case where θ is given as follows. Let $\delta \in H^0(\mathbb{P}^1, \mathcal{O}(1))$ be a nonzero section, set

$$\delta_\ell := \exp\Big(\frac{(2\ell+1)\pi i}{4}\Big)\cdot\delta, \quad \ell\in\{1,\dots,4\},$$

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and define $\theta := (\delta_1, \ldots, \delta_4) \in H^0(\mathbb{P}^1, \mathcal{F})$. For any $t \in \Delta$, we set $\mathcal{V}_t := f_\theta^{-1}(\mathbb{P}^1 \times \{t\})$, $f_t \colon \mathcal{V}_t \to \mathbb{P}^1 \times \{t\}$ as the restriction of f_θ, and we set $\overline{\mathrm{Graph}(\mu)}_t := \rho_\theta^{-1}(\mathcal{V}_t)$, $\rho_t \colon \overline{\mathrm{Graph}(\mu)}_t \to \mathcal{V}_t$ as the restriction of ρ_θ. We have the following commutative diagram

$$\begin{tikzcd}
\rho^{-1}(V) = \overline{\mathrm{Graph}(\mu)}_0 \arrow[r, hook] \arrow[d, "\rho_0=\rho"] & \overline{\mathrm{Graph}(\mu)}_\theta \arrow[d, "\rho_\theta"] & \overline{\mathrm{Graph}(\mu)}_t \arrow[l, hook'] \arrow[d, "\rho_t"] \\
V \arrow[r, hook] \arrow[d, "f_0=f"] & \mathcal{V}_\theta \arrow[d, "f_\theta"] & \mathcal{V}_t \arrow[l, hook'] \arrow[d, "f_t"] \\
\mathbb{P}^1 \times \{0\} \arrow[r, hook] & \mathbb{P}^1 \times \Delta & \mathbb{P}^1 \times \{t\} \arrow[l, hook']
\end{tikzcd}$$

Lemma 4.3.1. *Let δ be a global section of $\mathcal{O}(1) \to \mathbb{P}^1$ that vanishes only at one point. Then, for $t \neq 0$, the variety $\overline{\mathrm{Graph}(\mu)}_t$ has only one connected nodal complete curve of genus zero whose dual graph is of type A_3 and which is contracted by ρ_t (see the diagram above).*

Proof. Without lost of generality, we can assume that δ vanishes only at the point $[1:0]$. Let $W_0 := \{[z_0 : z_1] \in \mathbb{P}^1 \mid z_0 \neq 0\}$. As our bundles are trivial over W_0, the restriction of V over W_0 is given by $W_0 \times \mathbb{V}(xy - z^4) \subset W_0 \times \mathbb{C}^3$. The choice of the δ_ℓ implies that the 3-fold $\mathcal{V}_t$ is given by

$$W_0 \times \mathbb{V}\Big(xy - \prod_{\ell=1}^{4}(z + \delta_\ell t)\Big) = W_0 \times \mathbb{V}\big(xy - z^4 - (t\delta)^4\big) \subset W_0 \times \mathbb{C}^3.$$

By means of f_t, $\mathcal{V}_t$ is viewed as a family of surfaces parameterized by $\mathbb{P}^1$. As $t \neq 0$ and $\delta([1:0]) = 0$, the only singular surface of the family is the surface $f_t^{-1}([1:0] \times \{t\})$, which is a surface with an isolated A_3-singularity. Since $\rho_t \colon \overline{\mathrm{Graph}(\mu)}_t \to \mathcal{V}_t$ is a simultaneous resolution over $\mathbb{P}^1 \times \{t\}$, the fiber $\overline{\mathrm{Graph}(\mu)}_{([1:0],t)}$ is a smooth surface with only one complete connected curve of genus zero whose dual graph is of type A_3 and which is contracted by ρ_t. For any $[z_0 : z_1] \neq [1:0]$, the fiber $\overline{\mathrm{Graph}(\mu)}_{([z_0:z_1],t)}$ is isomorphic to the smooth surface $f_t^{-1}([z_0 : z_1] \times \{t\})$. Hence, the exceptional locus of the resolution $\rho_t \colon \overline{\mathrm{Graph}(\mu)}_t \to \mathcal{V}_t$ has only one connected nodal complete curve of genus zero whose dual graph is of type A_3 and which is contracted by ρ_t. ∎

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Let $\Gamma_1,\Gamma_2,\Gamma_3 \in H_2(\overline{\mathrm{Graph}(\mu)}_t;\mathbb{Z})$ be the homology classes of the components of the connected nodal complete curve of genus zero whose dual graph is of type A_3 and which is contracted by ρ_t. Let us assume that they are numbered in such a way that if Γ_i is the class of $\tilde{\Gamma}_i$, then the intersection $\tilde{\Gamma}_i \cap \tilde{\Gamma}_j$ is empty if $|i-j|>1$. Then Lemma 4.3.1 implies that, for $t \neq 0$, $\overline{\mathrm{Graph}(\mu)}_t$ satisfies the hypothesis of [8, Proposition 2.10]. Therefore, we deduce the following formula:

$$\begin{aligned}\deg\big[\overline{\mathcal{M}}_{0,0}(\overline{\mathrm{Graph}(\mu)}_t,\Gamma)\big]^{\mathrm{vir}} \\ =\begin{cases}1/d^3 & \text{if } \Gamma = d(\Gamma_\mu+\Gamma_{\mu+1}+\cdots+\Gamma_\nu), \text{ for } \mu\le\nu,\\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{6}$$

This formula together with Lemma 4.3.2 completes the proof of Theorem 3.3.1.

