J. Austral. Math. Soc. (Series A) **56** (1994), 391–402

CONSTRUCTING GRAPHS WHICH ARE 1/2-TRANSITIVE

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(Received 26 July 1991; revised 5 February 1992)

Communicated by L. Caccetta

Abstract

An infinite family of vertex- and edge-transitive, but not arc-transitive, graphs of degree 4 is constructed.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): 05 C 25.

1. Introduction

If G is a graph, then the *arcs* of G are obtained by taking one arc for each orientation of each edge of G so that there are twice as many arcs as edges. A *k-arc* of G is a directed walk of length k using the arcs of G except that one cannot follow an arc (u, v) by the arc (v, u) in the directed walk. A *k-path* P of G with end vertices u and v is a walk (undirected) of length k such that each of u and v is incident with one edge of P, and all other vertices of P are incident with two edges of P. A graph G is said to be *vertex-transitive*, *edge-transitive* and *arc-transitive* provided its automorphism group $\mathrm{Aut}(G)$ acts transitively on the vertices, edges and arcs of G, respectively. (The terms *symmetric* and *1-transitive* also have been used instead of arc-transitive.) In general, a graph is said to be *k-arc-transitive* if $\mathrm{Aut}(G)$ acts transitively on the k-arcs of G. Biggs [3] defines a graph G to be *k-transitive* if $\mathrm{Aut}(G)$ acts transitively on the k-arcs of G, but does not act transitively on the $(k + 1)$-arcs. Since we

This research was partially supported by the Natural Sciences and Engineering Research Council of Canada under Grant A-4792

https://doi.org/10.1017/S1446788700035564 Published online by Cambridge University Press

deal with graphs G for which $\mathrm{Aut}(G)$ is vertex- and edge-transitive but not arc-transitive and we wish to avoid that cumbersome phrase, in keeping with Biggs' definition (and viewing the vertices of G as 0-arcs), we shall call such graphs *1/2-transitive*. In general, a graph G is $(2m+1)/2$-transitive if $\mathrm{Aut}(G)$ acts transitively on the m-arcs and the $(m+1)$-paths of G, but does not act transitively on the $(m+1)$-arcs of G.

Of course, there exist vertex-transitive graphs which are not edge-transitive. Likewise, an edge-transitive graph is not necessarily regular, and thus not necessarily vertex-transitive. Even more, there are regular edge-transitive graphs which are not vertex-transitive [6]. The only general result linking vertex- and edge-transitivity to arc-transitivity is due to Tutte [15] who proved that a vertex- and edge-transitive graph of odd degree is necessarily arc-transitive. With regard to even degree, Bouwer [4] in 1970 constructed an infinite family of 1/2-transitive graphs. The smallest order graph in his family has 54 vertices. More recently, Holt [7] found one with 27 vertices.

The automorphism groups of Holt's graph and Bouwer's family of graphs act imprimitively on the vertex-set. This prompted Holton [8] to ask if the automorphism group of every 1/2-transitive graph is necessarily imprimitive. This question has been answered by Xu and Praeger [12] in the negative. They found several examples of 1/2-transitive graphs, the smallest of which has 253 vertices and is of degree 24, whose automorphism groups are primitive.

As mentioned above, Holt has produced a 1/2-transitive graph on 27 vertices. Using the facts that every vertex- and edge-transitive Cayley graph on an abelian group or with $2p$ vertices, p a prime, is also arc-transitive, McKay's list of all vertex-transitive graphs with 19 or fewer vertices [10], McKay and Royle's list of vertex-transitive graphs with 20 and 21 vertices [11], and Praeger and Royle's proof that there are no 1/2-transitive graphs with 24 vertices, we conclude that there are no 1/2-transitive graphs with fewer than 27 vertices. In his paper, Holt mentions that a referee informed him that Kornya found another example with 27 vertices. However, Xu has informed the authors (personal communication) that he has shown that Holt's graph is the only 1/2-transitive graph with 27 vertices and of degree 4.

The objective of this paper is to give an infinite family of 1/2-transitive graphs of degree 4. All of them are metacirculant graphs.

