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INDEPENDENCE ALGEBRAS, BASIS ALGEBRAS AND SEMIGROUPS OF QUOTIENTS

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Abstract We show that if A is a stable basis algebra satisfying the *distributivity condition*, then B is a reduct of an independence algebra A having the same rank. If this rank is finite, then the endomorphism monoid of B is a left order in the endomorphism monoid of A.

Keywords: semigroup; independence; basis; exchange property; endomorphism monoid; quotients

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1. Introduction

It is well known that, for any $n \in \mathbb{N}$, the ring $M_n(D)$ of $n \times n$ matrices over a division ring D is simple, that is, it has no non-trivial *ring* ideals. As a semigroup, however, $M_n(D)$ is not simple. Indeed, $M_n(D)$ has finitely many *semigroup* ideals I_k, $0 \leqslant k \leqslant n$, where

$$I_k = \{A \in M_n(D) : \mathrm{rank}(A) \leqslant k\}.$$

Clearly,

$$I_0 = \{0\} \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n = M_n(D);$$

moreover, the Rees quotients I_k/I_{k-1} of successive ideals, $1 \leqslant k \leqslant n$, are completely 0-simple. The ring $M_n(D)$ possesses further interesting 'semigroup' properties: by a result of Laffey [**16**], proved by J. A. Erdös [**4**] in the case when D is a field, every singular element of $M_n(D)$ is a product of idempotents.

The matrix ring $M_n(D)$ is of course isomorphic to the ring of linear maps of any n-dimensional vector space V over D, so that, as a semigroup, $M_n(D)$ is isomorphic to the endomorphism monoid $\mathrm{End}\, V$ of V. Vector spaces over division rings are particular examples of (universal) algebras belonging to the class of v^*-*algebras*. These first appeared in an article of Narkiewicz [**19**] and were inspired by Marczewski's study of notions of independence, initiated in [**18**] (see [**14**] and the survey article [**21**]). More recently, v^*-algebras have been referred to as *independence algebras* [**12**]. Such algebras may be defined via properties of the closure operator $\langle\cdot\rangle$, which takes a subset of an algebra to the subalgebra it generates. In an independence algebra, $\langle\cdot\rangle$ must satisfy the *exchange property*, which guarantees that we have a well-behaved notion of *rank* for subalgebras and hence for endomorphisms, generalizing that of the dimension of a vector space. Furthermore, independence algebras are *relatively free*. Precise definitions and further details may be found in § 2. We remark that free G-acts, for any group G, are further examples of independence algebras. A study of endomorphism monoids of independence algebras was initiated by the author in [**12**], where it is shown that, for an independence algebra A of finite rank, $\operatorname{End} A$ has the same ideal structure as $\operatorname{End} V$ for a finite-dimensional vector space V. Subsequently, Fountain and Lewin [**9**] proved that every 'singular' endomorphism of A is a product of idempotent endomorphisms.

The endomorphism monoid of an independence algebra A is regular. But, perhaps surprisingly, regularity of $\operatorname{End} A$ is not necessary for the above results concerning idempotent generation. For example, the results of Laffey [**16**] show that if A is a free module of finite rank n over a Euclidean domain, then the set of non-identity idempotents of $\operatorname{End} A$ generates the subsemigroup of endomorphisms of rank strictly less than n.

Fountain and Gould introduced in [**6**] a class of algebras called *stable basis algebras* that generalize free modules over Euclidean domains, in an attempt to put the results of Laffey, and later work of Fountain [**5**] and Ruitenberg [**20**], into a more general setting, an aim achieved in [**8**]. Stable basis algebras are, in particular, relatively free algebras in which the closure operator PC (pure closure) satisfies the exchange property. Certainly, independence algebras are stable basis algebras. Finitely generated free left modules over Bezout domains and finitely generated free left T-sets over any cancellative monoid T such that finitely generated left ideals of T are principal are examples of stable basis algebras. We recall that a *Bezout domain* is an integral domain (not necessarily commutative) in which all finitely generated left and right ideals are principal. As for independence algebras, rank is well defined for subalgebras and endomorphisms of basis algebras, where now the rank is defined via the operator PC. The endomorphism monoid of a stable basis algebra of finite rank has a $*$-ideal structure analogous to the ideal structure of the endomorphism monoid of a finite-rank independence algebra. Further details are given in §§ 2 and 3.

We remind the reader that if A and B are algebras such that the universe (that is, the underlying set) of B is contained in the universe of A, then B is a *reduct* of A if every basic operation of B is the restriction to B of a basic operation of A. Let R be a Bezout domain and let D be its division ring of (left) quotients. If F is a free module of finite rank n over R, then it is well known that F is a reduct of V, where V is a vector space over D: we have already observed that V is an independence algebra. On the other hand, if B is a stable basis algebra having only unary and nullary basic operations, then the results of [**6**] show that B is a reduct of an independence algebra. The first aim of this paper is to show that every stable basis algebra satisfying the *distributivity condition* is a reduct of an independence algebra. We remark that the distributivity condition is satisfied for all known examples of basis algebras that are not independence algebras and for the examples of independence algebras mentioned above. We enlarge upon this discussion in § 4.

Classical ring theory tells us that if R and D are as in the preceding paragraph, then $M_n(R)$ has a ring of (left) quotients $M_n(D)$, that is, $M_n(R)$ is a (left) order in $M_n(D)$. Of course, the endomorphism monoid of an arbitrary algebra, indeed of an arbitrary independence algebra A, need not be a ring, so it makes little sense to talk of left orders in $\operatorname{End} A$ in the sense of ring theory. Help is at hand, however, in the notion due originally to Fountain and Petrich [**11**] of a *semigroup* of (left) quotients, which we explain in § 3; if Q is a semigroup of (left) quotients of S, then we say that S is a *(left) order* in Q. The second aim of this paper is to show that if B is a stable basis algebra of finite rank n satisfying the distributivity condition, then $\operatorname{End} B$ is a left order in $\operatorname{End} A$, where A is the independence algebra we have constructed, of which B is a reduct.

If a semigroup S is a left order in a semigroup Q, then we hope that the structure of S is closely related to that of Q. This is certainly true if S has non-empty intersection with every $\mathcal{H}$-class of Q, a condition guaranteed if S is *straight* in Q (see § 3). Our final aim is to consider when $\operatorname{End} B$ is straight in $\operatorname{End} A$, where B is a stable basis algebra of finite rank n satisfying the distributivity condition and A is the independence algebra we have built from B. It might be anticipated that $\operatorname{End} B$ would *always* be straight in $\operatorname{End} A$, for we know from [**7**] that $\operatorname{End} B$ is certainly a straight left order in *some* semigroup. We know that, for any such B, the monoid T of non-constant unary term operations is a cancellative monoid that is *right reversible* (or *left Ore*), that is, for any $a, b \in T$ there exist $c, d \in T$ with $ca = db$. The property of *left reversibility* is defined dually. Perhaps surprisingly, we show that $\operatorname{End} B$ is straight in $\operatorname{End} A$ if and only if T acts by isomorphisms on the constant subalgebra of B and (if $n \geqslant 2$), T is also *left* reversible.

The structure of the paper is as follows. In § 2 we give a brief summary of the relevant definitions and properties associated with independence algebras and basis algebras. Section 3 contains the semigroup-theoretic results needed for this paper. In particular, we recall Green's relations and their $*$-generalizations and their realizations in $\operatorname{End} A$ and $\operatorname{End} B$, where A is an independence algebra and B is a stable basis algebra. The final three sections contain our results as outlined above. That is, given a stable basis algebra B satisfying the distributivity property, we construct in § 4 an independence algebra A of which B is a reduct. In § 5 we show that (if B has finite rank) $\operatorname{End} B$ is a left order in $\operatorname{End} A$ and we conclude in § 6 by addressing the question of when $\operatorname{End} B$ is straight in $\operatorname{End} A$.

2. Independence algebras and basis algebras

By an *algebra* A we mean an algebra in the sense of universal algebra. Since this paper is concerned with two special classes of algebras, namely independence and basis algebras, we briefly recall their construction and make note of some of their properties for later reference; further details may be found in [**7**, **8**, **12**]. We refer the reader to [**3**, **14**, **17**] for standard concepts of universal algebra. A *constant* in an algebra A is the image of a basic nullary operation.

Independence and basis algebras are approached via closure operators, defined below. In the case of independence algebras we use the standard subalgebra closure operator $\langle \cdot \rangle$,

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whereas for basis algebras we make use of the operator PC. These interrelated operators coincide for an independence algebra, but are distinct for a general basis algebra.

Let A be a set and let $C : \mathcal{P}(A) \to \mathcal{P}(A)$ be a function, where $\mathcal{P}(A)$ is the set of all subsets of A. Then C is a *closure operator* on A if C satisfies the following conditions for all $X, Y \in \mathcal{P}(A)$:

(i) $X \subseteq C(X)$;

(ii) if $X \subseteq Y$ then $C(X) \subseteq C(Y)$;

(iii) $C(X) = C(C(X))$.

A subset of A of the form $C(X)$ is said to be *closed*.

If A is any algebra, then $\langle\cdot\rangle$ is a closure operator on A, where, for all $X \subseteq A$, $\langle X\rangle$ is the subalgebra of A generated by X. Where there is more than one algebra in question and danger of ambiguity we denote the operator $\langle\cdot\rangle$ on A by $\langle\cdot\rangle_A$. We remark that if A has non-empty set of constants U, then $\langle\emptyset\rangle = \langle V\rangle = \langle U\rangle$ for any $V \subseteq U$, and $\langle\emptyset\rangle$ consists of those elements a for which there is a unary term operation with unique value a (see, for example, [**14**, p. 40, Corollary 3]). If A has no constants, then we make the convention that $\emptyset$ is a subalgebra, so that in this case $\langle\emptyset\rangle = \emptyset$. We say that A is *constant* if $A = \langle\emptyset\rangle$. More generally, it is clear that, for any subset X of an arbitrary algebra A, $\langle X\rangle$ is the set of terms that can be built from the elements of X. In view of this, it is easy to see that $\langle\cdot\rangle$ is always an algebraic closure operator, where a closure operator C on a set A is *algebraic* if, for all $X \subseteq A$,

$$C(X) = \bigcup\{C(Y) \mid Y \subseteq X,\ |Y| < \aleph_0\}.$$

A closure operator C on a set A satisfies the *exchange property* (EP) if, for all $X \subseteq A$ and $x, y \in A$,

$$\text{if } x \notin C(X) \text{ but } x \in C(X \cup \{y\}),\ \text{then } y \in C(X \cup \{x\}).$$

Perhaps the most familiar example of a closure operator with (EP) is $\langle\cdot\rangle$ on a vector space. Where there is no danger of ambiguity we say that an algebra A satisfies (EP) if $\langle\cdot\rangle$ does so.

Let C be a closure operator on a set A. A subset X of A is *C-independent* if $x \notin C(X \setminus \{x\})$ for all $x \in X$. In case A is an algebra and C is $\langle\cdot\rangle$, we say that a C-independent subset is *independent*. Clearly, for a vector space, a subset is independent if and only if it is linearly independent.

Algebraic closure operators that satisfy the exchange property are intimately connected with abstract dependence relations. Translating results of [**3**, § VII.2] to the language of algebraic closure operators yields the following result.

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Lemma 2.1 (Cohn [3, Lemma VII.2.2]). *Let C be an algebraic closure operator satisfying (EP) on a set A and let $Y \subseteq X \subseteq A$. Then the following conditions are equivalent:*

(i) *Y is a maximal C-independent subset of X;*

(ii) *Y is C-independent and $C(Y) = C(X)$;*

(iii) *Y is minimal with respect to $C(Y) = C(X)$.*

The next result is again classical, quoted here from [**6**].

Theorem 2.2 (Cohn [3] (cf. [6, Theorem 1.4])). *Let C be an algebraic closure operator satisfying (EP) on a set A, and let $X \subseteq Y \subseteq A$. If X is C-independent, then there is a C-independent subset Z with $X \subseteq Z \subseteq Y$ and $C(Z) = C(Y)$.*

Let C be an algebraic closure operator satisfying (EP) on a set A and let $Y \subseteq A$; the *C-rank* of Y is the cardinality of any maximal C-independent subset of Y. As explained in [**6**], classical results of universal algebra (see, for example, [**3**]) give that *the C-rank of Y is well defined*. Clearly, C-rank is monotonic and, from Lemma 2.1, for any $X \subseteq A$, $C\text{-rank}(X) = C\text{-rank}(C(X))$. Again we refer to our canonical example of a vector space, where the $\langle\cdot\rangle$-rank of a subspace is its familiar dimension. In this paper we frequently refer to $\langle\cdot\rangle$-rank more simply as *rank*.