Lemma 4.3.2. *Let Γ be as in the statement of Theorem 3.3.1. For any $t\in\Delta$, the following equality holds:*

$$\deg\big[\overline{\mathcal{M}}_{0,0}\big(\overline{\mathrm{Graph}(\mu)}_t,\Gamma\big)\big]^{\mathrm{vir}} = \deg[\overline{\mathcal{M}}_{0,0}(Z,\Gamma)]^{\mathrm{vir}}.$$

Proof. Since Γ is the homology class of a contracted curve, we have an isomorphism of moduli stacks (see [16, Lemma 7.1])

$$\overline{\mathcal{M}}_{0,0}(Z,\Gamma)\simeq\overline{\mathcal{M}}_{0,0}\big(\rho^{-1}(V),\Gamma\big). \tag{7}$$

In particular, the right-hand side moduli stack is proper with projective coarse moduli space. The isomorphism (7) identifies the tangent-obstruction theories used to define the Gromov-Witten invariants, hence the virtual fundamental classes $[\overline{\mathcal{M}}_{0,0}(\rho^{-1}(V),\Gamma)]^{\mathrm{vir}}$ and $[\overline{\mathcal{M}}_{0,0}(Z,\Gamma)]^{\mathrm{vir}}$ have the same degree. Then it is enough to prove that, for any $t\in\Delta$,

$$\deg\big[\overline{\mathcal{M}}_{0,0}(\overline{\mathrm{Graph}(\mu)}_t,\Gamma)\big]^{\mathrm{vir}} = \deg\big[\overline{\mathcal{M}}_{0,0}(\rho^{-1}(V),\Gamma)\big]^{\mathrm{vir}}. \tag{8}$$

Gromov-Witten invariants of projective varieties are invariant under deformation of the target variety. We now explain why this result holds for $\rho^{-1}(V)$ and $\overline{\mathrm{Graph}(\mu)}_t$, even if they are not projective.

Let $q_\theta\colon \overline{\mathrm{Graph}(\mu)}_\theta\to\Delta$ be the composition of $f_\theta\circ\rho_\theta$ in (5), followed by the projection $\mathbb{P}^1\times\Delta\to\Delta$. The morphism q_θ is smooth, as it is a composition of smooth morphisms. Moreover, q_θ factorizes through an embedding

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followed by a projective morphism. To see this, it is enough to prove the same statement for the morphism $F\circ\mathcal{R}\colon \overline{\mathrm{Graph}(\mu)}\to\mathcal{F}$ in (4). By construction, $\overline{\mathrm{Graph}(\mu)}$ is embedded in $(\mathcal{O}(1)\oplus\mathcal{O}(3)\oplus\mathcal{O}(1)\oplus\mathcal{F})\times\big(\bigtimes_{i=1}^{3}\mathbb{P}(\mathcal{O}(1)\oplus\mathcal{O}(i))\big)$; moreover, $F\circ\mathcal{R}$ is the restriction of the projection $(\mathcal{O}(1)\oplus\mathcal{O}(3)\oplus\mathcal{O}(1)\oplus\mathcal{F})\times\big(\bigtimes_{i=1}^{3}\mathbb{P}(\mathcal{O}(1)\oplus\mathcal{O}(i))\big)\to\mathcal{F}$.

Let us now consider the projection $\mathcal{O}(1)\oplus\mathcal{O}(3)\oplus\mathcal{O}(1)\oplus\mathcal{F}\to\mathcal{F}$. Because it has a vector bundle structure over $\mathcal{F}$, then it can be seen as a subbundle of the projective bundle $\mathbb{P}(\mathcal{O}(1)\oplus\mathcal{O}(3)\oplus\mathcal{O}(1)\oplus\mathcal{F}\oplus\mathcal{O}_{\mathcal{F}})\to\mathcal{F}$, and therefore we have that $F\circ\mathcal{R}$ factorizes as the composition of an embedding followed by a projective morphism.

To finish the proof, consider the moduli stack which parameterizes relative stable maps to $q_\theta\colon \overline{\mathrm{Graph}(\mu)}_\theta\to\Delta$ of homology class Γ and genus zero. We denote it by $\overline{\mathcal{M}}_{0,0}(\overline{\mathrm{Graph}(\mu)}_\theta/\Delta,\Gamma)$. As Γ is the class of curves which are contracted by the resolution ρ_θ and $q_\theta\colon \overline{\mathrm{Graph}(\mu)}_\theta\to\Delta$ factorizes through an embedding followed by a projective morphism, Abramovich and Vistoli's theorem [1, Theorem 1.4.1] implies that the moduli space $\overline{\mathcal{M}}_{0,0}(\overline{\mathrm{Graph}(\mu)}_\theta/\Delta,\Gamma)$ is a proper Deligne-Mumford stack. Since the class Γ is contracted by ρ_θ, for any $t\in\Delta$, the fiber at t of the natural morphism $\overline{\mathcal{M}}_{0,0}(\overline{\mathrm{Graph}(\mu)}_\theta/\Delta,\Gamma)\to\Delta$ is the proper Deligne-Mumford stack $\overline{\mathcal{M}}_{0,0}(\overline{\mathrm{Graph}(\mu)}_t,\Gamma)$. Then, by applying the same proof as in [14, Theorem 4.2] to this situation, we get (8). ∎