2. A family of graphs

Let $n \geq 2$. A permutation on a finite set is said to be *(m,n)-semiregular* if it has m cycles of length n in its disjoint cycle decomposition. We shall be sloppy and refer to the orbits of the group $\langle \alpha \rangle$ generated by α as the orbits of α. A graph G is an *(m,n)-metacirculant* if it has an (m, n)-semiregular automorphism α together with another automorphism β normalizing α and cyclically permuting the orbits of α such that as a permutation there is a cycle of length m in the disjoint cycle decomposition of β. Therefore, we may partition the vertex-set of an (m, n)-metacirculant into the orbits $X_0, X_1, \dots, X_{m-1}$ of α, where $X_{i+1} = \beta(X_i)$ for all $i \in Z_m$. We shall refer to the orbits of α as the *blocks* of the metacirculant graph. It should be pointed out that the blocks of a metacirculant graph need not be blocks of imprimitivity of the automorphism group of the graph.

If G is a graph and A and B are two vertex-disjoint subsets of the vertex-set $V(G)$ of G, we let $\langle A \rangle$ denote the subgraph induced by G on A, and let $\langle A, B \rangle$ denote the bipartite subgraph induced by G on $A \cup B$, that is, all edges of G with one endvertex in A and the other endvertex in B. If $S_0 \subseteq Z_n \backslash \{0\}$ is the symbol of the subcirculant $\langle X_0 \rangle$ and, for all $i \in Z_m \backslash \{0\}$, $T_i \subseteq Z_m$ is the symbol of the bipartite subgraph $\langle X_0, X_i \rangle$, then there exists an $r \in Z_m^*$, where Z_m^* denotes the multiplicative group of units in Z_m, such that for all $j \in Z_m$, the symbol of $\langle X_j \rangle$ is $r^j S_0$ and the symbol of the bipartite graph $\langle X_j, X_{j+1} \rangle$, $i \in Z_m$, is $r^j T_i$. Moreover, for all $i \in Z_m$, we have $T_{m-i} = -r^{m-i} T_i$. Thus, the metacirculant graph is completely determined by the $\lfloor (m + 4)/2 \rfloor$-tuple $(r; S_0, T_1, T_2, \dots, T_{\lfloor m/2 \rfloor})$ which is called a *symbol* of G. (For a more detailed discussion of metacirculants, the reader is referred to [1, 2].)

We are now ready to define a family of metacirculants of degree 4 containing infinitely many 1/2-transitive graphs. The smallest among them has 27 vertices.

Let $n \geq 5$ be an integer and $r \in Z_n^*$ be an element of order m or $2m$ such that $m \geq 3$. Then let $M(r; m, n)$ denote the (m, n)-metacirculant graph with symbol $(r; \emptyset, \{1, n-1\}, \emptyset, \dots, \emptyset)$. For example, the graph $M(2; 3, 9)$ can be thought of as having vertices $\{x_i^j : 0 \leq i \leq 2 \text{ and } 0 \leq j \leq 8\}$ with x_0^j adjacent to $x_1^{j\pm1}$, x_1^j adjacent to $x_2^{j\pm2}$, and x_2^j adjacent to $x_0^{j\pm4}$, where the superscripts are reduced modulo 9. This graph is in fact the Holt graph of [7]. This suggests there may be other 1/2-transitive $M(r; m, n)$ graphs, as indeed is the case, for other values of the parameters r, m and n.

Throughout the paper, α is the (m, n)-semiregular automorphism of $M = m(r; m, n)$ with orbits $X_0, X_1, \dots, X_{m-1}$, where $X_i = \{x_i^0, x_i^1, \dots, x_i^{n-1}\}$ and

$\alpha(x_i^j) = x_i^{j+1}$ for all $i \in Z_m$ and $j \in Z_n$. Similarly, β is the automorphism of M mapping according to the formula $\beta(x_i^j) = x_{i+1}^{rj}$, for $i \in Z_m$, $j \in Z_n$. Finally, another automorphism τ of $M(r; m, n)$ which will prove useful is defined by $\tau(x_i^j) = x_i^{-j}$.

DEFINITION. A cycle of $M(r; m, n)$ of length at least m is said to be *coiled* if every subpath with m vertices intersects each one of $X_0, X_1, \dots, X_{m-1}$. It is easy to see that a coiled cycle must have length a multiple of m.

DEFINITION. The *coiled girth* of $M(r; m, n)$ is the length of a shortest coiled cycle in $M(r; m, n)$.