We need one more concept in order to define independence algebras, which relates to free generators. Let A be an algebra. A subset X of A is *A-free* if any map from X to A can be extended to a morphism from $\langle X\rangle$ to A. As noted in [**14**], if $|A| > 1$, then every A-free subset is independent. We say that the *free basis* property (FB) holds for A, if every independent subset is A-free. An *independence algebra* is an algebra A satisfying (EP) and (FB). An independence algebra V is therefore certainly *relatively free*, that is, it is a free algebra in the variety it generates. As noted in § 1, vector spaces are the archetypical example of independence algebras. Notice that any constant algebra satisfies vacuously the conditions required to be an independence algebra.

The term 'independence algebra' was introduced in [**12**], where it is remarked that they are precisely the v^*-algebras of Narkiewicz [**14**, **19**]. The aim of [**12**] and subsequent papers such as [**1**, **9**, **10**] was to investigate the structure of the endomorphism monoid $\operatorname{End} A$ of an independence algebra A. Any such monoid has an ideal structure analogous to that of the monoid of linear maps of a vector space. In the case where the algebra has finite rank, Fountain and Lewin prove in [**9**] that, as was already known for vector spaces, every singular endomorphism can be written as a product of idempotent endomorphisms. By *singular*, we mean that the rank of the image of an endomorphism is strictly less than the rank of the algebra. As remarked in § 1, other algebras that are *not* independence algebras satisfy an analogous property, most particularly the endomorphism monoid of a finite-dimensional free abelian group. This phenomenon led to the development of the second class of algebras we consider, namely basis algebras.

Essentially, basis algebras are approached in an analogous way to independence algebras, but with the closure operator $\langle\cdot\rangle$ replaced by the operator PC.

For an element a of an algebra A and a subset X of A we write $a \prec X$ if

$$a \in \langle \emptyset \rangle \quad \text{or} \quad \langle a \rangle \cap \langle X \rangle \neq \langle \emptyset \rangle$$

and we set

$$\mathrm{PC}(X) = \{a \in A \mid a \prec X\}.$$

The operator PC need not be a closure operator [**6**, Theorem 1.6]. Where it is, it is algebraic and the closed subsets are subalgebras. In this case we refer to PC-independent subsets as *directly independent*. As remarked in [**6**], for any $X \subseteq A$ we have that $\langle X \rangle \subseteq \mathrm{PC}(X)$ and directly independent sets are independent, although the converse is not true in general. We say that A is a *weak exchange algebra* if PC is a closure operator satisfying (EP); in this case we say that A satisfies the *weak exchange property* (WEP). Subsets of A consequently have well-defined PC-rank. We require the following result from [**6**], most of which is classical and can be taken from [**3**].

Lemma 2.3 (Fountain and Gould [6, Corollary 1.11 (Corollary 1.12 of revised version)]). *Let X be a subset of a weak exchange algebra A. Then*

(i) $\text{PC-rank}(\langle X \rangle) = \text{PC-rank}(X) = \text{PC-rank}(\mathrm{PC}(X)) \leqslant |X|$;

(ii) *if X is finite and* $\text{PC-rank}(X) = |X|$, *then X is directly independent.*

From [**6**, Lemma 2.2 (Lemma 2.3 of revised version)], if A is a non-trivial algebra with constants, or without constants but having no constant unary term operations, then every A-free subset is directly independent. A *weak independence algebra A* is a weak exchange algebra in which every directly independent set is A-free.

The monoid T_1 of unary term operations of a weak independence algebra A is of particular importance to us. We let

$$T_C = \{\kappa^c : c \in \langle \emptyset \rangle\},$$

where κ^c denotes the constant map with image $c \in C$ and set

$$T = T_1 \setminus T_C.$$

Clearly, if A has no constants, then $T_C = \emptyset$ and $T = T_1$.

Proposition 2.4 (Fountain and Gould [6, Proposition 6.2 (Proposition 5.2 of revised version)]). *Let A be a weak independence algebra with constants such that A is not constant, that is, $A \neq \langle \emptyset \rangle$. Then, for $t \in T_1$, the following are equivalent:*

(i) *$t = \kappa^c$ for some $c \in A$;*

(ii) *$t(x) \in \langle \emptyset \rangle$ for all $x \in A$;*

(iii) *$t(x) \in \langle \emptyset \rangle$ for some $x \in A \setminus \langle \emptyset \rangle$.*

We say that a non-constant weak independence algebra A is *torsion-free* if each $t \in T$ is injective. We declare a constant algebra to be torsion-free.

Proposition 2.5 (Fountain and Gould [6, Proposition 6.4 (Corollary 5.5 of revised version)]). *Let A be a non-constant torsion-free weak independence algebra. Then T_C is a prime ideal of T_1, and T is a cancellative right reversible monoid.*

An A-free subset X of an algebra A is a *basis* of A if $X \cap \langle \emptyset \rangle = \emptyset$ and X generates A. It follows from the results of [**6**] that a basis of a subalgebra of a torsion-free weak independence algebra is exactly the same thing as a generating set that is directly independent.

In a weak independence algebra A a *pure* subalgebra is a subalgebra B that is PC-closed, that is, such that $B = \mathrm{PC}(B)$.

Lemma 2.6 (Fountain and Gould [6, Corollary 5.4 (Corollary 6.4 of revised version)]). *Let A be a torsion-free weak independence algebra. If X is a basis for A and $Y \subseteq X$, then $\langle Y \rangle$ is pure.*

A *basis algebra* A is a torsion-free weak independence algebra which satisfies the following condition:

(PEP) if P Q are pure subalgebras in A with $P \subseteq Q$, and X is a basis for P, then there is a basis Y for Q with $X \subseteq Y$.

It follows from this definition that every pure subalgebra of a basis algebra has a basis and, in particular, every basis algebra has a basis.

It is worth pausing to make some remarks concerning PC-rank and the cardinality of bases. Let B be a basis algebra and let $C = \langle X \rangle$ be a subalgebra of B. If X is a basis for C, then we have observed that X is a directly independent generating set for C. Since X generates C we see that X is *maximal* directly independent in C, in other words, X is a PC-basis for C. Consequently, $|X| = \text{PC-rank}(C)$ and $C \subseteq \mathrm{PC}(X) = \mathrm{PC}(C)$ with the inclusion being an equality if and only if C is pure.

Suppose now that A is a basis algebra, $\text{PC-rank}\, A = n$ and B is a pure subalgebra of A with $\text{PC-rank}\, B = n$. Since any basis of B can be extended to a basis of A, we must have that $A = B$.

Finally, we say that a basis algebra A is *stable* if every subalgebra of A having a generating set of cardinality at most PC-rank A has a basis.

Independence algebras are stable basis algebras. Indeed, in an independence algebra $\mathrm{PC}(X) = \langle X \rangle$, so that $\mathrm{rank}(X) = \text{PC-rank}(X)$ for any subset X. If R is a Bezout domain, then a free left R-module of finite rank is a stable basis algebra [**6**]. Our third canonical example of a stable basis algebra is a finitely generated free left T-act over a cancellative principal left ideal monoid T.

3. Left orders in semigroups: semigroups of left quotients

We present here a brief resumé of the semigroup theory needed for the remainder of this paper, and refer the reader to [**15**] for further details.

For any monoid S (that is, a semigroup with identity), the preorder $\leqslant_{\mathcal{L}}$ is defined by the rule that $a \leqslant_{\mathcal{L}} b$ if and only if $Sa \subseteq Sb$; the equivalence relation associated with $\leqslant_{\mathcal{L}}$

is Green's relation $\mathcal{L}$. The relations $\leqslant_{\mathcal{L}}$ and $\mathcal{L}$ are relations of *right divisibility*, for it is easy to see that, for elements $a, b \in S$, $a \leqslant_{\mathcal{L}} b$ if and only if $a = sb$ for some $s \in S$, and $a \,\mathcal{L}\, b$ if and only if $a = sb$, $b = ta$ for some $s, t \in S$. The preorder $\leqslant_{\mathcal{R}}$ and associated equivalence $\mathcal{R}$ are defined dually, using principal right ideals of S. The intersection of $\mathcal{L}$ and $\mathcal{R}$ is denoted by $\mathcal{H}$ and their join by $\mathcal{D}$; since $\mathcal{L}$ and $\mathcal{R}$ commute, $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$. The following crucial result is due to Green.

Theorem 3.1 (Green [15, Theorem 2.2.5]). *Let S be a monoid. For any $a \in S$, a lies in a subgroup of S if and only if $a \,\mathcal{H}\, a^2$.*

For the endomorphism monoid of an independence algebra A, $\leqslant_{\mathcal{L}}$ and $\leqslant_{\mathcal{R}}$, and so $\mathcal{L}$ and $\mathcal{R}$, have particularly pleasant realizations. We recall that, for any map $\alpha : X \to Y$, $\operatorname{Ker}\alpha$ is the equivalence relation on X defined by the rule that

$$x \operatorname{Ker}\alpha\, y \iff x\alpha = y\alpha.$$

Proposition 3.2 (Gould [12, Proposition 4.5]). *Let A be an independence algebra. Then, for any $\alpha, \beta \in \operatorname{End} A$,*

$$\alpha \leqslant_{\mathcal{L}} \beta \iff \operatorname{Im}\alpha \subseteq \operatorname{Im}\beta$$

and

$$\alpha \leqslant_{\mathcal{R}} \beta \iff \operatorname{Ker}\beta \subseteq \operatorname{Ker}\alpha.$$

Consequently,

$$\alpha \,\mathcal{L}\, \beta \iff \operatorname{Im}\alpha = \operatorname{Im}\beta$$

and

$$\alpha \,\mathcal{R}\, \beta \iff \operatorname{Ker}\alpha = \operatorname{Ker}\beta.$$

Let A be an independence algebra. We define the *rank* $\rho(\alpha)$ of $\alpha \in \operatorname{End} A$ to be the rank of $\operatorname{Im}\alpha$. The following result is a consequence of the ideal structure of $\operatorname{End} A$, as presented in [**12**].

Proposition 3.3 (Gould [12, Proposition 4.5, Theorem 4.9]). *Let A be an independence algebra of finite rank and let $\alpha, \beta \in \operatorname{End} A$. Then*

(i) *$\rho(\alpha) = \rho(\beta)$ if and only if $\alpha \mathcal{D} \beta$,*

(ii) *$\alpha \,\mathcal{L}\, \alpha^2$ if and only if $\alpha \,\mathcal{R}\, \alpha^2$ if and only if $\alpha \,\mathcal{H}\, \alpha^2$,*

(iii) *if $\alpha \leqslant_{\mathcal{L}} \beta$ and $\rho(\alpha) = \rho(\beta)$, then $\alpha \,\mathcal{L}\, \beta$, dually for $\leqslant_{\mathcal{R}}$ and $\mathcal{R}$.*

It follows from Proposition 3.3 that for an independence algebra A of finite rank, $\operatorname{End} A$ is *local*, that is, for any $\alpha \in \operatorname{End} A$, $\alpha \,\mathcal{L}\, I_A$ if and only if $\alpha \,\mathcal{R}\, I_A$, where we follow standard convention in denoting by I_A the identity map on A. For, if $\alpha \in \operatorname{End} A$ and $\alpha \,\mathcal{L}\, I_A$, then as $\mathcal{L}$ is right compatible with multiplication, $\alpha^2 \,\mathcal{L}\, \alpha$, whence $\alpha^2 \,\mathcal{H}\, \alpha$ so that, by Theorem 3.1, α lies in a subgroup G. Let ε be the identity of G. Then $\varepsilon \,\mathcal{L}\, I_A$, from which it follows that $\varepsilon = I_A$ and $\alpha \,\mathcal{H}\, I_A$. Dually, if $\alpha \,\mathcal{R}\, I_A$, then we again obtain that $\alpha \,\mathcal{H}\, I_A$.