§5. Application to the cohomological crepant resolution conjecture

5.1. The cohomological crepant resolution conjecture

Ruan's *crepant resolution conjecture* states that when $\rho\colon Z\to X$ is a crepant resolution of the coarse moduli space X of a Gorenstein orbifold $\mathcal{X}$, the (orbifold) quantum cohomology of $\mathcal{X}$ and Z are related by analytic continuation in the quantum parameters. This conjecture was formulated more precisely by Bryan and Graber [6] as an isomorphism of Frobenius manifolds (under some condition), and then further interpreted in full generality by Coates, Iritani, and Tseng [10] as a symplectic transformation between the Givental spaces associated to $\mathcal{X}$ and Z. This symplectic transformation encodes all information on the relationships between the genus zero Gromov-Witten theories of $\mathcal{X}$ and Z. We refer to Iritani [13] for details and references on this still-evolving conjecture. At a lower level, the conjecture implies the *cohomological crepant resolution conjecture*; that is, the quantum corrected

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cohomology ring of Z (deformed by Gromov-Witten invariants computed on curves contracted by ρ) is isomorphic to the orbifold (Chen-Ruan) cohomology ring of $\mathcal{X}$, after evaluation of the quantum parameters to roots of the unity. In the next sections, we check this conjecture on some weighted projective spaces.

We briefly recall the definition of the quantum corrected cohomology ring (see [18]). Assume that $\mathrm{M}_\rho(Z)$ is generated by a finite number of classes of rational curves which are linearly independent over $\mathbb{Q}$. This will be the case for weighted projective spaces by Lemma 3.1.1. Fix a set of such generators $\Gamma_1,\ldots,\Gamma_m$. Then, any $\Gamma \in \mathrm{M}_\rho(Z)$ can be written in a unique way as $\Gamma = \sum_{\ell=1}^m d_\ell \Gamma_\ell$ for some nonnegative integers d_ℓ. Assign a formal variable q_ℓ for each Γ_ℓ so that $\Gamma = \sum_{\ell=1}^m d_\ell\Gamma_\ell \in \mathrm{M}_\rho(Z)$ corresponds to the monomial $q_1^{d_1}\cdots q_m^{d_m}$. The *quantum 3-points function* is by definition

$$(9)\qquad (\alpha_1,\alpha_2,\alpha_3)_{\mathrm{q}}(q_1,\ldots,q_m) := \sum_{d_1,\ldots,d_m>0} \Psi_\Gamma^Z(\alpha_1,\alpha_2,\alpha_3)q_1^{d_1}\cdots q_m^{d_m},$$

where $\alpha_1,\alpha_2,\alpha_3 \in H^\star(X,\mathbb{C})$ and where $\Psi_\Gamma^Z(\alpha_1,\alpha_2,\alpha_3)$ is the Gromov-Witten invariant of Z of genus zero, homology class Γ, and three marked points. One makes the assumption that (9) defines an analytic function of the variables $q_1,\ldots,q_m$ on some region of the complex space $\mathbb{C}^m$. The *quantum corrected cup product* $\alpha_1 *_\rho \alpha_2$ of two classes $\alpha_1,\alpha_2 \in H^\star(Z;\mathbb{C})$ is then defined by requiring that, for all $\alpha_3 \in H^\star(Z;\mathbb{C})$, one has

$$\int_Z (\alpha_1 *_\rho \alpha_2)\alpha_3 = \int_Z \alpha_1\alpha_2\alpha_3 + (\alpha_1,\alpha_2,\alpha_3)_{\mathrm{q}}(q_1,\ldots,q_m).$$

The resulting associative, skew-symmetric, graded ring $(H^\star(Z;\mathbb{C}), *_\rho)$ is the *quantum corrected cohomology ring* with quantum parameters specialized at $(q_1,\ldots,q_m)$. It is also denoted by $H_\rho^\star(Z;\mathbb{C})(q_1,\ldots,q_m)$.

Let us fix the notation used for the computations below. Let $\rho\colon Z \to |\mathbb{P}(w)|$ be a crepant resolution defined by a subdivision Σ' of the fan Σ of $|\mathbb{P}(w)|$. Set $H := c_1(\mathcal{O}_{\mathbb{P}(w)}(1)) \in H^2(\mathbb{P}(w);\mathbb{C})$ and $h := \rho^\star H \in H^2(Z;\mathbb{C})$. For $i \in \{0,\ldots,n\}$, we denote by $b_i \in H^2(Z;\mathbb{C})$ the first Chern class of the line bundle associated to the torus-invariant divisor corresponding to the ray of Σ' generated by $\beta(v_i)$, and similarly $e_1,\ldots,e_d \in H^2(Z;\mathbb{C})$ for the rays in $\Sigma'(1)\setminus\Sigma(1)$. Since ρ is crepant, we have $h = (1/\sum_{i=0}^n w_i)(\sum_{i=0}^n b_i + \sum_{j=1}^d e_j)$ (see Fulton [11]). Since H is an ample line bundle (see [11, Section 3.4]), Lemma 3.1.1 shows that the Mori cone $\mathrm{M}_\rho(Z)$ is generated by the set

$$(10)\qquad \big\{[V(\nu)] \;\big|\; h\cap[V(\nu)] = 0, \nu \in \Sigma'(n-1)\setminus\Sigma(n-1)\big\}.$$