PROPOSITION 2.1. *The coiled girth of $M(r; m, n)$ is either m or $2m$.*

PROOF. Assume that $M(r; m, n)$ does not contain a coiled cycle of length m. Consider the closed trail

$$x_0^0 x_1^1 x_2^{1+r} \cdots x_{m-1}^{1+r+r^2+\dots+r^{m-2}} x_0^{1+r+r^2+\dots+r^{m-1}} x_1^{r+r^2+\dots+r^{m-1}} x_2^{r^2+\dots+r^{m-1}} \cdots x_0^0.$$

Since $M(r; m, n)$ has no coiled cycles of length m, all of the vertices of the closed trail are distinct. Thus, it is a coiled cycle of length $2m$ as required.

DEFINITION. We obtain natural edge-partitions of $M(r; m, n)$ using coiled girth cycles and α. If $M(r; m, n)$ has coiled girth m, let C be a coiled cycle of length m. If $C = x_0^{i_0} x_1^{i_1} \cdots x_{m-1}^{i_{m-1}}$, let $\alpha(C)$ denote the cycle obtained under the substitution of $\alpha(x_j^{i_j})$ for $x_j^{i_j}$, $0 \le j \le m-1$. Then $C, \alpha(C), \alpha^2(C), \dots, \alpha^{n-1}(C)$ is a 2-factor of $M(r; m, n)$. It is easy to see that the remaining edges also form a 2-factor made up of coiled cycles of length m. If C is a coiled cycle of length $2m$ such that of the two edges from X_j to X_{j+1} in C, one is of the form $x_j^{i_j} x_{j+1}^{i_j+r^j}$ and the other is of the form $x_j^{i_j} x_{j+1}^{i_j-r^j}$, then $C, \alpha(C), \alpha^2(C), \dots, \alpha^{n-1}(C)$ is a partition of the edge-set of $M(r; m, n)$ into $2m$-cycles. We call both of the above the *α-partition* of $M(r; m, n)$ induced by C.

DEFINITION. When $M(r; m, n)$ has coiled girth m and the α-partition of $M(r; m, n)$ induced by every coiled m-cycle yields the same 2-factorization of $M(r; m, n)$, then we shall say that $M(r; m, n)$ is *tightly coiled.* Otherwise, we shall say that the graph is *loosely coiled.* A similar definition applies in the case that the coiled girth of $M(r; m, n)$ is $2m$. However, in the latter case, $M(r; m, n)$ is always loosely coiled as we shall see soon.

We now set about proving some lemmas which will be useful in establishing that certain $M(r; m, n)$ metacirculants are 1/2-transitive.

LEMMA 2.2. *If $M = M(r; m, n)$ has coiled girth $2m$, then M is loosely coiled.*

PROOF. As seen in the proof of Proposition 2.1,

$$x_0^0 x_1^1 x_2^{1+r} \cdots x_{m-1}^{1+r+r^2+\cdots+r^{m-2}} x_0^{1+r+r^2+\cdots+r^{m-1}} x_1^{r+r^2+\cdots+r^{m-1}} x_2^{r^2+\cdots+r^{m-1}} \cdots x_0^0$$

is a coiled cycle C of length $2m$. Obtain another coiled cycle of length $2m$ by taking the first edge from x_0^0 to x_1^{n-1} and then taking the same kind of edge as in C until reaching X_0 again. This means the vertex in X_j will be $x_j^{-1+r+r^2+\cdots+r^{j-1}}$. When leaving X_0 the second time, use the edge to $x_1^{r+r^2+\cdots+r^{m-1}}$ from which point the vertices will be the same as in C.

LEMMA 2.3. *Let $M = M(r; m, n)$. If $\sigma \in \mathrm{Aut}(M)$ fixes two adjacent blocks of M pointwise, then σ is the identity.*

PROOF. Let σ fix X_i and X_{i+1} pointwise. Then X_{i-1} is fixed setwise by σ. The subgraph $\langle X_{i-1}, X_i \rangle$ is either a $2n$-cycle or two n-cycles because $r \in Z_n^*$. The automorphism σ fixes alternate vertices of the $2n$-cycle or the two n-cycles, and thus it fixes every vertex. This means that σ also fixes X_{i-1} pointwise. Continuing in this way establishes the result.