To make remarks corresponding to those in Propositions 3.2 and 3.3 for basis algebras, we must consider the $*$-generalizations of Green's relations.

Let S be a monoid. The relation $\leqslant_{\mathcal{L}^*}$ on S is defined by the rule that $a \leqslant_{\mathcal{L}^*} b$ if and only if, for any $x, y \in S$,

$$bx = by \quad \Longrightarrow \quad ax = ay.$$

It is clear that $\leqslant_{\mathcal{L}^*}$ is a preorder; the associated equivalence relation is denoted by $\mathcal{L}^*$. The relation $\mathcal{L}^*$ has another characterization, namely that $a \,\mathcal{L}^*\, b$ if and only if $a \,\mathcal{L}\, b$ in some oversemigroup of S; this may easily be justified by employing the left regular representation of S. The relations $\leqslant_{\mathcal{R}^*}$ and $\mathcal{R}^*$ are defined dually. We continue the analogy with the notation for Green's relations by putting $\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*$ and $\mathcal{D}^* = \mathcal{L}^* \vee \mathcal{R}^*$. Unlike the case for Green's relations, $\mathcal{L}^*$ and $\mathcal{R}^*$ do not, in general, commute.

For a stable basis algebra B we define the *rank* $\rho(\alpha)$ of an element $\alpha \in \operatorname{End} B$ to be the PC-rank of the image of α. As commented in § 2, there is no danger of ambiguity here, since in an independence algebra, rank and PC-rank coincide.

Proposition 3.4 (Fountain and Gould [7, Lemmas 4.1 and 4.5]). *Let B be a stable basis algebra. Then for any $\alpha, \beta \in \operatorname{End} B$,*

$$\begin{aligned} \alpha \leqslant_{\mathcal{L}^*} \beta \quad &\Longleftrightarrow \quad \mathrm{PC}(\operatorname{Im}\alpha) \subseteq \mathrm{PC}(\operatorname{Im}\beta) \\ \alpha \leqslant_{\mathcal{R}^*} \beta \quad &\Longleftrightarrow \quad \operatorname{Ker}\beta \subseteq \operatorname{Ker}\alpha. \end{aligned}$$

Consequently,

$$\alpha \,\mathcal{L}^*\, \beta \quad \Longleftrightarrow \quad \mathrm{PC}(\operatorname{Im}\alpha) = \mathrm{PC}(\operatorname{Im}\beta)$$

and

$$\alpha \mathcal{R}^* \beta \quad \Longleftrightarrow \quad \operatorname{Ker}\alpha = \operatorname{Ker}\beta.$$

Proposition 3.5 (Fountain and Gould [7, Theorems 4.9 and 7.4]). *Let A be a stable basis algebra of finite rank. Then $\mathcal{R}^* \circ \mathcal{L}^* = \mathcal{L}^* \circ \mathcal{R}^*$ so that, consequently, $\mathcal{D}^* = \mathcal{L}^* \circ \mathcal{R}^*$. Further, for any $\alpha, \beta \in \operatorname{End} B$:*

(i) *$\rho(\alpha) = \rho(\beta)$ if and only if $\alpha \mathcal{D}^* \beta$;*

(ii) *$\alpha \,\mathcal{L}^*\, \alpha^2$ if and only if $\alpha \mathcal{R}^* \alpha^2$ if and only if $\alpha \,\mathcal{H}^*\, \alpha^2$;*

(iii) *if $\alpha \leqslant_{\mathcal{L}^*} \beta$ and $\rho(\alpha) = \rho(\beta)$, then $\alpha \,\mathcal{L}^*\, \beta$; dually for $\leqslant_{\mathcal{R}^*}$ and $\mathcal{R}^*$.*

A $*$-*ideal* of a monoid S is an ideal that is a union of $\mathcal{L}^*$-classes and $\mathcal{R}^*$-classes. In a stable basis algebra B of finite rank n there are $n+1$ $*$-ideals, namely $I_k, 0 \leqslant k \leqslant n$, where

$$I_k = \{\alpha \in \operatorname{End} B : \operatorname{rank}\alpha \leqslant k\}.$$

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The Rees quotients I_k/I_{k-1} are the non-regular analogue of completely 0-simple semigroups, being isomorphic to Rees matrix semigroups over cancellative monoids. We do not pursue these ideas here, but the interested reader may find further details in [**7**].

Unlike the case for independence algebras, in a stable basis algebra B not all subalgebras, hence not all images of endomorphisms, are pure. Moreover, in general, not every subalgebra of B will have a basis.

Proposition 3.6 (Fountain and Gould [7, Corollary 4.4]). *Let B be a stable basis algebra and let $\alpha \in \operatorname{End} B$. Then $\operatorname{Im}\alpha$ has a basis. Moreover, if α is idempotent, then $\operatorname{Im}\alpha$ is pure in B so that $\operatorname{Im}\alpha$ has a basis that can be extended to a basis for B.*

We end this section by giving the necessary background on the concept of a semigroup of left quotients, as introduced by Fountain and Petrich in [**11**]. We refer the reader to [**13**] for further details. Let S be a semigroup. An element a of S is *square-cancellable* if $a\,\mathcal{H}^*\,a^2$. By a remark following the definition of Green's $*$-relations above, together with Theorem 3.1, we see that being square-cancellable is a natural necessary condition for an element of S to lie in a subgroup of an oversemigroup. Suppose now that S is a subsemigroup of a semigroup Q. Then Q is a *semigroup of left quotients* of S and S is a *left order* in Q if every $q \in Q$ can be written as $q = a^\sharp b$, where $a, b \in S$ and $a^\sharp$ denotes the inverse of a in a subgroup of Q, and if, in addition, every square-cancellable element of S lies in a subgroup of Q. In the case that every $q \in Q$ can be written as $q = a^\sharp b$ in Q, where $a, b \in S$ and $a\,\mathcal{R}\,b$, then we say that S is *straight* in Q.

If Q is a group, then our definition of a semigroup of left quotients coincides with the classical notion of group of left quotients. The next theorem is due to Ore and Dubreil.

Theorem 3.7 (Ore and Dubreil [2, Theorem 1.24]). *A monoid T has a group of left quotients if and only if it is cancellative and right reversible.*

We will make repeated use of the classical 'Common Denominator Theorem' for a group of left quotients, easily proved via an inductive argument.

Theorem 3.8. *Let T be a left order in a group G. Then for any $g_1, \dots, g_n \in G$, there exist $a, b_1, \dots, b_n \in T$ such that $g_i = a^{-1}b_i$ for $1 \leqslant i \leqslant n$.*

4. A stable basis algebra *B* as a reduct of an independence algebra *A*

Throughout this section B denotes a non-constant stable basis algebra with monoid of non-constant unary term operations T having identity 1 (so that 1 is the identity map I_B on B). We may regard B as a left T-act, that is, there is a map $T \times B \to B$, where $(\alpha, b) \mapsto \alpha b = \alpha(b)$, such that $1b = b$ and $\alpha(\beta b) = (\alpha\beta)b$ for all $b \in B$ and $\alpha, \beta \in T$. We assume in addition that B satisfies the *distributivity condition*, which says that for all $\alpha \in T$ and n-ary basic term operations s, where $n \geqslant 2$, we have

$$\alpha(s(x_1, \dots, x_n)) = s(\alpha(x_1), \dots, \alpha(x_n))$$

for all $x_1, \dots, x_n \in B$. Our aim is to show that B is the reduct of an independence algebra A.

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Before proceeding, we make some remarks concerning the distributivity condition. Certainly, free modules over rings, and S-sets over monoids (hence our canonical examples of stable basis algebras), satisfy this condition. We have observed that all independence algebras are stable basis algebras. Independence algebras, under the original name of v^*-algebras, were completely determined in the 1960s; we refer the reader to [**21**] for the details. One may then ask which independence algebras (other than vector spaces and free G-sets over a group G) have the distributivity property. What we are trying to ascertain is whether or not it is possible, given an arbitrary independence algebra, to pick a generating set of basic term operations which will have the property that the elements of T will distribute over the chosen n-ary basic term operations for $n \geqslant 2$. For the four-element *exceptional algebra* and the *affine* independence algebras it is possible to do so. Other independence algebras are either essentially unary–nullary, when the distributivity condition holds trivially, or the S-*homogeneous* algebras or Q-*homogeneous* algebras, where S is a monoid and Q a quasi-field. It is an open problem whether all S-*homogeneous* and Q-*homogeneous* algebras satisfy the distributivity condition.

From Proposition 2.5 the monoid T is cancellative and right reversible. From Theorem 3.7 we know that T has a group of left quotients G.

Set $\Sigma = T \times B$ and define $\sim$ on Σ by the rule that $(\alpha, a) \sim (\beta, b)$ if and only if there exist $\gamma, \delta \in T$ with

$$\gamma\alpha = \delta\beta \quad \text{and} \quad \gamma a = \delta b.$$

Lemma 4.1. *The relation $\sim$ is an equivalence relation on Σ.*

Proof. Clearly, $\sim$ is symmetric, and as we certainly have $1\alpha = 1\alpha$ and $1a = 1a$ for any $(\alpha, a) \in \Sigma$, $\sim$ is reflexive. It remains to show that $\sim$ is transitive.

To this end, let $(\alpha, a), (\beta, b), (\gamma, c) \in \Sigma$ and suppose that

$$(\alpha, a) \sim (\beta, b) \sim (\gamma, c).$$

Then there exist $\delta, \epsilon, \mu, \nu$ with

$$\delta\alpha = \epsilon\beta, \qquad \delta a = \epsilon b,$$
$$\mu\beta = \nu\gamma, \qquad \mu b = \nu c.$$

Since T is right reversible, there are elements $\rho, \pi \in T$ with

$$\rho\epsilon = \pi\mu.$$

Then

$$\rho\delta\alpha = \rho\epsilon\beta = \pi\mu\beta = \pi\nu\gamma$$

and, similarly, $\rho\delta a = \pi\nu c$. Thus, $(\alpha, a) \sim (\gamma, c)$, and $\sim$ is an equivalence as required. □

We denote the $\sim$-equivalence class of $(\alpha, a) \in \Sigma$ by $[\alpha, a]$. Let $A = \Sigma/{\sim}$.

Lemma 4.2. *Suppose that $[1, a], [\beta, b] \in A$ and*

$$[1, a] = [\beta, b].$$

Then $\beta a = b$.

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Proof. From the definition of $\sim$ we have that

$$\alpha 1 = \gamma\beta \quad \text{and} \quad \alpha a = \gamma b$$

for some $\alpha, \gamma \in T$. Hence,

$$\gamma\beta a = \gamma b$$

and as γ is injective we have that $\beta a = b$. □

We now proceed to define basic operations on A, under which it becomes the independence algebra we require.

The nullary operations on A are straightforward. For each nullary operation c_B on B, with image c, we define a nullary operation c_A on A with image $[1, c]$. Similarly, for any basic unary operation $v_B^c = \kappa^c \in T_C$, where $c \in \langle\emptyset\rangle$, we define a basic unary operation v_A^c of A by the rule that $v_A^c([\alpha, a]) = [1, c]$, for any $[\alpha, a] \in A$.

The next lemma will help us to show that the remaining n-ary operations on A for $n \geqslant 1$, as given below, are well defined.

Lemma 4.3. *Suppose that $\alpha, \beta, \gamma, \delta, \mu, \nu \in T$ and $a, b \in B$ are such that*

$$\gamma\alpha = \delta\beta, \quad \gamma a = \delta b \quad \text{and} \quad \mu\alpha = \nu\beta.$$

Then

$$\mu a = \nu b.$$

Proof. We have that $\gamma^{-1}\delta = \alpha\beta^{-1} = \mu^{-1}\nu$, so that $\mu\gamma^{-1}\delta = \nu$. Since T is right reversible, we can choose $\theta, \varphi \in T$ with

$$\theta\mu = \varphi\gamma.$$

Then $\mu\gamma^{-1} = \theta^{-1}\varphi$ and so

$$\varphi\delta = \theta\nu.$$

Calculating, we have that

$$\theta\mu a = \varphi\gamma a = \varphi\delta b = \theta\nu b.$$

Now θ is injective and so $\mu a = \nu b$ as required. □

For each $\xi \in G$ we define a unary operation u_A^ξ by the rule

$$u_A^\xi([\gamma, c]) = [\mu, \nu c],$$

where $\xi\gamma^{-1} = \mu^{-1}\nu$ and $\mu, \nu \in T$.