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5.2. The case of $|\mathbb{P}(1,3,4,4)|$

Consider the crepant resolution $\rho\colon Z \to |\mathbb{P}(1,3,4,4)|$ defined in Section 3.2.

THEOREM 5.2.1. *For $(q_1,q_2,q_3,q_4) \in \{(\mathrm{i},\mathrm{i},\mathrm{i},0),(-\mathrm{i},-\mathrm{i},-\mathrm{i},0)\}$, there is an explicit ring isomorphism*

$$H^\star_\rho(Z;\mathbb{C})(q_1,q_2,q_3,q_4) \cong H^\star_{\mathrm{CR}}\big(\mathbb{P}(1,3,4,4);\mathbb{C}\big)$$

which is an isometry with respect to the Poincaré pairing on $H^\star_\rho(Z;\mathbb{C})(q_1,q_2,q_3,q_4)$ and with respect to the Chen-Ruan pairing on $H^\star_{\mathrm{CR}}(\mathbb{P}(1,3,4,4);\mathbb{C})$.

Proof. A toric computation shows that the cohomology ring of Z is isomorphic to the quotient of the polynomial ring $\mathbb{C}[h,e_1,e_2,e_3,e_4]$ by the ideal generated by

$$\begin{aligned}
3he_4, e_1e_3, e_1e_4, e_2e_4, e_3e_4,\\
e_1^2 - 10he_1 - 4he_2 - 2he_3 + 24h^2,\\
e_1e_2 + 3he_1 + 2he_2 + he_3 - 12h^2,\\
e_2^2 - 6he_1 - 12he_2 - 2he_3 + 24h^2,\\
e_2e_3 + 3he_1 + 6he_2 + he_3 - 12h^2,\\
e_3^2 - 6he_1 - 12he_2 - 14he_3 + 24h^2,\\
16h^2e_1, 16h^2e_2, 16h^2e_3, 16h^3 - \frac{1}{27}e_4^3.
\end{aligned}$$

We fix the following basis of the vector space $H^\star(Z;\mathbb{C})$:

$$1, h, e_1, e_2, e_3, e_4, h^2, he_1, he_2, he_3, e_4^2, h^3.$$

Let $\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4 \in M_\rho(Z)$ be the generators defined in Section 3.3 and whose equations are $\Gamma_1 := \mathrm{PD}(4he_1)$, $\Gamma_2 := \mathrm{PD}(4he_2)$, $\Gamma_3 := \mathrm{PD}(4he_3)$, and $\Gamma_4 := \mathrm{PD}(-(1/3)e_4^2)$. We now give a presentation of the quantum corrected cohomology ring $H^\star_\rho(Z;\mathbb{C})(q_1,q_2,q_3,0)$. First, notice that any curve of homology class $d_4\Gamma_4$ is disjoint from any other curve of class $d_1\Gamma_1 + d_2\Gamma_2 + d_3\Gamma_3$; in other words, $\overline{\mathcal{M}}_{0,0}(Z,\Gamma)$ is empty if $\Gamma = \sum_{\ell=1}^4 d_\ell\Gamma_\ell$ with $d_4\cdot(d_1 + d_2 + d_3) \neq 0$. From the degree axiom, it follows that we need to consider only Gromov-Witten invariants $\Psi^Z_\Gamma(\alpha_1,\alpha_2,\alpha_3)$ with $\alpha_i \in H^2(Z;\mathbb{C})$, $i \in \{1,2,3\}$. Finally, by applying the divisor axiom, we deduce the following expression

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for the quantum 3-points function:

$$\begin{aligned}
&\langle \alpha_1,\alpha_2,\alpha_3\rangle_{\mathrm{q}}(q_1,q_2,q_3,q_4)\\
&= \sum_{d_1,d_2,d_3>0} \Big(\prod_{i=1}^{3}\int_{\sum_{\ell=1}^{3} d_\ell\Gamma_\ell}\alpha_i\Big)\deg\Big[\overline{\mathcal{M}}_{0,0}\Big(Z,\sum_{\ell=1}^{3} d_\ell\Gamma_\ell\Big)\Big]^{\mathrm{vir}} q_1^{d_1}q_2^{d_2}q_3^{d_3}\\
&\quad + \sum_{d_4>0}\Big(\prod_{i=1}^{3}\int_{d_4\Gamma_4}\alpha_i\Big)\deg[\overline{\mathcal{M}}_{0,0}(Z,d_4\Gamma_4)]^{\mathrm{vir}} q_4^{d_4}.
\end{aligned}$$

Since $\int_{\Gamma_\ell} h = 0$ for any $\ell\in\{1,2,3,4\}$, one has $h *_\rho \alpha = h\alpha$ for any $\alpha\in H^\star(Z;\mathbb{C})$, and similarly

$$e_i *_\rho e_4 = \begin{cases} e_ie_4 = 0 & \text{if } i\neq 4,\\ \epsilon(q_4)e_4^2 & \text{otherwise,}\end{cases}$$

for some function $\epsilon(q_4)$ such that $\epsilon(0)=1$.