LEMMA 2.4. *Let $M = M(r; m, n)$. If $\sigma \in \mathrm{Aut}(M)$ fixes some block of M pointwise, then σ is the identity.*

PROOF. Without loss of generality we may assume X_1 is the block of M which is fixed pointwise by σ. The neighbors of x_1^0 are x_2^r, x_2^{-r}, x_0^1 and x_0^{n-1}, and the neighbors of x_1^2 are $x_2^{r+2}, x_2^{-r+2}, x_0^1$ and x_0^3. Since both x_1^0 and x_1^2 are fixed by σ, either x_0^1 is also fixed by σ or $\{x_2^r, x_2^{-r}, x_0^{n-1}\} \cap \{x_2^{r+2}, x_2^{-r+2}, x_0^3\}$ is non-empty. In the latter case, $x_0^{n-1} \neq x_0^3$ because $n \geq 5$. This forces either $x_2^r = x_2^{-r+2}$ or $x_2^{-r} = x_2^{r+2}$. In the first case $r = (n+2)/2$ must hold and in the second case $r = (n-2)/2$ must hold. Neither are possible when n is odd. If n is a multiple of 4, then $r^2 = 1$ in Z_n^* contradicting the fact that r has order at least 3. If n is even and not a multiple of 4, then r is even contradicting the fact that $r \in Z_n^*$. This implies x_0^1 is fixed by σ. Continuing in this way we obtain that X_0 is fixed pointwise. The preceding lemma yields the desired result.

LEMMA 2.5. *Let* $M = M(r; m, n)$ *and suppose that whenever* $\sigma \in \mathrm{Aut}(M)$ *fixes two adjacent vertices of* M, σ *is the identity. Then either* $\mathrm{Aut}(M) = \langle \alpha, \beta, \tau \rangle$ *or* $|\mathrm{Aut}(M)| = 2|\langle \alpha, \beta, \tau \rangle|$.

PROOF. By hypothesis, the stabilizer of an edge of M is either the identity or has order 2. Thus, $|\mathrm{Aut}(M)| = 2mn$ or $4mn$ and the result follows.

LEMMA 2.6. *Let* $M = M(r; m, n)$, *with* m *and* n *odd, have coiled girth* m *and be loosely coiled. If* $\sigma \in \mathrm{Aut}(M)$ *fixes two adjacent vertices of* M, *then* σ *is the identity.*

PROOF. Suppose $x \in X_i$ and $y \in X_{i+1}$ are two adjacent vertices of M fixed by σ. Let y' be the other neighbor of x in X_{i+1} and let z and z' be the two neighbors of x in X_{i-1}. Since m is odd, neither of the two triples zxz' and yxy' are in m-cycles. On the other hand, since M is loosely coiled, each of the four triples zxy, zxy', $z'xy$ and $z'xy'$ are in m-cycles. Therefore, σ must also fix y' in addition to fixing x and y. For the same reason, σ must fix the other neighbor of y' in X_i. Continuing in this way, we see that σ fixes all the vertices of X_i and X_{i+1}. By Lemma 2.3, the conclusion follows.

THEOREM 2.7. *Let* $M = M(r; m, n)$, *with* m *and* n *odd, have coiled girth* m. *If* M *is loosely coiled,* $n > 7$ *and* $m \geq 3$, *then* M *is* $1/2$*-transitive.*

PROOF. If $\mathrm{Aut}(M) = \langle \alpha, \beta, \tau \rangle$, then M is $1/2$-transitive. Assume $\mathrm{Aut}(M) \neq \langle \alpha, \beta, \tau \rangle$ and M is arc-transitive. Then there is a $\sigma \in \mathrm{Aut}(M)$ interchanging two adjacent vertices, say $x \in X_i$ and $y \in X_{i+1}$. Let $a, a' \in X_{i-1}$ and $y' \in X_{i+1}$ be the remaining neighbors of x, and let $u, u' \in X_{i+2}$ and $x' \in X_i$ be the remaining three neighbors of y. The triples uyx' and $u'yx'$ are contained in an m-cycle but uyu' is not. Similarly, axy' and $a'xy'$ are contained in m-cycles but yxy' is not. We know σ interchanges $\{u, u', x'\}$ and $\{a, a', y'\}$ so that σ must interchange x' and y'. Hence, σ interchanges X_i and X_{i+1} which implies σ interchanges X_{i-1} and X_{i+2}, X_{i-2} and X_{i+3}, and so on. Thus, $\mathrm{Aut}(M)$ acts imprimitively with the orbits of α as blocks. Since σ^2 fixes x and y, σ is an involution by Lemma 2.6.