Lemma 4.4. *The operation u_A^ξ is well defined.*

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Proof. Suppose that $(\gamma_1, c_1) \sim (\gamma_2, c_2)$ and μ_i, ν_i, $i = 1, 2$, are chosen such that

$$\xi\gamma_1^{-1} = \mu_1^{-1}\nu_1 \quad \text{and} \quad \xi\gamma_2^{-1} = \mu_2^{-1}\nu_2.$$

We aim to show that $(\mu_1, \nu_1 c_1) \sim (\mu_2, \nu_2 c_2)$.

Since $(\gamma_1, c_1) \sim (\gamma_2, c_2)$ there exist $\rho, \kappa \in T$ with

$$\rho\gamma_1 = \kappa\gamma_2 \quad \text{and} \quad \rho c_1 = \kappa c_2.$$

By the right reversibility of T we can choose $\eta, \pi \in T$ with

$$\eta\mu_1 = \pi\mu_2.$$

We have that

$$\xi = \mu_1^{-1}\nu_1\gamma_1 = \mu_2^{-1}\nu_2\gamma_2 \quad \text{and} \quad \mu_1\mu_2^{-1} = \eta^{-1}\pi,$$

so that

$$\eta\nu_1\gamma_1 = \pi\nu_2\gamma_2.$$

Since in addition we have that

$$\rho\gamma_1 = \kappa\gamma_2 \quad \text{and} \quad \rho c_1 = \kappa c_2,$$

we call upon Lemma 4.3 to deduce that $\eta\nu_1 c_1 = \pi\nu_2 c_2$. Thus, $(\mu_1, \nu_1 c_1) \sim (\mu_2, \nu_2 c_2)$ as required. □

If $t_B = \alpha \in T$ is a unary operation on B, then we declare t_A to be u_A^α.

For each n-ary basic operation t_B on B, where $n \geqslant 2$, we define an n-ary operation t_A on A by the rule that

$$t_A([\alpha_1, a_1], \dots, [\alpha_n, a_n]) = [\alpha, t_B(\beta_1 a_1, \dots, \beta_n a_n)],$$

where $\alpha, \beta_i \in T$ $(1 \leqslant i \leqslant n)$ are chosen by Lemma 3.8 such that $\alpha_i^{-1} = \alpha^{-1}\beta_i$ $(1 \leqslant i \leqslant n)$.

Lemma 4.5. *The operation t_A is well defined.*

Proof. Suppose that $(\alpha_i, a_i), (\alpha_i', a_i') \in \Sigma$ and

$$(\alpha_i, a_i) \sim (\alpha_i', a_i'), \quad 1 \leqslant i \leqslant n,$$

and α, α', β_i, β_i' are chosen such that

$$\alpha_i^{-1} = \alpha^{-1}\beta_i \quad \text{and} \quad (\alpha_i')^{-1} = (\alpha')^{-1}\beta_i'$$

for $1 \leqslant i \leqslant n$. We must show that

$$(\alpha, t_B(\beta_1 a_1, \dots, \beta_n a_n)) \sim (\alpha', t_B(\beta_1' a_1', \dots, \beta_n' a_n')).$$

First, we choose $\gamma, \delta \in T$ with

$$\gamma\alpha = \delta\alpha'.$$

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Since for $1 \leqslant i \leqslant n$ we have that $(\alpha_i, a_i) \sim (\alpha'_i, a'_i)$, there are elements ρ_i, τ_i such that

$$\rho_i \alpha_i = \tau_i \alpha'_i \quad \text{and} \quad \rho_i a_i = \tau_i a'_i, \quad 1 \leqslant i \leqslant n.$$

Notice that for any $i \in \{1, \dots, n\}$ we have that

$$\alpha(\alpha')^{-1} = \gamma^{-1}\delta = \beta_i \alpha_i (\beta'_i \alpha'_i)^{-1}$$

and so

$$\gamma \beta_i \alpha_i = \delta \beta'_i \alpha'_i.$$

Calling upon Lemma 4.3 we deduce that

$$\gamma \beta_i a_i = \delta \beta'_i a'_i, \quad 1 \leqslant i \leqslant n.$$

Using the distributivity condition, we have that

$$\begin{aligned} \gamma t_B(\beta_1 a_1, \dots, \beta_n a_n) &= t_B(\gamma \beta_1 a_1, \dots, \gamma \beta_n a_n) \\ &= t_B(\delta \beta'_1 a'_1, \dots, \delta \beta'_n a'_n) \\ &= \delta t_B(\beta'_1 a'_1, \dots, \beta'_n a'_n) \end{aligned}$$

and so

$$(\alpha, t_B(\beta_1 a_1, \dots, \beta_n a_n)) \sim (\alpha', t_B(\beta'_1 a'_1, \dots, \beta'_n a'_n)),$$

as required. □

We show via a series of lemmas that A, together with the basic nullary, unary and n-ary ($n \geqslant 2$) operations defined as above, is an independence algebra. As a first step we gather together some useful elementary observations.

Lemma 4.6.

(i) *For any $\alpha, \beta \in T$ and $b \in B$ we have*

$$u_A^{\alpha}([1, b]) = [1, \alpha b], u_A^{\alpha^{-1}}([1, b]) = [\alpha, b] \quad \textit{and} \quad [\alpha\beta, \alpha b] = [\beta, b].$$

(ii) *For any $\theta \in G$, $(u_A^{\theta})^{-1} = u_A^{\theta^{-1}}$.*

(iii) *If C is a subalgebra of A, then for any $a \in B$ and $\alpha, \beta, \gamma, \delta \in T$ we have*

$$[\alpha, \beta a] \in C \quad \Longleftrightarrow \quad [\gamma, \delta a] \in C.$$

Proof. (i) We have that

$$\alpha 1^{-1} = \alpha 1 = 1\alpha = 1^{-1}\alpha$$

so that, by definition,

$$u_A^{\alpha}([1, b]) = [1, \alpha b].$$

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Similarly,

$$\alpha^{-1}1^{-1} = \alpha^{-1}1$$

and so

$$u_A^{\alpha^{-1}}([1, b]) = [\alpha, 1b] = [\alpha, b].$$

Finally,

$$1(\alpha\beta) = \alpha\beta \quad \text{and} \quad 1(\alpha b) = \alpha b,$$

so that the third statement follows from the definition of $\sim$.

(ii) Let $[\alpha, a] \in A$ and $\theta \in G$. Then $\theta^{-1}\alpha^{-1} = \mu^{-1}\nu$ for some $\mu, \nu \in T$, so that

$$u_A^{\theta^{-1}}([\alpha, a]) = [\mu, \nu a].$$

Now $\theta\mu^{-1} = \alpha^{-1}\nu^{-1} = (\nu\alpha)^{-1} = (\nu\alpha)^{-1}1$, so that

$$u_A^{\theta} u_A^{\theta^{-1}}([\alpha, a]) = u_A^{\theta}([\mu, \nu a]) = [\nu\alpha, \nu a] = [\alpha, a]$$

by (i). It follows that u_A^{θ} and $u_A^{\theta^{-1}}$ are mutually inverse.

(iii) Let C be a subalgebra of A and suppose that $[\gamma, \delta a] \in C$. Suppose that $\alpha, \beta \in T$ and let $\theta = \alpha^{-1}\beta\delta^{-1}\gamma$. Then $\theta\gamma^{-1} = \mu^{-1}\nu$, say, where $\mu, \nu \in T$, so that

$$u_A^{\theta}([\gamma, \delta a]) = [\mu, \nu\delta a].$$

We have that $\alpha^{-1}\beta\delta^{-1} = \mu^{-1}\nu$, so that $\alpha^{-1}\beta = \mu^{-1}\nu\delta$. Writing $\mu\alpha^{-1}$ as $\tau^{-1}\kappa$, we have that $\tau\mu = \kappa\alpha$ and $\tau\nu\delta a = \kappa\beta a$, whence

$$[\alpha, \beta a] = u_A^{\theta}([\gamma, \delta a]) \in C.$$

□

Lemma 4.7. *For any term operation t_B and $z_1, \dots, z_n \in B$,*

$$t_A([1, z_1], \dots, [1, z_n]) = [1, t_B(z_1, \dots, z_n)].$$

Proof. For a basic nullary operation c_B, we have that

$$c_A(\emptyset) = [1, c] = [1, c_B(\emptyset)].$$

Similarly, for a basic unary operation of the form $v_B^c = \kappa^c \in T_C$, we have that

$$v_A^c([1, a]) = [1, c] = [1, v_B^c(a)];$$

for basic unary operations of the form u_A^{α}, where $\alpha \in T$, we call upon Lemma 4.6 (i). For a basic n-ary operation t_B for $n \geqslant 2$, the statement follows immediately from the definition of t_A. The result can now be argued using induction on the number of basic term operations needed to build an arbitrary term operation t_B. □

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For a subset D of B we write $[T, D]$ as shorthand for

$$\{[\alpha, d] \mid \alpha \in T, d \in D\}.$$

Lemma 4.8. *Let $X = \{[\alpha_i, a_i] \mid i \in I\} \subseteq A$ and set $Y = \{a_i \mid i \in I\}$. Then*

$$\langle X\rangle_A = [T, \langle Y\rangle_B].$$

Proof. It is clear that

$$\langle X\rangle_A \subseteq [T, \langle Y\rangle_B].$$

For the converse, we show by induction on the number of basic term operations needed to build $y \in \langle Y\rangle_B$ from elements of Y that $[1, y] \in \langle X\rangle_A$.

If $y \in Y$, then $y = a_i$ for some $i \in I$ and pair $[\alpha_i, a_i] \in X$. Since $[\alpha_i, y] \in X$, Lemma 4.6 gives that $[1, y] \in \langle X\rangle_A$.

If y is the image of a nullary operation y_B, then $[1, y]$ is the image of the nullary operation y_A, so that $[1, y] \in \langle X\rangle_A$. Similarly, if $y = v^c_B(u)$ where $c \in \langle\emptyset\rangle_B$ and $u \in \langle Y\rangle_B$ with $[1, u] \in \langle X\rangle_A$, then

$$[1, y] = [1, c] = v^c_A([1, u]) \in \langle X\rangle_A.$$

If $y = \alpha z$, where $\alpha \in T$, $z \in \langle Y\rangle_B$ and $[1, z] \in \langle X\rangle_A$, then Lemma 4.6 gives directly that $[1, y] \in \langle X\rangle_A$.

Finally, suppose that $y = t_B(z_1, \dots, z_n)$ for some basic n-ary operation t_B ($n \geqslant 2$) and $z_1, \dots, z_n \in \langle Y\rangle_B$ with $[1, z_i] \in \langle X\rangle_A$ for $1 \leqslant i \leqslant n$. Then Lemma 4.7 yields that

$$[1, y] = [1, t_B(z_1, \dots, z_n)] = t_A([1, z_1], \dots, [1, z_n]) \in \langle X\rangle_A.$$

The result now follows from Lemma 4.6. □

Lemma 4.9. *The algebra A satisfies (EP).*

Proof. Let $X = \{[\alpha_i, a_i] \mid i \in I\}$, and suppose that

$$[\alpha, a] \in \langle X \cup \{[\beta, b]\}\rangle_A$$

but that

$$[\alpha, a] \notin \langle X\rangle_A.$$

By Lemma 4.8,

$$[\alpha, a] = [\gamma, t_B(b_1, \dots, b_n, b)]$$

for some $b_1, \dots, b_n \in Y = \{a_i \mid i \in I\}$. By definition of $\sim$ we have that

$$\mu\alpha = \nu\gamma \quad \text{and} \quad \mu a = \nu t_B(b_1, \dots, b_n, b)$$

for some $\mu, \nu \in T$.