Since in the isomorphism of rings that we will define we put $q_4=0$, we only consider classes $\Gamma = d_1\Gamma_1 + d_2\Gamma_2 + d_3\Gamma_3$ for $d_i\in\mathbb{N}$. We set $\Gamma_{\mu\nu} := \Gamma_\mu+\cdots+\Gamma_\nu$ for $1\le\mu\le\nu\le 3$. Using Theorem 3.3.1, we get

$$\deg[\overline{\mathcal{M}}_{0,0}(Z,\Gamma)]^{\mathrm{vir}} = \begin{cases} 1/d^3 & \text{if } \Gamma = d\Gamma_{\mu\nu} \text{ for } 1\le\mu\le\nu\le 3 \text{ and } d\in\mathbb{N}^*,\\ 0 & \text{otherwise.}\end{cases}$$

Hence, the remaining part of the multiplicative table of $H^\star_\rho(Z;\mathbb{C})(q_1,q_2,q_3,0)$ is as follows:

$$\begin{aligned}
e_1 *_\rho e_1 = -24h^2 &+ \Big(10 + 16\frac{q_1}{1-q_1} + 4\frac{q_1q_2}{1-q_1q_2} + 4\frac{q_1q_2q_3}{1-q_1q_2q_3}\Big)he_1\\
&+ \Big(4 + 4\frac{q_2}{1-q_2} + 4\frac{q_1q_2}{1-q_1q_2} + 4\frac{q_2q_3}{1-q_2q_3} + 4\frac{q_1q_2q_3}{1-q_1q_2q_3}\Big)he_2\\
&+ \Big(2 + 4\frac{q_2q_3}{1-q_2q_3} + 4\frac{q_1q_2q_3}{1-q_1q_2q_3}\Big)he_3,
\end{aligned}$$

$$\begin{aligned}
e_1 *_\rho e_2 = 12h^2 &+ \Big(-3 - 8\frac{q_1}{1-q_1} + 4\frac{q_1q_2}{1-q_1q_2}\Big)he_1\\
&+ \Big(-2 - 8\frac{q_2}{1-q_2} + 4\frac{q_1q_2}{1-q_1q_2} - 4\frac{q_2q_3}{1-q_2q_3}\Big)he_2\\
&+ \Big(-1 - 4\frac{q_2q_3}{1-q_2q_3}\Big)he_3,
\end{aligned}$$

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$$\begin{aligned}
e_1 *_\rho e_3 = &\Big(-4\frac{q_1q_2}{1-q_1q_2}+4\frac{q_1q_2q_3}{1-q_1q_2q_3}\Big)he_1\\
&+\Big(4\frac{q_2}{1-q_2}-4\frac{q_1q_2}{1-q_1q_2}-4\frac{q_2q_3}{1-q_2q_3}+4\frac{q_1q_2q_3}{1-q_1q_2q_3}\Big)he_2\\
&+\Big(-4\frac{q_2q_3}{1-q_2q_3}+4\frac{q_1q_2q_3}{1-q_1q_2q_3}\Big)he_3,
\end{aligned}$$

$$\begin{aligned}
e_2 *_\rho e_2 = -24h^2 &+ \Big(6+4\frac{q_1}{1-q_1}+4\frac{q_1q_2}{1-q_1q_2}\Big)he_1\\
&+\Big(12+16\frac{q_2}{1-q_2}+4\frac{q_1q_2}{1-q_1q_2}+4\frac{q_2q_3}{1-q_2q_3}\Big)he_2\\
&+\Big(2+4\frac{q_3}{1-q_3}+4\frac{q_2q_3}{1-q_2q_3}\Big)he_3,
\end{aligned}$$

$$\begin{aligned}
e_2 *_\rho e_3 = 12h^2 &+ \Big(-3-4\frac{q_1q_2}{1-q_1q_2}\Big)he_1\\
&+\Big(-6-8\frac{q_2}{1-q_2}-4\frac{q_1q_2}{1-q_1q_2}+4\frac{q_2q_3}{1-q_2q_3}\Big)he_2\\
&+\Big(-1-8\frac{q_3}{1-q_3}+4\frac{q_2q_3}{1-q_2q_3}\Big)he_3,
\end{aligned}$$

$$\begin{aligned}
e_3 *_\rho e_3 = -24h^2 &+ \Big(6+4\frac{q_1q_2}{1-q_1q_2}+4\frac{q_1q_2q_3}{1-q_1q_2q_3}\Big)he_1\\
&+\Big(12+4\frac{q_2}{1-q_2}+4\frac{q_1q_2}{1-q_1q_2}+4\frac{q_2q_3}{1-q_2q_3}+4\frac{q_1q_2q_3}{1-q_1q_2q_3}\Big)he_2\\
&+\Big(14+16\frac{q_3}{1-q_3}+4\frac{q_2q_3}{1-q_2q_3}+4\frac{q_1q_2q_3}{1-q_1q_2q_3}\Big)he_3.
\end{aligned}$$