Note that the automorphism group of $\langle X_i \cup X_{i+1} \rangle$ is dihedral of order $4n$. Hence, the restriction of $\sigma\alpha\sigma$ to $\langle X_i \cup X_{i+1} \rangle$ is the restriction of α^{-1}. Thus, the restriction of $(\sigma\alpha)^2$ to $\langle X_i \cup X_{i+1} \rangle$ is 1 implying that $(\sigma\alpha)^2 = 1$ by Lemma 2.1. This together with the fact that $\tau\alpha\tau = \alpha^{-1}$ implies that $\sigma\tau$ centralizes α. Clearly, the action of $\sigma\tau$ on the orbits of α is identical to the action of σ. Thus, there is an orbit Z of α which is fixed by $\sigma\tau$. The restriction of $\sigma\tau$ to Z is then the same as that of some α^i. Let $\gamma = \sigma\tau\alpha^{-i}$. Note that γ restricted to Z is 1 and that γ interchanges the neighboring orbits, say U and W, of Z. However, the structure of the graph M implies that whenever two vertices of Z have a common neighbor in U, they do not have a common neighbor in W. Therefore, an automorphism such as γ cannot exist. This means that $\mathrm{Aut}(M) = \langle \alpha, \beta, \tau \rangle$ and M is $1/2$-transitive.

COROLLARY 2.8. *Let p be a prime and r a divisor of $p-1$ whose order m in Z_p^* is odd and composite. Then $M = M(r; m, p)$ is $1/2$-transitive. In particular, there are infinitely many $1/2$-transitive graphs of degree 4.*

PROOF. The graph M has coiled girth m because $1 + r + r^2 + \cdots + r^{m-1} \equiv 0(\mathrm{mod}\, p)$ implying that $x_0^0 x_1^1 x_2^{1+r} \cdots x_{m-1}^{1+r+\cdots+r^{m-2}} x_0^0$ is an m-cycle C. It suffices to show that M is loosely coiled. Let H be the subgroup of Z_p^* of order m generated by r. Let d be a nontrivial divisor of $p-1$. Then the subgroup H_1 of H generated by r^d has order $(p-1)/d$. The sum of the entries in any subgroup of Z_p^* is congruent to zero modulo p. In fact, the sum of the powers of r over each of the cosets of H_1 in H is also congruent to zero modulo p. Thus, we can either add or subtract all the elements of a given coset to give us other m-cycles of M which induce 2-factorizations of M different than that induced by C. Hence, M is loosely coiled and the corollary follows from Theorem 2.7.

3. Three blocks

In the previous section, the general case for m and n odd is covered when $M(r; m, n)$ has coiled girth m and is loosely coiled. We take care of the case when $M(r; m, n)$ is tightly coiled or has coiled girth $2m$ in this section, but only for $m = 3$. The following lemma has the same conclusion as Lemma 2.6, but the proof is completely different. Notice that if $r^3 \equiv -1(\mathrm{mod}\, n)$, then $(-r)^3 \equiv 1(\mathrm{mod}\, n)$. Since $M(r; 3, n) \cong M(-r; 3, n)$, we may as well assume r has order 3.

LEMMA 3.1. *Let $M = M(r; 3, n)$, $n \geq 9$ and n odd, let $r^3 \equiv 1(\mathrm{mod}\, n)$, and let M have coiled girth 3. If $\sigma \in \mathrm{Aut}(M)$ fixes two adjacent vertices of M, then σ is the identity.*

https://doi.org/10.1017/S1446788700035564 Published online by Cambridge University Press

PROOF. Since $r^3 \equiv 1(\bmod n)$, $(r-1)(r^2+r+1) \equiv 0(\bmod n)$. This implies that either $r^2+r+1 \equiv 0(\bmod n)$ or r^2+r+1 is a zero divisor in Z_n. Since 2 is not a zero divisor in Z_n when n is odd, $r^2+r+1 \not\equiv 2(\bmod n)$ implying that $r^2+r-1 \not\equiv 0(\bmod n)$. Similarly, $r^2-r+1 \not\equiv 0(\bmod n)$ because r is a unit in Z_n. Therefore, since M has coiled girth 3, either $r^2+r+1 \equiv 0(\bmod n)$ or $r^2-r-1 \equiv 0(\bmod n)$ must hold.

