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THE LOCAL h-POLYNOMIALS OF CLUSTER SUBDIVISIONS HAVE ONLY REAL ZEROS

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Abstract

Athanasiadis ['A survey of subdivisions and local h-vectors', in *The Mathematical Legacy of Richard P. Stanley* (American Mathematical Society, Providence, RI, 2017), 39–51] asked whether the local h-polynomials of type A cluster subdivisions have only real zeros. We confirm this conjecture and prove that the local h-polynomials for all the Cartan–Killing types have only real roots. Our proofs use multiplier sequences and Chebyshev polynomials of the second kind.

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1. Introduction

In this paper, we answer a question of Athanasiadis [1] and show that the local h-polynomials of type A cluster subdivisions have only real zeros. We prove this result for all the Cartan–Killing types.

We first give an overview of local h-polynomials. The local h-polynomials were introduced by Stanley [12] in his study of the face enumeration of subdivisions of complexes. Let V be an n-element vertex set. Given a simplicial subdivision Γ of the abstract simplex 2^V, the local h-polynomial $\ell_V(\Gamma, x)$ is defined as an alternating sum of the h-polynomials of the restrictions of Γ to the faces of 2^V, namely,

$$\ell_V(\Gamma, x) = \sum_{F\subseteq V} (-1)^{n-|F|}\, h(\Gamma_F, x),$$

where $h(\Gamma_F, x)$ is the h-polynomial of Γ_F. Stanley [12] showed that $\ell_V(\Gamma, x)$ has nonnegative and symmetric coefficients, so that it can be expressed as

$$\ell_V(\Gamma, x) = \sum_{i=0}^{\lfloor n/2\rfloor} \xi_i\, x^i (1+x)^{n-2i}. \tag{1.1}$$

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In his survey, Athanasiadis [1] asked whether local h-polynomials for several families of subdivisions have only real zeros.

This paper is concerned with cluster subdivisions. Let I be an n-element set and $\Phi = \{a_i : i \in I\}$ be a root system. The cluster complex $\Delta(\Phi)$, studied by Fomin and Zelevinsky [6, 7], is a simplicial complex on the vertex set of positive roots and negative simple roots. The positive cluster complex $\Delta_+(\Phi)$ is the restriction of $\Delta(\Phi)$ to the positive roots. It naturally defines a geometric subdivision of the simplex on the vertex set of simple roots of Φ, the so-called *cluster subdivision* $\Gamma(\Phi)$. The local h-polynomial $\ell_I(\Gamma(\Phi), x)$ is given by

$$\ell_I(\Gamma(\Phi), x) = \sum_{J\subseteq I} (-1)^{|I\setminus J|} h(\Delta_+(\Phi_J), x),$$

where Φ_J is the parabolic root subsystem of Φ with respect to J.

Although closed form expressions for the local h-polynomials of type A and type B are not yet known, the following result of Athanasiadis and Savvidou [2] gives explicit formulae for the numbers ξ_i defined by (1.1).

Lemma 1.1 [2, Theorem 1.2]. *Let Φ be an irreducible root system of rank n and Cartan–Killing type $\mathcal{X}$ and let $\xi_i(\Phi)$ be the integers uniquely defined by (1.1). Then $\xi_0(\Phi) = 0$ and*

$$\xi_i(\Phi) = \begin{cases} \dfrac{1}{n-i+1}\dbinom{n}{i}\dbinom{n-i-1}{i-1} & \text{if } \mathcal{X} = A_n, \\ \dbinom{n}{i}\dbinom{n-i-1}{i-1} & \text{if } \mathcal{X} = B_n, \\ \dfrac{n-2}{i}\dbinom{2i-2}{i-1}\dbinom{n-2}{2i-2} & \text{if } \mathcal{X} = D_n \end{cases}$$

for $1 \le i \le \lfloor n/2 \rfloor$. Moreover,

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \xi_i(\Phi)x^i = \begin{cases} (m-2)x & \text{if } \mathcal{X} = I_2(m), \\ 8x & \text{if } \mathcal{X} = H_3, \\ 42x + 40x^2 & \text{if } \mathcal{X} = H_4, \\ 10x + 9x^2 & \text{if } \mathcal{X} = F_4, \\ 7x + 35x^2 + 13x^3 & \text{if } \mathcal{X} = E_6, \\ 16x + 124x^2 + 112x^3 & \text{if } \mathcal{X} = E_7, \\ 44x + 484x^2 + 784x^3 + 120x^4 & \text{if } \mathcal{X} = E_8. \end{cases}$$

Athanasiadis [1] asked the following specific question.