To compute the Chen-Ruan cohomology ring $H^\star_{\mathrm{CR}}(\mathcal{X};\mathbb{C})$ (here $\mathcal{X}=\mathbb{P}(1,3,4,4)$), we follow Boissière, Mann, and Perroni [3]. The twisted sectors are indexed by the set $\mathrm{T} := \big\{\exp(2\pi \mathrm{i}\gamma) \mid \gamma \in \{0,1/3,2/3,1/4,1/2,3/4\}\big\}$. For $g\in T$, written $g=\exp(2\pi\mathrm{i}\gamma)$ with $\gamma\in\{0,1/3,2/3,1/4,1/2,3/4\}$, the *age* of g is given by the formula

$$\operatorname{age}(g)=\{\gamma\}+\{3\gamma\}+\{4\gamma\}+\{4\gamma\},$$

where $\{\cdot\}$ denotes the fractional part. For any $g\in\mathrm{T}$, $\mathcal{X}_{(g)}$ is a weighted projective space. Setting $\mathrm{I}(g):=\{i\in\{0,1,2,3\}\mid g^{w_i}=1\}$, one has $\mathcal{X}_{(g)}=$

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$\mathbb{P}(w_{\mathrm{I}(g)})$, where $w_{\mathrm{I}(g)} = (w_i)_{i\in \mathrm{I}(g)}$. The inertia stack is the disjoint union of the twisted sectors

$$\mathrm{I}\mathcal{X} = \bigsqcup_{g\in \mathrm{T}} \mathbb{P}(w_{\mathrm{I}(g)}).$$

As a vector space, the Chen-Ruan cohomology is the cohomology of the inertia stack; that is, the graded structure is obtained by shifting the degree of the cohomology of any twisted sector by twice the corresponding age. We have

$$\begin{aligned} H^p_{\mathrm{CR}}(\mathcal{X};\mathbb{C}) &= \bigoplus_{g\in \mathrm{T}} H^{p-2\,\mathrm{age}(g)}\big(\mathbb{P}(w_{\mathrm{I}(g)});\mathbb{C}\big) \\ &= H^p\big(\mathbb{P}(1,3,4,4);\mathbb{C}\big) \oplus H^{p-2}\big(\mathbb{P}(3);\mathbb{C}\big) \oplus H^{p-4}\big(\mathbb{P}(3);\mathbb{C}\big) \\ &\quad \oplus H^{p-2}\big(\mathbb{P}(4,4);\mathbb{C}\big) \oplus H^{p-2}\big(\mathbb{P}(4,4);\mathbb{C}\big) \oplus H^{p-2}\big(\mathbb{P}(4,4);\mathbb{C}\big). \end{aligned} \tag{11}$$

A basis of $H^\star_{\mathrm{CR}}(\mathcal{X};\mathbb{C})$ is easily obtained in the following way, set

$$H, E_1, E_2, E_3, E_4 \in H^\star_{\mathrm{CR}}(\mathcal{X};\mathbb{C})$$

as the image of $c_1(\mathcal{O}_{\mathcal{X}}(1)) \in H^2(\mathcal{X};\mathbb{C})$, $1 \in H^0(\mathcal{X}_{(\exp(\pi\mathrm{i}/2))};\mathbb{C})$, $1 \in H^0(\mathcal{X}_{(\exp(\pi\mathrm{i}))};\mathbb{C})$, $1 \in H^0(\mathcal{X}_{(\exp(\pi\mathrm{i}3/2))};\mathbb{C})$, and $1 \in H^0(\mathcal{X}_{(\exp(2\pi\mathrm{i}/3))};\mathbb{C})$, respectively, under the inclusion $H^{\star-2\,\mathrm{age}(g)}(\mathbb{P}(w_{\mathrm{I}(g)})) \to H^\star_{\mathrm{CR}}(\mathcal{X})$ determined by the decomposition (11). As a $\mathbb{C}$-algebra, the Chen-Ruan cohomology ring is generated by H, E_1, E_2, E_3, E_4 with the following relations (see [3]):

$$\begin{gathered} HE_4, E_1E_1 - 3HE_2, E_1E_2 - 3HE_3, E_1E_3 - 3H^2, \\ E_2E_2 - 3H^2, E_2E_3 - HE_1, E_3E_3 - HE_2, 16H^3 - E_4^3, \\ H^2E_1, H^2E_2, H^2E_3, E_1E_4, E_2E_4, E_3E_4. \end{gathered}$$

We see that the following elements form a basis of $H^\star_{\mathrm{CR}}(\mathcal{X};\mathbb{C})$ which we fix for the rest of the proof:

$$1, H, E_1, E_2, E_3, E_4, H^2, HE_1, HE_2, HE_3, E_4^2, H^3.$$

(Note that the elements of our basis are different from those used in [3] by a combinatorial factor.)