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If $a \in \langle\emptyset\rangle_B$, then, by Lemma 4.8, $[\alpha, a] \in \langle\emptyset\rangle_A \subseteq \langle X\rangle_A$, which is a contradiction. Thus, $a \notin \langle\emptyset\rangle_B$ and by Proposition 2.4 we have that

$$\mu a = \nu t_B(b_1, \dots, b_n, b) \notin \langle\emptyset\rangle_B,$$

so that

$$a \prec \{b_1, \dots, b_n, b\}.$$

If $a \prec \{b_1, \dots, b_n\}$, then, as $a \notin \langle\emptyset\rangle_B$, we must have that

$$\pi a = s_B(b_1, \dots, b_n) \notin \langle\emptyset\rangle_B$$

for some term operation s_B. It follows that $[1, \pi a] \in \langle X\rangle_A$, so that from Lemma 4.6 $[\alpha, a] \in \langle X\rangle_A$, again a contradiction.

We deduce that $a \not\prec \{b_1, \dots, b_n\}$ and so, as B has (WEP), $b \prec \{b_1, \dots, b_n, a\}$.

If $b \in \langle\emptyset\rangle_B$, then we must have that $[\beta, b] \in \langle X \cup \{[\alpha, a]\}\rangle_A$. On the other hand, if $\kappa b = v_B(b_1, \dots, b_n, a) \notin \langle\emptyset\rangle_B$, then from Lemma 4.8 we have that $[1, \kappa b] \in \langle X \cup \{[\alpha, a]\}\rangle_A$ and so finally, from Lemma 4.6, $[\beta, b] \in \langle X \cup \{[\alpha, a]\}\rangle_A$. Thus, A satisfies (EP). □

Lemma 4.10. *Let $X = \{[\alpha_i, a_i] \mid i \in I\}$, where we suppose that $[\alpha_i, a_i] \neq [\alpha_j, a_j]$ for $i \neq j$, and let $Y = \{a_i \mid i \in I\}$. Then X is independent in A if and only if $a_i \neq a_j$ for all $i, j \in I$ with $i \neq j$ and Y is directly independent in B.*

Proof. Suppose first that X is independent. We first observe that if $i \neq j$, then $a_i \neq a_j$. For if $a_i = a_j$, then by Lemma 4.6

$$[\alpha_i, a_i] \in \langle\{[\alpha_j, a_j]\}\rangle_A \subseteq \langle X \setminus \{[\alpha_i, a_i]\}\rangle_A,$$

contradicting the independence of X.

Suppose now that, for some $i \in I$,

$$a_i \prec Y \setminus \{a_i\},$$

so that either $a_i \in \langle\emptyset\rangle_B$ or $\gamma a_i = t_B(y_1, \dots, y_n) \notin \langle\emptyset\rangle_B$ for some $\gamma \in T$, $y_1, \dots, y_n \in Y \setminus \{a_i\}$ and term t_B. The first possibility would lead to the contradiction that $[\alpha_i, a_i] \in \langle\emptyset\rangle_A \subseteq \langle X \setminus \{[\alpha_i, a_i]\}\rangle_A$. The second would lead via a now familiar argument using Lemmas 4.8 and 4.6 to $[1, \gamma a_i] \in \langle X \setminus \{[\alpha_i, a_i]\}\rangle_A$ and then to $[\alpha_i, a_i] \in \langle X \setminus \{[\alpha_i, a_i]\}\rangle_A$. We deduce that Y is directly independent.

Suppose conversely that $a_i \neq a_j$ for all $i, j \in I$ with $i \neq j$, and that Y is directly independent. If

$$[\alpha_i, a_i] \in \langle X \setminus \{[\alpha_i, a_i]\}\rangle_A$$

for some $i \in I$, then by Lemma 4.8 we have that

$$[\alpha_i, a_i] = [\beta, t_B(y_1, \dots, y_n)]$$

for some $\beta \in T$ and term t_B with $y_1, \dots, y_n \in Y \setminus \{a_i\}$. Using the definition of $\sim$ we must have that

$$\gamma\alpha_i = \delta\beta \quad \text{and} \quad \gamma a_i = \delta t_B(y_1, \dots, y_n)$$

for some $\gamma, \delta \in T$. Since Y is directly independent, we have that $a_i \notin \langle \emptyset \rangle$ so that, by Proposition 2.4,

$$\gamma a_i = \delta t_B(y_1, \dots, y_n) \notin \langle \emptyset \rangle.$$

But this says that

$$a_i \prec Y \setminus \{a_i\},$$

contradicting the fact that Y is directly independent. We deduce that X is independent. □

In order to conclude that A is an independence algebra, it remains to show that A satisfies (FB). To this end we need one further subsidiary lemma.

Lemma 4.11. *Let $[\alpha_i, a_i], [\beta_i, b_i] \in A$ for $1 \leqslant i \leqslant n$. Then there exist $\gamma_i, \delta_i \in T$, $1 \leqslant i \leqslant n$, such that, for any n-ary term operation t_A, there exist an n-ary term operation s_B and an element $\varepsilon \in T$ (depending on t_A) such that*

$$t_A([\alpha_1, a_1], \dots, [\alpha_n, a_n]) = [\varepsilon, s_B(\gamma_1 a_1, \dots, \gamma_n a_n)]$$

and

$$t_A([\beta_1, b_1], \dots, [\beta_n, b_n]) = [\varepsilon, s_B(\delta_1 b_1, \dots, \delta_n b_n)].$$

Proof. We employ Lemma 3.8 to find elements $\varepsilon, \gamma_1, \dots, \gamma_n, \delta_1, \dots, \delta_n$ of T such that

$$\alpha_i^{-1} = \varepsilon^{-1}\gamma_i \quad \text{and} \quad \beta_i^{-1} = \varepsilon^{-1}\delta_i, \quad 1 \leqslant i \leqslant n.$$

Consequently,

$$[\alpha_i, a_i] = [\varepsilon, \gamma_i a_i] \quad \text{and} \quad [\beta_i, b_i] = [\varepsilon, \delta_i b_i], \quad 1 \leqslant i \leqslant n.$$

Suppose that t_A is the ith projection p_A^i. We therefore have that

$$t^A([\alpha_1, a_1], \dots, [\alpha_n, a_n]) = [\alpha_i, a_i] = [\varepsilon, \gamma_i a_i] = [\varepsilon, p_B^i(\gamma_1 a_1, \dots, \gamma_n a_n)]$$

and, similarly,

$$t^A([\beta_1, b_1], \dots, [\beta_n, b_n]) = [\varepsilon, p_B^i(\delta_1 b_1, \dots, \delta_n b_n)].$$

We proceed by induction on the number of basic term operations of A needed to construct t_A. Suppose that t is constructed in $m \geqslant 2$ steps, and the result is true for all term operations constructed in fewer moves. If t_A is a constant term operation, then the result is clear with $\varepsilon = 1$.

Suppose now that $t_A = v_A^c s_A$, where $c \in \langle \emptyset \rangle_B$; by the inductive assumption we can find $\pi \in T$ and term function w_B of B such that

$$s_A([\alpha_1, a_1], \dots, [\alpha_n, a_n]) = [\pi, w_B(\gamma_1 a_1, \dots, \gamma_n a_n)]$$

and

$$s_A([\beta_1, b_1], \dots, [\beta_n, b_n]) = [\pi, w_B(\delta_1 b_1, \dots, \delta_n b_n)].$$

Now

$$\begin{aligned} t_A([\alpha_1, a_1], \dots, [\alpha_n a_n]) &= v_A^c([\pi, w_B(\gamma_1 a_1, \dots, \gamma_n a_n)]) \\ &= [1, c] \\ &= [1, v_B^c p_B^1(\gamma_1 a_1, \dots, \gamma_n a_n)] \end{aligned}$$

and, similarly,

$$t_A([\beta_1, b_1], \dots, [\beta_n, b_n]) = [1, v_B^c p_B^1(\delta_1 b_1, \dots, \delta_n b_n)].$$

On the other hand, suppose that $t_A = u_A^\rho s_A$ for some $\rho \in G$; let π, w_B be as above. Then $\rho\pi^{-1} = \mu^{-1}\nu$ for some $\mu, \nu \in T$, giving that

$$\begin{aligned} t_A([\alpha_1, a_1], \dots, [\alpha_n, a_n]) &= u_A^\rho([\pi, w_B(\gamma_1 a_1, \dots, \gamma_n a_n)]) \\ &= [\mu, \nu w_B(\gamma_1 a_1, \dots, \gamma_n a_n)] \end{aligned}$$

and

$$t_A([\beta_1, b_1], \dots, [\beta_n, b_n]) = [\mu, \nu w_B(\delta_1 b_1, \dots, \delta_n b_n)].$$

Finally, we suppose that

$$t_A = s_A(w_A^1, \dots, w_A^m)$$

for some $m \geqslant 2$ and basic m-ary operation s_B of B. By our inductive assumption we can find $\pi_1, \dots, \pi_m \in T$ and n-ary term operations $v_B^1, \dots, v_B^m$ of B such that, for $1 \leqslant i \leqslant m$,

$$w_A^i([\alpha_1, a_1], \dots, [\alpha_n, a_n]) = [\pi_i, v_B^i(\gamma_1 a_1, \dots, \gamma_n a_n)]$$

and

$$w_A^i([\beta_1, b_1], \dots, [\beta_n, b_n]) = [\pi_i, v_B^i(\delta_1 b_1, \dots, \delta_n b_n)].$$

Choose π, ρ_i with

$$\pi_i^{-1} = \pi^{-1}\rho_i \quad \text{for } 1 \leqslant i \leqslant m,$$

so that

$$\begin{aligned} t_A([\alpha_1, a_1], \dots, [\alpha_n, a_n]) &= s_A([\pi_1, v_B^1(\gamma_1 a_1, \dots, \gamma_n a_n)], \dots, [\pi_m, v_B^m(\gamma_1 a_1, \dots, \gamma_n a_n)]) \\ &= [\pi, s_B(\rho_1 v_B^1(\gamma_1 a_1, \dots, \gamma_n a_n), \dots, \rho_m v_B^m(\gamma_1 a_1, \dots, \gamma_n a_n))] \end{aligned}$$

and, similarly,

$$t_A([\beta_1, b_1], \dots, [\beta_n, b_n]) = [\pi, s_B(\rho_1 v_B^1(\delta_1 b_1, \dots, \delta_n b_n), \dots, \rho_m v_B^m(\delta_1 b_1, \dots, \delta_n b_n))]$$

as required. □

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Lemma 4.12. *The algebra A satisfies (FB).*

Proof. It only remains to show that every independent subset of A is A-free. To this end let

$$X = \{[\alpha_i, a_i] \mid i \in I\}$$

be an independent subset of A, where we assume that for $i \neq j$, $[\alpha_i, a_i] \neq [\alpha_j, a_j]$. By Lemma 4.10, $a_i \neq a_j$ for $i \neq j$, and $Y = \{a_i \mid i \in I\}$ is directly independent.

Suppose that $\theta : X \to A$ is a map such that

$$[\alpha_i, a_i]\theta = [\beta_i, b_i].$$

We define

$$\bar{\theta} : \langle X\rangle_A \to A$$

by

$$t_A([\alpha_1, a_1], \dots, [\alpha_n, a_n])\bar{\theta} = t_A([\beta_1, b_1], \dots, [\beta_n, b_n]).$$

If $\bar{\theta}$ is well defined, it is clear that it is a morphism and extends θ.