Question 1.2. *Do the local h-polynomials of type A cluster subdivisions of the simplex have only real zeros?*

In this paper, we answer this question and prove that the local h-polynomials for all the cluster types have only real roots.

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Theorem 1.3. *For any irreducible root system, the local h-polynomial of the cluster subdivision of the simplex has only real zeros.*

The remainder of this paper is organised as follows. In Section 2, we give an overview of the theory of multiplier sequences. In Section 3, we present our proof of Theorem 1.3.

2. Preliminaries

A sequence of real numbers $\{\lambda_k\}_{k=0}^{\infty}$ is a multiplier sequence if, for every polynomial $\sum_{k=0}^{n} a_k z^k$ whose zeros are all real, the polynomial $\sum_{k=0}^{n} \lambda_k a_k z^k$ is either identically zero or has only real zeros. In this section, we list some properties of multiplier sequences which will be used later. For a complete introduction to multiplier sequences, we refer the reader to [4, 5, 11].

An entire function $\phi(x) = \sum_{i=0}^{\infty} \gamma_k x^k/k!$ is in the *Laguerre–Pólya class*, written $\phi \in \mathscr{L}\text{–}\mathscr{P}$, if it has the form

$$\phi(x) = cx^m e^{-ax^2+bx} \prod_{k=1}^{\omega} \Big(1 + \frac{x}{x_k}\Big) e^{-x/x_k},$$

where $0 \le \omega \le \infty$, m is a nonnegative integer, $a \ge 0$, $b, c, x_k \in \mathbb{R}$, $x_k \ne 0$ and $\sum_{k=1}^{\omega} 1/x_k^2 < \infty$. Let $\mathscr{L}\text{–}\mathscr{P}^+$ denote the set of functions in the Laguerre–Pólya class with nonnegative coefficients and $\mathscr{L}\text{–}\mathscr{P}(-\infty, 0]$ denote the set of functions in the Laguerre–Pólya class that have only nonpositive zeros. A remarkable property is that an entire function is in the Laguerre–Pólya class if and only if it is a locally uniform limit of real polynomials which have only real zeros.

A complete characterisation of multiplier sequences was given by Pólya and Schur.

Theorem 2.1 (Pólya–Schur [10]). *Let $\{\lambda_k\}_{k=0}^{\infty}$ be a sequence of real numbers. The following statements are equivalent:*

(i) *$\{\lambda_k\}_{k=0}^{\infty}$ is a multiplier sequence;*

(ii) *for any nonnegative integer n, either the polynomial $\sum_{k=0}^{n} \binom{n}{k} \lambda_k x^k$ has only real zeros of the same sign or it is identically zero;*

(iii) *either $\sum_{k=0}^{\infty} \lambda_k x^k/k!$ or $\sum_{k=0}^{\infty} (-1)^k \lambda_k x^k/k!$ belongs to $\mathscr{L}\text{–}\mathscr{P}^+$.*

For convenience, we let $1/k!$ be zero whenever k is a negative integer. By Theorem 2.1, we obtain the following result.

Lemma 2.2. *For any positive integer n, the sequence $\{1/(n-k)!\}_{k=0}^{\infty}$ is a multiplier sequence.*

Proof. Clearly, the function

$$\sum_{k=0}^{\infty} \frac{1}{(n-k)!} \frac{x^k}{k!} = \frac{1}{n!}(1+x)^n$$

has only real zeros. This completes the proof by Theorem 2.1. □

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The following result of Laguerre can be used to produce multiplier sequences.