First consider the case that $r^2+r+1 \equiv 0(\bmod n)$. Without loss of generality, assume that σ fixes x_0^0 and x_1^1. We now determine the vertices at increasing distances from $\{x_0^0, x_1^1\}$ and use them to prove the result in this case. The only problem that may arise is that for small n and certain values of r some of the apparently different vertices may be the same. The vertex x_2^{r+1} is the only vertex adjacent to both x_0^0 and x_1^1 so it too is fixed by σ. The other neighbors of x_0^0 are $x_2^{r^2}$ and $x_1^{r+r^2}$, and the other neighbors of x_1^1 are x_0^2 and $x_2^{r^2+2}$. The vertices x_0^2, $x_1^{r+r^2}$, x_2^{r+1}, and $x_2^{r^2+r}$ are all distinct because $n \geq 9$, n is odd and our assumption about r^2+r+1.

We now determine the vertices at distance 2 from $\{x_0^0, x_1^1\}$. The remaining neighbors of $x_2^{r^2}$ are $x_0^{r^2-r-1}$ and $x_1^{r^2-r}$, of $x_1^{r^2+r}$ are x_0^{n-2} and x_2^{r-1}, of x_2^{r+1} are x_0^{2r+2} and x_1^{2r+1}, of x_0^2 are x_1^3 and x_2^{r+3}, and of $x_2^{r^2+2}$ are $x_0^{r^2-r+1}$ and $x_1^{r^2-r+2}$. We now show that we may assume all these vertices are distinct and that there is only one adjacency amongst them. This is based on the following results. The element $r \neq 0$; if $r = 2$, then $r^3 \equiv 1(\bmod n)$ implies $n = 7$ which is a contradiction; if $r = n-2$, then $n = 9$ so that M is Holt's graph, which is known to be 1/2-transitive; $r \neq 1$; $r \neq -1$; if $r = (n-1)/2$, then $r^2+r+1 \equiv 0(\bmod n)$ implies that $r^2 \equiv r(\bmod n)$ which is impossible; if $r = (n+1)/2$, then $r^2 \equiv (n-3)/2 = r-2(\bmod n)$ which implies $r^3 \equiv (n-5)/2(\bmod n)$ which, in turn, implies $n = 7$; and if $r = (n-3)/2$, then $r^2 \equiv (n+1)/2 = r+2(\bmod n)$ again leading to the contradiction that $n = 7$.

Following are a few examples showing how the above occur. Can $x_0^{n-2} = x_0^{2r+2}$, that is, can $n-2 \equiv 2r+2(\bmod n)$? Since n is odd, we must have $2r = 2n-4$ or equivalently $r = n-2$. As another example consider whether or not $x_0^{2r+2} = x_0^{r^2-r+1}$ is possible. If the two are equal, then $4r \equiv -2(\bmod n)$. This implies that $r = (n-1)/2$ which is one of the above. Checking all other possibilities leads to one of the above or that $n < 9$.

We now may assume that the vertices at distance 2 from $\{x_0^0, x_1^1\}$ listed above are distinct. We now consider adjacencies amongst these vertices. Notice that there is an edge joining $x_0^{r^2-r+1}$ and $x_1^{r^2-r}$. Now consider the vertex $x_0^{r^2-r+1}$ as an example. Its other neighbor in X_2 is $x_2^{r^2-2r-2}$. For example, can $x_2^{r^2-2r-2} = x_2^{r+3}$? If so, then $r+3 \equiv r^2-2r-2 \equiv -3r-3(\bmod n)$. This implies that $4r \equiv$

$-6 (\mathrm{mod}\, n)$ which, in turn, implies that $r = (n-3)/2$. This means the vertex $x_2^{r^2-2r-2}$ cannot lie at distance 2 from $\{x_0^0, x_1^1\}$. Checking all other possibilities leads us to be able to assume the above edge between $x_0^{r^2-r+1}$ and $x_1^{r^2-r}$ is the only edge joining any two vertices at distance 2 from $\{x_0^0, x_1^1\}$.

This implies that σ cannot interchange $x_2^{r^2+2}$ and x_0^2 since there is a path of length 4 from $x_2^{r^2+2}$ to x_0^0, but not from x_0^2 to x_0^0. Similarly, σ cannot interchange $x_2^{r^2}$ and x_1^{n-1}. In particular, this implies that σ fixes x_1^{n-1} and x_0^2. Now repeat the same argument with the adjacent vertices x_0^2 and x_1^1 leading to x_1^3 being fixed by σ. Because n is odd, continuing in this way leads to σ fixing every vertex of X_0 and X_1. By Lemma 2.3, σ is the identity. This completes the case for $r^2+r+1 \equiv 0 (\mathrm{mod}\, n)$.