Suppose now that

$$t_A([\alpha_1, a_1], \dots, [\alpha_n, a_n]) = s_A([\alpha_1, a_1], \dots, [\alpha_n, a_n]) \quad \text{for some } [\alpha_1, a_1], \dots, [\alpha_n, a_n] \in X.$$

From Lemma 4.11 there exist $\mu, \nu, \gamma_1, \dots, \gamma_n, \delta_1, \dots, \delta_n \in T$ and n-ary term operations u_B and v_B such that

$$\begin{aligned} t_A([\alpha_1, a_1], \dots, [\alpha_n, a_n]) &= [\mu, u_B(\gamma_1 a_1, \dots, \gamma_n a_n)], \\ t_A([\beta_1, b_1], \dots, [\beta_n, b_n]) &= [\mu, u_B(\delta_1 b_1, \dots, \delta_n b_n)], \\ s_A([\alpha_1, a_1], \dots, [\alpha_n, a_n]) &= [\nu, v_B(\gamma_1 a_1, \dots, \gamma_n a_n)] \end{aligned}$$

and

$$s_A([\beta_1, b_1], \dots, [\beta_n, b_n]) = [\nu, v_B(\delta_1 b_1, \dots, \delta_n b_n)].$$

We have that

$$[\mu, u_B(\gamma_1 a_1, \dots, \gamma_n a_n)] = [\nu, v_B(\gamma_1 a_1, \dots, \gamma_n a_n)],$$

and so there exist $\pi, \tau \in T$ with

$$\pi\mu = \tau\nu \quad \text{and} \quad \pi u_B(\gamma_1 a_1, \dots, \gamma_n a_n) = \tau v_B(\gamma_1 a_1, \dots, \gamma_n a_n).$$

From [**7**, Lemma 2.8], $\{\gamma_1 a_1, \dots, \gamma_n a_n\}$ is a directly independent subset of B of cardinality n. Now B is a basis algebra, so that $\{\gamma_1 a_1, \dots, \gamma_n a_n\}$ is therefore A-free, and the function $\gamma_i a_i \mapsto \delta_i b_i$, $1 \leqslant i \leqslant n$, lifts to a morphism from $\langle\{\gamma_1 a_1, \dots, \gamma_n a_n\}\rangle_B$ to B. It follows that

$$\pi u_B(\delta_1 b_1, \dots \delta_n b_n) = \tau v_B(\delta_1 b_1, \dots, \delta_n b_n)$$

and, consequently,

$$t_A([\beta_1, b_1], \dots, [\beta_n, b_n]) = s_A([\beta_1, b_1], \dots, [\beta_n, b_n]),$$

so that $\bar{\theta}$ is well defined. □

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Having constructed the independence algebra A, we now define $\iota : B \to A$ by $b\iota = [1, b]$, for any $b \in B$.

Lemma 4.13. *The function ι is one–one and embeds B as a reduct of A.*

Proof. If $b\iota = c\iota$ for some $b, c \in B$, then from $[1, b] = [1, c]$ we must have that

$$\alpha 1 = \beta 1 \quad \text{and} \quad \alpha b = \beta c$$

for some $\alpha, \beta \in T$. Hence, $\alpha = \beta$ so that $\alpha b = \alpha c$. By torsion-freeness we deduce that $b = c$ and that ι is one–one. That ι embeds B as a reduct of A follows immediately from Lemma 4.7. □

We can now present the main result of this section.

Theorem 4.14. *Let B be a stable basis algebra satisfying the distributivity condition. Then B is a reduct of an independence algebra A. Moreover, the rank of B is equal to the rank of A.*

Proof. We suppose that B is non-constant, else the result is clearly true with $A = B$. With A constructed as above, it only remains to show that A and B have the same rank. We can say rather more than this. Let $Y \subseteq B$. Observe that $Y\iota = \{[1, y] \mid y \in Y\}$ and immediately from Lemma 4.10 we have that $Z \subseteq Y$ is directly independent if and only if $Z\iota \subseteq Y\iota$ is independent, so that PC-rank $Y = \operatorname{rank} Y\iota$.

Let X be a maximal directly independent subset of B, so that $|X| = \text{PC-rank}\, B$. Then $X\iota$ is a maximal independent subset of $B\iota$, so that by Lemma 2.1, X is a maximal independent subset of $\langle B\iota \rangle = A$, and $\operatorname{rank} A = |X\iota| = |X|$. □

Corollary 4.15. *Let B be a non-constant stable basis algebra satisfying the distributivity condition, and let T be the monoid of non-constant unary term operations on B. Then the following conditions are equivalent:*

(i) *T is a group;*

(ii) *B is an independence algebra;*

(iii) End B *is regular.*

Proof. (i) ⇒ (ii). If T is a group, then T coincides with its group of left quotients G. Let A be constructed as above and let $\iota : B \to A$ be the given embedding. For any $[\alpha, a] \in A$ we observe that $[\alpha, a] = [1, \alpha^{-1} a]$, so that $B\iota = A$. Identifying B with its image in A, we notice that the term operations of A and $B\iota$ coincide (although in general A has more *basic* operations than B) so that, for any $X \subseteq A = B$, $\langle X \rangle_A = \langle X \rangle_B$ and clearly B is an independence algebra.

(ii) ⇒ (iii). This follows from [**12**, Proposition 4.7].

(iii) ⇒ (i). Let X be a basis for B, fix $x \in X$ and let $\alpha \in T$. Define $\theta : B \to B$ by $y\theta = \alpha(x)$ for all $y \in X$. Since End B is regular by assumption, there is an endomorphism φ of B such that $\theta\varphi\theta = \theta$.

We have that

$$\alpha(x) = x\theta = x\theta\varphi\theta = (\alpha(x))\varphi\theta = \alpha(x\varphi\theta),$$

so that $x = x\varphi\theta$ since α is injective. Let $x\varphi = t(y_1, \dots, y_n)$ for some term t and $y_1, \dots, y_n \in X$. Consequently,

$$x = t(y_1, \dots, y_n)\theta = t(y_1\theta, \dots, y_n\theta) = t(\alpha(x), \dots, \alpha(x)) = \beta(\alpha(x))$$

for some $\beta \in T$. But $\{x\}$ is B-free, so we deduce that $b = \beta\alpha(b)$ for all $b \in B$.

We have shown that β is a left inverse for α in the monoid T; but T is cancellative, so that α and β are mutually inverse. Consequently, T is a group. □

We end this section with an illustrative example. Let T be a cancellative monoid such that its finitely generated left ideals are principal, and let B be the free left T-act on a finite set X. We have commented that B is a finite-rank stable basis algebra. The monoid T is (isomorphic to) the monoid of (non-constant) unary term operations on B; we know from our general theory that T must be right reversible; this is also easy to see directly, since for any $a, b \in T$ we have that Ta and Tb are comparable. Vacuously, B has the distributivity property. The independence algebra constructed in Theorem 4.14 is the free left G-act on X, where G is the group of left quotients of T.

5. End B is a left order in End A

Throughout this section we let B be a non-constant stable basis algebra satisfying the distributivity condition, and let A be the independence algebra constructed as in § 4. We show that if B has finite rank, then End B is a left order in End A. For our preliminary lemmas, however, we need impose no condition on the rank of B.

Lemma 5.1. *The endomorphism monoid of B can be embedded in the endomorphism monoid of A.*

Proof. Let Y be a basis for B. It follows from Lemma 4.10 that

$$X = Y\iota = \{[1, y] \mid y \in Y\}$$

is independent in A. Moreover, from Lemma 4.8, X generates A and is thus a basis for A.

Let $\theta \in$ End B and define $\bar{\theta} \in$ End A by the rule that

$$[1, y]\bar{\theta} = [1, y\theta]$$

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for all $[1, y] \in X$. Let $b \in B$; as Y is a basis for B, we have that $b = t_B(y_1, \dots, y_n)$ for some $y_i \in Y$. In view of Lemma 4.7,

$$\begin{aligned}
[1, b]\bar{\theta} &= [1, t_B(y_1, \dots, y_n)]\bar{\theta} \\
&= t_A([1, y_1], \dots, [1, y_n])\bar{\theta} \\
&= t_A([1, y_1]\bar{\theta}, \dots, [1, y_n]\bar{\theta}) \\
&= t_A([1, y_1\theta], \dots, [1, y_n\theta]) \\
&= [1, t_B(y_1\theta, \dots, y_n\theta)] \\
&= [1, t(y_1, \dots, y_n)\theta] \\
&= [1, b\theta].
\end{aligned}$$

Indeed, we can say a little more than this. If $[\alpha, a] \in A$, then

$$\begin{aligned}
[\alpha, a]\bar{\theta} &= (u_A^{\alpha^{-1}}([1, a]))\bar{\theta} \\
&= u_A^{\alpha^{-1}}([1, a]\bar{\theta}) \\
&= u_A^{\alpha^{-1}}([1, a\theta]) \\
&= [\alpha, a\theta].
\end{aligned}$$

Suppose now that $\bar{\theta} = \bar{\varphi}$. Then for any $b \in B$, $[1, b]\bar{\theta} = [1, b]\bar{\varphi}$, so that $[1, b\theta] = [1, b\varphi]$ and so $b\theta = b\varphi$ since ι is an embedding. Hence, $\theta = \varphi$.

We now define $\Phi : \operatorname{End} B \to \operatorname{End} A$ by the rule that

$$\theta\Phi = \bar{\theta}.$$

By the above, Φ is an injection, and clearly $I_B\Phi = I_A$.

Let $\theta, \varphi \in \operatorname{End} B$. For any $y \in Y$ we have that

$$[1, y]\bar{\theta}\bar{\varphi} = [1, y\theta]\bar{\varphi} = [1, y\theta\varphi] = [1, y]\overline{\theta\varphi}$$

so that $\bar{\theta}\bar{\varphi}$ and $\overline{\theta\varphi}$ agree on a basis. Consequently, $\bar{\theta}\bar{\varphi} = \overline{\theta\varphi}$ and Φ is an embedding as required. □

In what follows, for $\psi \in \operatorname{End} B$, $\bar{\psi}$ will denote the endomorphism of A constructed as in Lemma 5.1.

Lemma 5.2.

(i) *Let* $\theta \in \operatorname{End} B$ *and let* Y *be a* PC-*basis for* $\operatorname{Im}\theta$. *Then*

$$X = \{[1, y] \mid y \in Y\}$$

is a basis for $\operatorname{Im}\bar{\theta}$.

(ii) *If* $\theta, \varphi \in \operatorname{End} B$, *then* $\theta \mathcal{L}^* \varphi$ *in* $\operatorname{End} B$ *if and only if* $\bar{\theta} \mathcal{L} \bar{\varphi}$ *in* $\operatorname{End} A$.

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Proof. (i) Since Y is directly independent, X is independent by Lemma 4.10. Let $C = \langle X \rangle_A$.

For any $y = y'\theta \in Y$ we have from Lemma 5.1 that

$$[1, y] = [1, y'\theta] = [1, y']\bar{\theta} \in \operatorname{Im} \bar{\theta},$$

so that $X \subseteq \operatorname{Im} \bar{\theta}$ and consequently, $C \subseteq \operatorname{Im} \bar{\theta}$.

Let Z be a basis for B so that $Z\iota$ is a basis for A. Certainly,

$$\operatorname{Im} \bar{\theta} = \langle \{[1, z] \mid z \in Z\} \rangle_A \bar{\theta} = \langle \{[1, z]\bar{\theta} \mid z \in Z\} \rangle_A = \langle \{[1, z\theta] \mid z \in Z\} \rangle_A.$$

For any $z \in Z$ we have that

$$z\theta \prec Y,$$

so that

$$z\theta \in \langle \emptyset \rangle \quad \text{or} \quad \alpha(z\theta) = s_B(y_1, \dots, y_n) \notin \langle \emptyset \rangle$$

for some $y_1, \dots, y_n \in Y$ and term operations α and s_B. In the first case $[1, z\theta] \in C$, and in the second case we must have that $z\theta \notin \langle \emptyset \rangle$ and $\alpha \in T$, so that

$$[1, \alpha z\theta] = [1, s_B(y_1, \dots, y_n)] = s_A([1, y_1], \dots, [1, y_n]) \in C$$

using Lemma 4.7. But then $[1, z\theta] \in C$ and we deduce that $\operatorname{Im} \bar{\theta} \subseteq C$. Hence, $C = \operatorname{Im} \bar{\theta}$ as required.