For cohomology classes α_1 and α_2, the product $\alpha_1 *_\rho \alpha_2 \in H^\star_\rho(Z;\mathbb{C})(q_1, q_2, q_3, 0)$ differs from the usual cup product only when $\alpha_1, \alpha_2 \in \{e_1, e_2, e_3\}$. We

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now set $q_1 = q_2 = q_3 = \mathrm{i}$, and we compute $e_i *_\rho e_j$ in the choosen basis of $H^\star(Z;\mathbb{C})$ to get

$$\begin{aligned}
e_1 *_\rho e_1 &= -24h^2 + (-2+6\mathrm{i})he_1 - 4he_2 + (-2-2\mathrm{i})he_3,\\
e_1 *_\rho e_2 &= 12h^2 + (-1-4\mathrm{i})he_1 + (2-4\mathrm{i})he_2 + he_3,\\
e_1 *_\rho e_3 &= -2\mathrm{i}he_1 - 2\mathrm{i}he_3,\\
e_2 *_\rho e_2 &= -24h^2 + (2+2\mathrm{i})he_1 + 8\mathrm{i}he_2 + (-2+2\mathrm{i})he_3,\\
e_2 *_\rho e_3 &= 12h^2 - he_1 + (-2-4\mathrm{i})he_2 + (1-4\mathrm{i})he_3,\\
e_3 *_\rho e_3 &= -24h^2 + (2-2\mathrm{i})he_1 + 4he_2 + (2+6\mathrm{i})he_3.
\end{aligned}$$

We now define a linear map

$$H^\star_\rho(Z;\mathbb{C})(\mathrm{i},\mathrm{i},\mathrm{i},0) \to H^\star_{\mathrm{CR}}\big(\mathbb{P}(1,3,4,4);\mathbb{C}\big) \tag{12}$$

as follows. We send

$$\begin{pmatrix} h\\ e_1\\ e_2\\ e_3\\ e_4 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & -\sqrt{2} & -2\mathrm{i} & \sqrt{2} & 0\\ 0 & -\mathrm{i}\sqrt{2} & 2\mathrm{i} & -\mathrm{i}\sqrt{2} & 0\\ 0 & \sqrt{2} & -2\mathrm{i} & -\sqrt{2} & 0\\ 0 & 0 & 0 & 0 & 3\exp(\frac{2\pi\mathrm{i}}{3}) \end{pmatrix} \begin{pmatrix} H\\ E_1\\ E_2\\ E_3\\ E_4 \end{pmatrix}$$

(the image of the other elements of the basis is uniquely determined by requiring that (12) be a ring isomorphism). A direct computation shows that (12) is a ring isomorphism and that it is an isometry with respect to the inner products given by the Poincaré duality and the Chen-Ruan pairing, respectively.

The case $q_1 = q_2 = q_3 = -\mathrm{i}$ and $q_4 = 0$ is analogous to the previous one. We define a linear map

$$H^\star_\rho(Z;\mathbb{C})(-\mathrm{i},-\mathrm{i},-\mathrm{i},0) \to H^\star_{\mathrm{CR}}\big(\mathbb{P}(1,3,4,4);\mathbb{C}\big) \tag{13}$$

by sending

$$\begin{pmatrix} h\\ e_1\\ e_2\\ e_3\\ e_4 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & -\sqrt{2} & 2\mathrm{i} & \sqrt{2} & 0\\ 0 & \mathrm{i}\sqrt{2} & -2\mathrm{i} & \mathrm{i}\sqrt{2} & 0\\ 0 & \sqrt{2} & 2\mathrm{i} & -\sqrt{2} & 0\\ 0 & 0 & 0 & 0 & 3\exp(\frac{2\pi\mathrm{i}}{3}) \end{pmatrix} \begin{pmatrix} H\\ E_1\\ E_2\\ E_3\\ E_4 \end{pmatrix}$$

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and extending to the remaining part of the basis (in the unique way) such that the resulting map is a ring isomorphism. Also, in this case a direct computation shows that (13) is a ring isomorphism and that it respects the inner pairings. ∎

REMARK 5.2.2. The values $q_1 = q_2 = q_3 = \pm i$ were proposed in [16, Conjecture 1.9]: thus Theorem 5.2.1 agrees with this conjecture. The change of variables is inspired by those of Nahm and Wendland [15] (see also [7], [9]).

The fact that one quantum parameter can be put to zero to get the conjectured isomorphism is strange in regard to the conjecture, and somehow unsatisfactory. The problem concerns the computation of the function $\epsilon(q_4)$ in the proof. In [2], we solve this problem so that the fourth quantum parameter can be put to 1.

5.3. The case of $|\mathbb{P}(1,\ldots,1,n)|$

The quantum parameter put to zero in the case of $\mathbb{P}(1,3,4,4)$ corresponds to the isolated singularity $(1/3)(1,1,1)$. Surprisingly, this phenomenon can be observed in any dimension by considering the n-dimensional weighted projective space $\mathbb{P}(1,\ldots,1,n)$ whose coarse moduli space has an isolated singularity of type $(1/n)(1,\ldots,1)$ at the point $[0:\ldots:0:1]$. In this example, a crepant resolution can be constructed for any n.