The case for $r^2 - r - 1 \equiv 0\ (\mathrm{mod}\, n)$ is done in the same way. The only adjacency between two vertices at distance 2 from $\{x_0^0, x_1^1\}$ is between x_0^{-2r} and x_1^{-2r-1}. A new contradiction is used in this case. Several times $-r-3 \equiv r+3 (\mathrm{mod}\, n)$ arises. This forces $r = -3$ which implies $n = 11$ because of $r^2 - r - 1 \equiv 0 (\mathrm{mod}\, n)$. But then $r^3 \not\equiv 1 (\mathrm{mod}\, 11)$ which is a contradiction.

LEMMA 3.2. *Let $M = M(r; 3, n)$, $n \geq 9$ and n odd, let $r^3 \equiv 1 (\mathrm{mod}\, n)$, and let M have coiled girth 6. If $\sigma \in \mathrm{Aut}(M)$ fixes two adjacent vertices of M, then σ is the identity.*

PROOF. This proof hinges on the difference between coiled 6-cycles and non-coiled 6-cycles. If C is a non-coiled 6-cycle, then it must contain three successive vertices u, v, w such that $u \in X_{i+1}$, $v \in X_i$ and $w \in X_{i+1}$, and such that if the 6-cycle is $uvwyztu$, then $y \in X_i$, $z \in X_{i-1}$ and $t \in X_i$ does not happen. Call the 2-path uvw an *anchor* of C. Because of the action of α and β, it is easy to see that if M has a non-coiled 6-cycle, then there is a non-coiled 6-cycle which has the 2-path $x_1^{-1}x_0^0x_1^1$ as an anchor.

Without loss of generality, let $\sigma \in \mathrm{Aut}(M)$ fix x_0^0 and x_1^1. It is easy to see that each of the 2-paths $x_2^{r^2}x_0^0x_1^1$, $x_2^{r^2}x_0^0x_1^{-1}$, $x_2^{-r^2}x_0^0x_1^1$, and $x_2^{-r^2}x_0^0x_1^{-1}$ is contained in a coiled 6-cycle. If the 2-path $x_1^{-1}x_0^0x_1^1$ is not contained in a 6-cycle (that is, every 6-cycle is coiled), then σ must also fix x_1^{-1}. If the 2-path $x_1^{-1}x_0^0x_1^1$ is not in a 6-cycle, then neither is the 2-path $x_0^{-2}x_1^{-1}x_0^0$. Hence, σ also must fix x_0^{-2}. Continuing in this way leads to the conclusion that σ fixes every vertex of X_0 and X_1. This implies that σ is the identity.

Now consider the case that there are non-coiled 6-cycles. We now describe the possible non-coiled 6-cycles with anchor $x_1^{-1}x_0^0x_1^1$. One possibility is $x_0^0x_1^1x_0^2x_1^3x_0^4x_1^5x_0^6$. However, this possibility would force $n = 6$ which is a contradiction. A second possibility is $x_0^0 x_1^1 x_0^2 x_1^3 x_2^1 x_1^{-1} x_0^0$ which implies either $r = 2$ or $r = -2$. The former implies $n = 7$, which is a contradiction, and the latter implies $n = 9$ and M is the Holt graph.

A third possibility involves completing the 4-path $x_2^{-1-r} x_1^{-1} x_0^0 x_1^1 x_2^{1+r}$ in X_0. However, if x_2^{-1-r} and x_2^{1+r} have a common neighbor in X_0, it must be x_0^0. This implies M has coiled girth 3 which is a contradiction. Likewise, the possibility that the 4-path $x_2^{-1+r} x_1^{-1} x_0^0 x_1^1 x_2^{1-r}$ completes to a 6-cycle in X_0 leads to the same contradiction.