THEOREM 2.3 [5]. *If $\phi(x) \in \mathscr{L}\text{–}\mathscr{P}(-\infty, 0]$, then $\{\phi(k)\}_{k=0}^{\infty}$ is a multiplier sequence.*

The following identity for the gamma function, due to Weierstrass,

$$\Gamma(x) = \frac{1}{x}\exp(-\gamma x)\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)^{-1}\exp\left(\frac{x}{n}\right),$$

where $\gamma \approx 0.577216\ldots$ is the Euler–Mascheroni constant, shows that $1/\Gamma(x)$ belongs to $\mathscr{L}\text{–}\mathscr{P}(-\infty, 0]$. Hence, Theorem 2.3 gives the following result.

LEMMA 2.4. *The sequence $\{1/k!\}_{k=0}^{\infty} = \{1/\Gamma(k+1)\}_{k=0}^{\infty}$ is a multiplier sequence.*

We give another multiplier sequence which will be used in the next section.

LEMMA 2.5. *For any positive integer n, the sequence $\{1/i!(n-i)!\}_{i\geq 0}$ is a multiplier sequence.*

PROOF. From the definition, the Hadamard product (termwise product) of two multiplier sequences is also a multiplier sequence. The result therefore follows from Lemmas 2.2 and 2.4. □

We will also use the following elementary fact.

LEMMA 2.6 [9, Observation 4.2]. *If a polynomial $\ell(x)$ has symmetric coefficients,*

$$\ell(x) = \sum_{i=1}^{\lfloor n/2\rfloor} \xi_i\, x^i (1+x)^{n-2i}$$

has only negative real zeros if and only if this property also holds for the polynomial

$$\xi(x) = \sum_{i=1}^{\lfloor n/2\rfloor} \xi_i\, x^i.$$

3. Local h-polynomials of cluster subdivisions

In this section, we give the proof of Theorem 1.3. From Lemmas 1.1 and 2.6, it is easy to check that the local h-polynomials for the exceptional groups have only real zeros. It remains to discuss the cases for type A, type B and type D.

3.1. Type A. In this subsection, we deal with the local h-polynomial

$$\ell_I(\Gamma(A_n), x) = \sum_{i=1}^{\lfloor n/2\rfloor} \frac{1}{n-i+1}\binom{n}{i}\binom{n-i-1}{i-1} x^i (1+x)^{n-2i}.$$

In view of Lemma 2.6, we turn our attention to the polynomial

$$\xi_I(\Gamma(A_n), x) = \sum_{i=1}^{\lfloor n/2\rfloor} \frac{n!}{i!(n-i+1)!}\binom{n-i-1}{i-1} x^i.$$

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THEOREM 3.1. *For any positive integer n, the polynomial $\xi_I(\Gamma(A_n), x)$ has only real zeros.*

We first consider the polynomial

$$\sum_{i=1}^{\lfloor n/2\rfloor}\binom{n-i-1}{i-1}x^i.$$

Since

$$\sum_{i=1}^{\lfloor n/2\rfloor}\binom{n-i-1}{i-1}x^i=\sum_{i=1}^{\lfloor n/2\rfloor}\binom{n-2-(i-1)}{i-1}x^i=x\sum_{j=0}^{\lfloor n/2\rfloor-1}\binom{n-2-j}{j}x^j,$$

we focus on the polynomial

$$H_n(x)=\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n-j}{j}x^j.$$

LEMMA 3.2. *For any positive integer n, the polynomial $H_n(x)$ has only negative and simple zeros.*

PROOF. The polynomial $H_n(x)$ is closely related to the Chebyshev polynomial of the second kind,

$$U_n(y)=\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n-k}{k}(2y)^{n-2k}.$$

Replacing y by $1/2y$,

$$y^nU_n\Big(\frac{1}{2y}\Big)=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}(-y^2)^k. \tag{3.1}$$

From its trigonometric definition, the Chebyshev polynomial of the second kind satisfies

$$U_n(\cos\theta)=\frac{\sin(n+1)\theta}{\sin\theta}.$$

Hence, the zeros of $U_n(y)$ are $\cos(\pi k/(n+1))$, where $k=1,2,\ldots,n$. From (3.1), it follows that the zeros of $H_n(x)$ are $-\frac{1}{4}\sec^2(\pi k/(n+1))$, where $k=1,2,\ldots,\lfloor n/2\rfloor$. Therefore, the zeros of $H_n(x)$ are real and simple. This completes the proof. □

Now we are able to prove Theorem 3.1.