We identify the stacky fan (N,β,Σ) defined in (1) with $\big(\mathbb{Z}^n, \{\Lambda(\beta(v_i))\}_{i=0}^n, \Sigma\big)$ by means of the isomorphism $\Lambda : N \to \mathbb{Z}^n$ defined by sending v_0 to $(-1,\ldots,-1,-n)$ and v_i to the ith vector of the standard basis of $\mathbb{Z}^n$, for $i \in \{1,\ldots,n\}$. The crepant resolution is defined as follows. Consider the ray generated by

$$\mathrm{P} := (0,\ldots,0,-1) = \frac{1}{n}\sum_{i=0}^{n-1} \Lambda\big(\beta(v_i)\big),$$

and let Σ' be the fan obtained from Σ (by refinement) by replacing the cone generated by $\Lambda(\beta(v_0)),\ldots,\Lambda(\beta(v_{n-1}))$ with the cones generated by P and the rays $\Lambda(\beta(v_0)),\ldots,\widehat{\Lambda(\beta(v_i))},\ldots,\Lambda(\beta(v_{n-1}))$ for any $i \in \{0,\ldots,n-1\}$. We draw as an example the polytope for the case $n=3$ in Figure 2. Define Z to be the toric variety associated to Σ', and define $\rho\colon Z \to |\mathbb{P}(1,\ldots,1,n)|$ to be the morphism associated to the identity in $\mathbb{Z}^n$. We have the following.

PROPOSITION 5.3.1. *For any $n \geq 2$, there is an explicit ring isomorphism*

$$\mathrm{H}^\star(Z;\mathbb{C}) \cong \mathrm{H}^\star_{\mathrm{CR}}\big(\mathbb{P}(1,\ldots,1,n);\mathbb{C}\big).$$

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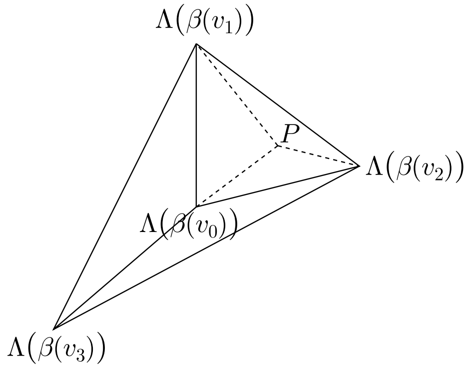

Figure 2: Polytope of $\mathbb{P}(1,1,1,3)$ and a crepant resolution

Proof. One computes that $H^\star(Z;\mathbb{C}) \cong \mathbb{C}[b_0,\dots,b_n,e]/I$, where I is generated by

$$-b_0+b_i \quad \text{for } 1 \le i \le n-1,$$
$$-nb_0-e+b_n, eb_n, b_0\cdots b_{n-1}.$$

With $h := (1/2n)(b_0+\cdots+b_n+e) = b_0+(1/n)e$, one gets a better presentation as

$$H^\star(Z;\mathbb{C}) \cong \mathbb{C}[h,e]\Big/\Big\langle h^n+(-1)^n\Big(\frac{e}{n}\Big)^n, he\Big\rangle.$$

The Mori cone $\mathrm{M}_\rho(Z)$ is generated by one class $\Gamma_1 := \mathrm{PD}((h-(e/n))^{n-2}e)$. We will set the quantum parameter q_1 to zero so that we do not have to compute any nontrivial Gromov-Witten invariants.

Concerning the Chen-Ruan cohomology ring of $\mathcal{X} := \mathbb{P}(1,\dots,1,n)$, the twisted sectors are indexed by the set $\mathrm{T} = \{\exp((2\pi\mathrm{i}k/n)) \mid k \in \{0,\dots,n-1\}\}$. For $g \in \mathrm{T}\setminus\{1\}$, one has $\mathcal{X}_{(g)} \cong \mathbb{P}(n)$, whereas $\mathcal{X}_{(1)} \cong \mathcal{X}$. As a vector space, we have

$$H^\star_{\mathrm{CR}}(\mathcal{X};\mathbb{C}) := \bigoplus_{g\in T} H^{\star-2\,\mathrm{age}(g)}(\mathcal{X}_{(g)}). \tag{14}$$

Let

$$H, E_1 \in H^2_{\mathrm{CR}}(\mathcal{X};\mathbb{C})$$

be the image of $c_1(\mathcal{O}_{\mathcal{X}}(1)) \in H^2(\mathcal{X};\mathbb{C})$ and $1 \in H^0(\mathcal{X}_{(\exp(2\pi\mathrm{i}/n))};\mathbb{C})$, respectively, with respect to the inclusion $H^{\star-2\,\mathrm{age}(g)}(\mathcal{X}_{(g)}) \to H^\star_{\mathrm{CR}}(\mathcal{X})$ determined

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by (14). Then we have the following presentation:

$$H^{\star}_{\mathrm{CR}}\big(\mathbb{P}(1,\ldots,1,n);\mathbb{C}\big) \cong \mathbb{C}[H,E_1]/\langle H^n - (E_1)^n, HE_1\rangle.$$

The ring isomorphism

$$H^{\star}_{\mathrm{CR}}\big(\mathbb{P}(1,\ldots,1,n);\mathbb{C}\big) \xrightarrow{\sim} H^{\star}(Z;\mathbb{C})$$

is obtained by mapping $H \mapsto h$ and $E_1 \mapsto -\exp(\mathrm{i}\pi/n)(e/n)$. ∎

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