The two remaining possibilities come from completing the two preceding 4-paths in X_1. One resulting possibility is the 6-cycle $x_0^0 x_1^1 x_2^{1+r} x_1^0 x_2^{-1-r} x_1^{-1} x_0^0$ which implies that $r = (n-1)/2$. The other resulting possibility is the 6-cycle $x_0^0 x_1^1 x_2^{1-r} x_1^0 x_2^{-1+r} x_1^{-1} x_0^0$ which implies that $r = (n+1)/2$. We see that the 2-path $x_1^1 x_0^0 x_1^{-1}$ lies in precisely two non-coiled 6-cycles in both cases. Thus, if $r \neq (n-1)/2$ and $r \neq (n+1)/2$, there are no non-coiled 6-cycles and the first part of the proof establishes the result. Suppose that $r = (n-1)/2$. Then the 2-paths $x_2^{-r^2} x_0^0 x_1^1$ and $x_2^{r^2} x_0^0 x_1^{-1}$ lie in three 6-cycles and the remaining 2-paths with x_0^0 as central vertex lie in two 6-cycles. Thus, σ must also fix the vertex $x_2^{-r^2}$. The same argument applied to the 2-paths centered at $x_2^{-r^2}$ implies that σ must also fix $x_1^{-r-r^2}$. Continuing in this way, σ also fixes $x_0^{-1-r-r^2}$, $x_2^{-1-r-2r^2}$, $x_1^{-1-2r-2r^2}$, $x_0^{-2-2r-2r^2}$, and so on until eventually we reach x_1^1 at which point we have completed a cycle C in M. If $d = \gcd(n, 1 + r + r^2)$, then the vertices of X_0 lying in C are $x_0^0, x_0^d, x_0^{2d}, \dots, x_0^{n-d}$. It is still possible that σ interchanges $x_2^{r^2}$ and x_1^{-1}. If so, then σ must also interchange $x_0^{1+r+r^2}$ and $x_0^{-1-r-r^2}$. However, the latter two vertices are distinct and lie on C. This contradicts the fact that σ fixes every vertex of C and we conclude that σ also fixes x_1^{-1}, x_2^{-1-r}, $x_0^{-1-r-r^2}$, $x_1^{-2-r-r^2}, \dots, x_2^{r^2}$. By repeating the preceding argument with x_1^1 replacing x_0^0 and so on, we eventually achieve that σ must fix every vertex of M.

We are left with the case that $r = (n+1)/2$. However, this case does not arise as it is an easy number theoretic exercise to show that if $((n+1)/2)^3 \equiv 1 \pmod{n}$, then either $n = 3$ or $n = 7$. This contradicts our assumption that $n \geq 9$.

THEOREM 3.3. *Let $M = M(r; 3, n)$, $n \geq 9$ and n odd, and let $r^3 \equiv 1 \pmod{n}$. Then M is $1/2$-transitive.*

PROOF. We know that either $\mathrm{Aut}(M) = \langle \alpha, \beta, \tau \rangle$ or $|\mathrm{Aut}(M)| = 12n$ because of Lemmas 3.1 and 3.2. In the former case, M is $1/2$-transitive as required. In the latter case, if M is arc-transitive, there must be an automorphism $\sigma \in \mathrm{Aut}(M)$ such that $\sigma \notin \langle \alpha, \beta, \tau \rangle$, $\mathrm{Aut}(M) = \langle \alpha, \beta, \tau, \sigma \rangle$ and σ interchanges the vertices

x_0^0 and x_1^1. Orient the edge $x_0^0x_1^1$ from x_0^0 to x_1^1 obtaining the arc (x_0^0, x_1^1). The group $\langle\alpha, \beta, \tau\rangle$ is regular on edges, so orienting the edge $g(x_0^0)g(x_1^1)$ from $g(x_0^0)$ to $g(x_1^1)$ for each $g \in \langle\alpha, \beta, \tau\rangle$ gives an orientation of M which we denote by $M*$. The digraph $M*$ is arc-transitive under the group $\langle\alpha, \beta, \tau\rangle$ and the latter group acts regularly on the arcs of $M*$. Since σ interchanges x_0^0 and x_1^1, σ must be orientation-reversing on $M*$. We now carefully examine the action of σ.

Since σ interchanges x_0^0 and x_1^1 and is orientation reversing, σ must also interchange x_1^{-1} and x_0^2. Then it must interchange x_1^{-3} and x_0^4. Continuing in this way, we see that σ must interchange x_1^{-k} and x_0^{k+1} for all $k \in Z_n$. Now x_1^1 and x_1^{1+2r} have the common neighbor x_2^{1+r} in X_2. Thus, x_0^0 and x_0^{-2r} must have a common neighbor in X_2. If they have a common neighbor, it must be x_2^{-r}. But the neighbors of x_0^0 are $x_2^{r^2}$ and $x_2^{-r^2}$. Now $-r$ and $-r^2$ clearly cannot be the same. Likewise, if $-r$ and r^2 are the same, then $r = -1$ is forced which is impossible. Therefore no such σ exists and M is not arc-transitive as required.

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