(ii) In view of the comments following the definition of $\mathcal{L}^*$ and $\mathcal{R}^*$, we need only show that if $\theta \,\mathcal{L}^*\, \varphi$ in $\operatorname{End} B$, then $\bar{\theta} \,\mathcal{L}\, \bar{\varphi}$ in $\operatorname{End} A$. Suppose therefore that $\theta \,\mathcal{L}^*\, \varphi$, so that, by Proposition 3.4, $\mathrm{PC}(\operatorname{Im} \theta) = \mathrm{PC}(\operatorname{Im} \varphi)$. Let Y and Z be bases (and hence PC-bases) for $\operatorname{Im} \theta$ and $\operatorname{Im} \varphi$, respectively, so that

$$\mathrm{PC}(Y) = \mathrm{PC}(\operatorname{Im} \theta) = \mathrm{PC}(\operatorname{Im} \varphi) = \mathrm{PC}(Z).$$

By (i) we have that

$$\operatorname{Im} \bar{\theta} = \langle \{[1, y] \mid y \in Y\} \rangle_A \quad \text{and} \quad \operatorname{Im} \bar{\varphi} = \langle \{[1, z] \mid z \in Z\} \rangle_A.$$

For any $z \in Z$ we have that

$$z \in \operatorname{Im} \varphi \subseteq \mathrm{PC}(\operatorname{Im} \varphi) = \mathrm{PC}(Y)$$

so that $z \prec Y$. Hence $z \in \langle \emptyset \rangle_B$, or $\alpha z = u(y_1, \dots, y_n) \notin \langle \emptyset \rangle_B$ for some $\alpha \in T, y_1, \dots, y_n \in Y$ and term function u_B. In the first case, $[1, z] \in \langle \emptyset \rangle_A \subseteq \operatorname{Im} \bar{\theta}$, and in the second,

$$[1, \alpha z] = [1, u(y_1, \dots, y_n)] \in \operatorname{Im} \bar{\theta},$$

by Lemma 4.8. But then $[1, z] \in \operatorname{Im} \bar{\theta}$, whence $\operatorname{Im} \bar{\varphi} \subseteq \operatorname{Im} \bar{\theta}$. Together with the dual argument, we obtain that $\operatorname{Im} \bar{\theta} = \operatorname{Im} \bar{\varphi}$, so that $\bar{\theta} \,\mathcal{L}\, \bar{\varphi}$ by Proposition 3.2. $\square$

We can now state the second of the two main results of this paper.

Theorem 5.3. *Let B be a stable basis algebra satisfying the distributivity condition, with finite* PC*-rank $n \geqslant 1$. Then B is a reduct of an independence algebra A such that* $\operatorname{End} B$ *is a left order in* $\operatorname{End} A$.

Proof. Let A be constructed as given in § 4. It only remains to show that $\operatorname{End} B$ is a left order in $\operatorname{End} A$.

Let $Y = \{b_1, \dots, b_n\}$ be a basis for B, so that as in Lemma 5.1,

$$X = Y\iota = \{[1, b_1], \dots, [1, b_n]\}$$

is a basis for A. Let $\theta \in \operatorname{End} A$; by a now standard argument using the common denominator theorem we can write

$$[1, b_i]\theta = [\alpha, a_i],$$

for some $\alpha \in T$ and $a_i \in B$, $1 \leqslant i \leqslant n$.

Define $\kappa \in \operatorname{End} B$ by $b_i\kappa = \alpha b_i$, $1 \leqslant i \leqslant n$. Then $\bar{\kappa} \in \operatorname{End} A$ is given by

$$[1, b_i]\bar{\kappa} = [1, \alpha b_i], \quad 1 \leqslant i \leqslant n.$$

We now define $\tau \in \operatorname{End} A$ by the rule that

$$[1, b_i]\tau = [\alpha, b_i], \quad 1 \leqslant i \leqslant n.$$

We claim that $\bar{\kappa}$ and τ are mutually inverse in $\operatorname{End} A$. To see this we calculate that, for $i \in \{1, \dots, n\}$,

$$[1, b_i]\bar{\kappa}\tau = [1, \alpha b_i]\tau = (u_A^{\alpha}([1, b_i]))\tau = u_A^{\alpha}([1, b_i]\tau) = u_A^{\alpha}([\alpha, b_i]) = [1, b_i],$$

so that, consequently, $\bar{\kappa}\tau = I_A$. The monoid $\operatorname{End} A$ is local, so we obtain that $\bar{\kappa}$ and τ are mutually inverse.

Finally, we define $\varphi \in \operatorname{End} B$ by the rule that

$$b_i\varphi = a_i, \quad 1 \leqslant i \leqslant n.$$

Then $\bar{\varphi} \in \operatorname{End} A$ and

$$\begin{aligned}
[1, b_i]\tau\bar{\varphi} &= [\alpha, b_i]\bar{\varphi} \\
&= (u_A^{\alpha^{-1}}([1, b_i]))\bar{\varphi} \\
&= u_A^{\alpha^{-1}}([1, b_i]\bar{\varphi}) \\
&= u_A^{\alpha^{-1}}([1, b_i\varphi]) \\
&= u_A^{\alpha^{-1}}([1, a_i]) \\
&= [\alpha, a_i] \\
&= [1, b_i]\theta.
\end{aligned}$$

Consequently, $\theta = \tau\bar{\varphi} = \bar{\kappa}^{-1}\bar{\varphi}$.

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It remains to show that every square-cancellable element of End B lies in a subgroup of End A or, more properly, that if $\theta \in \operatorname{End} B$ and $\theta \, \mathcal{H}^* \, \theta^2$, then $\bar{\theta}$ lies in a subgroup of End A.

Suppose then that $\theta \in \operatorname{End} B$ is square-cancellable, so that, in particular, $\theta \, \mathcal{L}^* \, \theta^2$. By Lemma 5.2,

$$\bar{\theta} \, \mathcal{L} \, \overline{\theta^2} = \bar{\theta}^2.$$

Proposition 3.3 tells us that $\bar{\theta} \, \mathcal{H} \, \bar{\theta}^2$, whence, by Theorem 3.1, $\bar{\theta}$ lies in a subgroup of End A. □

6. When is End B straight in End A?

The main success achieved in characterizing left orders in semigroups and the most natural examples of left orders has, to date, been in the case where the left orders are straight (see § 3). Rather surprisingly, not all of our left orders of the form End B, where B is a finite-rank stable basis algebra satisfying the distributivity condition, need be straight. We prove in this section that, for such a B, End B is straight in End A, where A is the independence algebra constructed as in § 4, if and only the monoid T of non-constant unary term operations satisfies a rather natural property that we call the 'constant isomorphism' condition, and, if $n \geqslant 2$, T is *left* reversible. This result is all the more curious, since Theorem 6.12 of [**7**] tells us End B *is* a straight left order in *some* semigroup.

We remark that, for any $\alpha \in T$,

$$\alpha|_{\langle \emptyset \rangle_B} : \langle \emptyset \rangle_B \to \langle \emptyset \rangle_B$$

is a one–one map since B is torsion-free. We say that B satisfies the *constant isomorphism* condition (CI) if

$$\alpha|_{\langle \emptyset \rangle_B} : \langle \emptyset \rangle_B \to \langle \emptyset \rangle_B$$

is *onto*, hence an *isomorphism* of the constant subalgebra.

We begin our argument with a subsidiary result.

Lemma 6.1. *Let A be an independence algebra with basis $\{x_1, \dots, x_n\}$. Let $k \in \{0, \dots, n\}$ and let $\theta \in \operatorname{End} A$ be defined by the rule*

$$x_i\theta = x_i, \quad 1 \leqslant i \leqslant k,$$

and

$$x_j\theta = u_j \in \langle \{x_1, \dots, x_k\} \rangle, \quad k+1 \leqslant j \leqslant n.$$

Then

$$\operatorname{Ker} \theta = \langle \{(x_j, u_j) \mid k+1 \leqslant j \leqslant n\} \rangle.$$

Proof. Let $\rho = \langle \{(x_j, u_j) \mid k+1 \leqslant j \leqslant n\} \rangle$; clearly, $\rho \subseteq \operatorname{Ker} \theta$. On the other hand, if $v(x_1, \dots, x_n), w(x_1, \dots, x_n) \in A$ and

$$v(x_1, \dots, x_n)\theta = w(x_1, \dots, x_n)\theta,$$

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then

$$v(x_1,\dots,x_n)\,\rho\,v(x_1,\dots,x_k,u_{k+1},\dots,u_n)=w(x_1,\dots,x_k,u_{k+1},\dots,u_n)\,\rho\,w(x_1,\dots,x_n),$$

so that $\operatorname{Ker}\theta\subseteq\rho$. □

Our first characterization of straightness is technical; we will simplify later to the conditions given at the beginning of this section.

Proposition 6.2. *Let B be a stable basis algebra of finite rank $n\geqslant 1$, satisfying the distributivity condition, let A be the independence algebra constructed as in § 4, and let $\bar{\ }:\operatorname{End}B\to\operatorname{End}A$ be the embedding as given in § 5. Then $\operatorname{End}B$ is a straight left order in $\operatorname{End}A$ if and only if, for any $k\in\{0,\dots,n\}$, k-ary term operations $t_B^{k+1},\dots,t_B^n$ and $\alpha\in T$, there exist directly independent $a_1,\dots,a_k\in B$ with*

$$t_B^j(a_1,\dots,a_k)\in\alpha(B),\quad k+1\leqslant j\leqslant n.$$

Proof. Let $\{b_1,\dots,b_n\}$ be a basis for B, so that, as in Lemma 5.1, $\{[1,b_1],\dots,[1,b_n]\}$ is a basis for A.

Suppose first that $\operatorname{End}B$ is straight in $\operatorname{End}A$ and let k, α and $t_B^{k+1}\dots,t_B^n$ be as given. Define $\theta\in\operatorname{End}A$ by the rule that

$$[1,b_i]\theta=[1,b_i],\quad 1\leqslant i\leqslant k,$$

and

$$[1,b_j]\theta=[\alpha,t_B^j(b_1,\dots,b_k)],\quad k+1\leqslant j\leqslant n.$$

Since $[\alpha,t_B^j(b_1,\dots,b_k)]\in\langle\{[1,b_i]\mid 1\leqslant i\leqslant k\}\rangle_A$ we have that

$$\operatorname{Im}\theta=\langle\{[1,b_i]\mid 1\leqslant i\leqslant k\}\rangle_A.$$

Furthermore, θ restricts to the identity on $\langle\{[1,b_i]\mid 1\leqslant i\leqslant k\}\rangle_A$, so that, for $k+1\leqslant j\leqslant n$,

$$[1,b_j]\theta=[\alpha,t_B^j(b_1,\dots,b_k)]\theta$$

and moreover, by Lemma 6.1,

$$\operatorname{Ker}\theta=\langle\{([1,b_j],[\alpha,t_B^j(b_1,\dots,b_k)])\mid k+1\leqslant j\leqslant n\}\rangle.$$

By assumption, $\operatorname{End}B$ is straight in $\operatorname{End}A$, so that, by [**13**, Proposition 3.1], $\theta\,\mathcal{H}\,\bar{\varphi}$ for some $\varphi\in\operatorname{End}B$. Set $b_i\varphi=a_i$ so that $[1,b_i]\bar{\varphi}=[1,a_i]$, $1\leqslant i\leqslant n$. Since $\operatorname{Ker}\theta=\operatorname{Ker}\bar{\varphi}$, we have that, for $k+1\leqslant j\leqslant n$,

$$\begin{aligned}[1,a_j]&=[1,b_j]\bar{\varphi}\\&=[\alpha,t_B^j(b_1,\dots,b_k)]\bar{\varphi}\\&=[\alpha,t_B^j(a_1,\dots,a_k)],\end{aligned}$$

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using Lemma 5.1, so that

$$\operatorname{Im}\bar{\varphi} = \langle\{[1, a_1], \dots, [1, a_k]\}\rangle_A.$$

We know that $k = \operatorname{rank}\theta = \operatorname{rank}\bar{\varphi}$, whence $[1, a_1], \dots, [1, a_k]$ are independent. From Lemma 4.10, $a_1, \dots, a_k$ are directly independent. For $k + 1 \leqslant j \leqslant n$ we use Lemma 4.2 to deduce from $[1, a_j] = [\alpha, t^j_B(a_1, \dots, a_k)]$ that $t^j(a_1, \dots, a_k) = \alpha a_j$.

Conversely, we suppose that the given condition holds. We begin by considering an endomorphism $\theta : A \to A$ of rank k defined by the rule that

$$\begin{aligned} [1, b_i]\theta &= [1, b_i], & 1 \leqslant i \leqslant k, \\ [1, b_j]\theta &= v^j_A([1, b_1], \dots, [1, b_k]), & k + 1 \leqslant j \leqslant n. \end{aligned}$$

Notice that, from Lemma 6.1,

$$\operatorname{Ker}\theta = \langle\{[([1, b_j], v^j_A([1, b_1], \dots, [1, b_k])) \mid k + 1 \leqslant j \leqslant n\}\rangle.$$

In view of Lemmas 3.8 and 4.8, we can find term operations t^j_B, $k+1 \leqslant j \leqslant n$ and $\alpha \in T$ such that

$$[1, b_j]\theta = [\alpha, t^j_B(b_1, \dots, b_k)], \quad k + 1 \leqslant j \leqslant n.$$

We now invoke our hypothesis to choose directly independent $a_1, \dots, a_k$ in B such that

$$t^j_B(a_1, \dots, a_k) = \alpha a_j, \quad k + 1 \leqslant j \leqslant n,$$

for some $a_{k+1}, \dots, a_n \in B$. Defining $\varphi : B \to B$ by the rule that $b_i\varphi = a_i$, $1 \leqslant i \leqslant n$, we claim that $\bar{\varphi}\,\mathcal{R}\,\theta$.