PROOF OF THEOREM 3.1. By Lemma 3.2, the polynomial

$$\sum_{i=1}^{\lfloor n/2\rfloor}\binom{n-i-1}{i-1}x^i=xH_{n-2}(x)$$

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has only real zeros. By Lemma 2.5, the sequence $\{1/i!(n-i+1)!\}_{i\geq 0}$ is a multiplier sequence. Hence,

$$\xi_I(\Gamma(A_n),x)=\sum_{i=1}^{\lfloor n/2\rfloor}\frac{n!}{i!(n-i+1)!}\binom{n-i-1}{i-1}x^i$$

has only real zeros. This completes the proof. □

3.2. Type B. We next consider the polynomials

$$\ell_I(\Gamma(B_n),x)=\sum_{i=1}^{\lfloor n/2\rfloor}\binom{n}{i}\binom{n-i-1}{i-1}x^i(1+x)^{n-2i}$$

for any positive integer n.

THEOREM 3.3. *For any positive integer n, the polynomial $\ell_I(\Gamma(B_n),x)$ has only real zeros.*

PROOF. By Lemma 3.2, the polynomial

$$\sum_{i=1}^{\lfloor n/2\rfloor}\binom{n-i-1}{i-1}x^i=xH_{n-2}(x)$$

has only real zeros. By Lemma 2.5, the sequence $\{1/i!(n-i)!\}_{i\geq 0}$ is a multiplier sequence. For any positive integer n, the polynomial

$$\xi_I(\Gamma(B_n),x)=\sum_{i=1}^{\lfloor n/2\rfloor}\binom{n}{i}\binom{n-i-1}{i-1}x^i=n!\sum_{i=1}^{\lfloor n/2\rfloor}\frac{1}{i!(n-i)!}\binom{n-i-1}{i-1}x^i$$

has only real zeros and so, by Lemma 2.6, $\ell_I(\Gamma(B_n),x)$ has only real zeros. □

3.3. Type D. Finally we consider the polynomials

$$\ell_I(\Gamma(D_n),x)=\sum_{i=1}^{\lfloor n/2\rfloor}\frac{n-2}{i}\binom{2i-2}{i-1}\binom{n-2}{2i-2}x^i(1+x)^{n-2i}$$

for any positive integer $n\geq 2$. From an identity for *Narayana polynomials* [3]:

$$\sum_{i=0}^{\lfloor n/2\rfloor}\frac{1}{i+1}\binom{2i}{i}\binom{n}{2i}x^i(1+x)^{n-2i}=\sum_{i=0}^{n}\frac{1}{n+1}\binom{n+1}{i}\binom{n+1}{i+1}x^i,$$

it follows that

$$\begin{aligned}\ell_I(\Gamma(D_n),x)&=\sum_{i=1}^{\lfloor n/2\rfloor}\frac{n-2}{i}\binom{2i-2}{i-1}\binom{n-2}{2i-2}x^i(1+x)^{n-2i}\\&=(n-2)x\sum_{i=0}^{\lfloor (n-2)/2\rfloor}\frac{1}{i+1}\binom{2i}{i}\binom{n-2}{2i}x^i(1+x)^{n-2-2i}\\&=(n-2)x\sum_{i=0}^{n-2}\frac{1}{n-1}\binom{n-1}{i}\binom{n-1}{i+1}x^i.\end{aligned}$$

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The Narayana polynomials have only real roots (see [3, 8]) and this gives the conclusion for Type D.

THEOREM 3.4. *For any positive integer $n \geq 2$, the polynomial $\ell_I(\Gamma(D_n), x)$ has only real zeros.*

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