Making use of an observation in Lemma 5.1,

$$[\alpha, t^j_B(b_1, \dots, b_k)]\bar{\varphi} = [\alpha, t^j_B(a_1, \dots, a_k)].$$

On the other hand, by Lemma 4.2,

$$[1, b_j]\bar{\varphi} = [1, a_j] = [\alpha, t^j_B(a_1, \dots, a_k)],$$

and we deduce that $\operatorname{Ker}\theta \subseteq \operatorname{Ker}\bar{\varphi}$. Clearly, $\operatorname{Im}\bar{\varphi} = \langle\{[1, a_1], \dots, [1, a_k]\}\rangle_A$, so that, as $\{a_1, \dots, a_k\}$ are directly independent by assumption, $\bar{\varphi}$ has rank k. Proposition 3.3 now gives that $\theta\,\mathcal{R}\,\bar{\varphi}$.

Now choose an arbitrary $\psi \in \operatorname{End} A$ with rank k. Without loss of generality we may assume that

$$\operatorname{Im}\psi = \langle\{[1, b_1]\psi, \dots, [1, b_k]\psi\}\rangle_A,$$

where $[1, b_1]\psi, \dots, [1, b_k]\psi$ are independent and, for $k + 1 \leqslant j \leqslant n$,

$$[1, b_j]\psi \in \langle\{[1, b_1]\psi, \dots, [1, b_k]\psi\}\rangle_A.$$

Define $\mu : \operatorname{Im}\psi \to A$ by the rule that

$$[1, b_i]\psi\mu = [1, b_i].$$

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By [**12**, Lemma 3.7], μ is a one–one morphism, so that $\operatorname{Ker}\psi\mu = \operatorname{Ker}\psi$ and $\psi\mu\,\mathcal{R}\,\psi$. From the above we know that $\psi\mu\,\mathcal{R}\,\bar{\varphi}$ for some $\varphi \in \operatorname{End} B$, so that $\psi\,\mathcal{R}\,\bar{\varphi}$. We conclude that every $\mathcal{R}$-class of $\operatorname{End} A$ contains an element in (the image of) $\operatorname{End} B$.

For the remainder of the proof, suppose that $\theta \in \operatorname{End} A$ and $\varphi \in \operatorname{End} B$ with $\theta\,\mathcal{R}\,\bar{\varphi}$. We aim to show that there exists $\kappa \in \operatorname{End} B$ with $\theta\,\mathcal{H}\,\bar{\kappa}$.

By Proposition 3.3 and Lemma 5.2, θ, φ and $\bar{\varphi}$ all have the same rank. From [**7**, Theorem 4.9], $\varphi\,\mathcal{L}^*\,\varepsilon$ for some $\varepsilon = \varepsilon^2 \in \operatorname{End} B$, so that, from Lemma 5.2, $\bar{\varphi}\,\mathcal{L}\,\bar{\varepsilon}$. Now [**7**, Lemma 4.7] tells us that $\operatorname{Im}\varepsilon$ is pure in B, so that, as B is a basis algebra, $\operatorname{Im}\varepsilon$ has a basis Y that can be extended to a basis $Y \cup Z$ of B. Certainly, Y is a PC-basis for $\operatorname{Im}\varepsilon$ so that, from Lemma 5.2,

$$X = \{[1, y] \mid y \in Y\}$$

is a basis for $\operatorname{Im}\bar{\varepsilon} = \operatorname{Im}\bar{\varphi}$.

Since θ and $\bar{\varphi}$ have the same rank, we have a basis $T = \{[\alpha_y, a_y] \mid y \in Y\}$ for $\operatorname{Im}\theta$, where $[\alpha_y, a_y] \neq [a_{y'}, a_{y'}]$ for $y \neq y'$. Clearly, $U = \{[1, a_y] \mid y \in Y\}$ generates $\operatorname{Im}\theta$ and, by two applications of Lemma 4.8, U is independent and hence a basis for $\operatorname{Im}\theta$.

Define $\xi \in \operatorname{End} B$ by fixing its value on the basis $Y \cup Z$ by

$$y\xi = a_y, \quad z\xi = b \quad \text{for } y \in Y,\ z \in Z \text{ and fixed } b \in B.$$

Then $\operatorname{Im}\bar{\varphi}\bar{\xi} = \operatorname{Im}\theta$, whence $\bar{\varphi}\bar{\xi} = \overline{\varphi\xi}\,\mathcal{L}\,\theta$. Furthermore, $\bar{\xi}|_X$ is one–one and $X\bar{\xi} = U$ is independent, so that, by [**12**, Lemma 3.7], $\bar{\xi}|_{\langle X\rangle_A}$ is one–one. Thus, $\operatorname{Ker}\overline{\varphi\xi} = \operatorname{Ker}\theta$, giving that $\overline{\varphi\xi}\,\mathcal{R}\,\theta$. We have therefore shown that $\overline{\varphi\xi}\,\mathcal{H}\,\theta$ so that, by [**13**, Proposition 3.1], $\operatorname{End} B$ is straight in $\operatorname{End} A$ as claimed. □

We can now prove our final result.

Theorem 6.3. *Let B be a stable basis algebra of finite rank $n \geqslant 1$ satisfying the distributivity condition, and let T be the monoid of non-constant unary term operations. Let A be the independence algebra constructed as in § 4, and let* $^{-} : \operatorname{End} B \to \operatorname{End} A$ *be the embedding as given in § 5. Then $\operatorname{End} B$ is a straight left order in $\operatorname{End} A$ if and only if B satisfies (CI) and if $n \geqslant 2$, then T is left reversible.*

Proof. Suppose first that $\operatorname{End} B$ is straight in $\operatorname{End} A$. Let $\alpha \in T$; we are required to argue that $\alpha|_{\langle\emptyset\rangle_B} : \langle\emptyset\rangle_B \to \langle\emptyset\rangle_B$ is onto. To this end, let $c \in \langle\emptyset\rangle_B$. Let $k = 0$ and consider the nullary term operation $w_B^c : B \to B$ with image c. Set $t_i = w_B^c$ for $1 \leqslant i \leqslant n$; by Proposition 6.2, $c = w_B^c(\emptyset) = \alpha b$ for some $b \in B$. But we are then forced to have $b \in \langle\emptyset\rangle_B$, so that $\alpha|_{\langle\emptyset\rangle} : \langle\emptyset\rangle_B \to \langle\emptyset\rangle_B$ is onto.

Suppose now that $n \geqslant 2$. In order to show that T is left reversible, let $\alpha, \beta \in T$, set $k = 1$ and let $t_B^2 = \cdots = t_B^n$ be the unary term operations given by

$$t_B^j(x) = \beta(x).$$

From Proposition 6.2, there is an element $a_1 \in B$ with $\{a_1\}$ directly independent such that

$$\beta(a_1) = t_B^2(a_1) = \alpha(a_2)$$

for some $a_2 \in B$. It follows that $\beta(a_1) \notin \langle\emptyset\rangle_B$, so that $a_2 \notin \langle\emptyset\rangle_B$ and $a_2 \prec \{a_1\}$. Let $\{d\}$ be a basis for $\mathrm{PC}(\{a_1\})$, so that

$$\langle\{d\}\rangle_B = \mathrm{PC}(\{a_1\}).$$

We therefore have that

$$a_1 = \gamma(d) \quad \text{and} \quad a_2 = \delta(d)$$

for some unary term operations γ, δ. Clearly, $\gamma, \delta \in T$ and we have that

$$\beta\gamma(d) = \alpha\delta(d).$$

Now $\{d\}$ is B-free, whence $\beta\gamma(b) = \alpha\delta(b)$ *for all* $b \in B$. But this says that $\beta\gamma = \alpha\delta$ and T is left reversible as required.

Conversely, we suppose that B has (CI) and, in the case where $n \geqslant 2$, that T is left reversible. We show that the condition given in Proposition 6.2 holds.

Consider first the case where $k = 0$. Let $\alpha \in T$ and let $t_B^1, \dots, t_B^n$ be nullary term operations on B. Let $c_j \in \langle\emptyset\rangle_B$ be the image of t_B^j for $1 \leqslant j \leqslant n$. By our assumption that B has (CI), there exist $d_1, \dots, d_n \in \langle\emptyset\rangle_B$ such that, for $1 \leqslant j \leqslant n$, $t_B^j(\emptyset) = c_j = \alpha d_j$.

Suppose now that $k \geqslant 1$. Notice that if $k = n$, then the condition of Proposition 6.2 is vacuously satisfied. We assume, therefore, that $k < n$ so that $n \geqslant 2$ and T is therefore left reversible.

At this stage it is convenient to consider the free term algebra $\mathcal{T}_k$ on $\{x_1, \dots, x_k\}$ having the same signature as B. We write $\langle\emptyset\rangle$ for $\langle\emptyset\rangle_{\mathcal{T}_k}$ and denote elements of $\langle\emptyset\rangle$ by $\bar{c}$, where c is the corresponding element of $\langle\emptyset\rangle_B$, and write $\bar{\alpha}$ for a basic unary operation with interpretation α in T. For an arbitrary term u of $\mathcal{T}_k$ we denote by u_B its interpretation in B.

Let $u^1, \dots, u^m$ be a finite list of k-ary term operations in $\mathcal{T}_k$. Assume for the moment that this list is of the form

$$p^1, \dots, p^k, v^1, \dots, v^h,$$

where for $1 \leqslant \ell \leqslant k$, p^ℓ is the ℓth projection and, for $1 \leqslant i \leqslant h$, $v^i = \kappa^{\overline{c_i}}$, where (with some abuse of notation) $\kappa^{\overline{c_i}}$ is the k-ary constant term operation having constant image $\overline{c_i} \in \langle\emptyset\rangle$. Since B has (CI), we can find elements $d_i \in \langle\emptyset\rangle_B$ such that $c_i = \alpha d_i$, $1 \leqslant i \leqslant h$. Let $\{b_1, \dots, b_n\}$ be a basis for B and set $a_\ell = \alpha b_\ell$, $1 \leqslant \ell \leqslant k$. Then

$$p_B^\ell(a_1, \dots, a_k) = p_B^\ell(\alpha b_1, \dots, \alpha b_k) = \alpha b_\ell \quad \text{and} \quad v_B^i(a_1, \dots, a_k) = c_i = \alpha d_i$$

for $1 \leqslant \ell \leqslant k$ and $1 \leqslant i \leqslant h$. Moreover, $\{a_1, \dots, a_k\}$ is directly independent, by [**7**, Lemma 2.8].

We now consider an arbitrary finite list $\mathfrak{L}$,

$$u^1, \dots, u^m,$$

of k-ary term operations of $\mathcal{T}_k$ and show by induction on

$$N(\mathfrak{L}) = \sum_{j=1}^{m} N(u^j),$$

where $N(u^j)$ is the number of basic operations needed to construct u^j from projections and elements of $\langle\emptyset\rangle$, that for any $\alpha \in T$ there are directly independent elements $a_1, \ldots, a_k \in B$ such that

$$u^j_B(a_1, \ldots, a_k) \in \alpha(B), \quad 1 \leqslant i \leqslant m,$$

whence the condition of Proposition 6.2 follows immediately.

The case for $N = 0$ has been successfully argued. Suppose now that $N(\mathfrak{L}) > 0$ and the result is true for all lists $\mathfrak{L}'$ with $N(\mathfrak{L}') < N(\mathfrak{L})$. Fix $\alpha \in T$. Since $N(\mathfrak{L}) > 0$, we must be able to find an element of the list that without loss of generality we may take to be u^m, such that u^m is neither a projection nor of the form $\kappa^{\bar{c}}$. There are three cases to consider.

We first look at the situation where $u^m = \bar{\delta}s(y_1, \ldots, y_k)$, $\bar{\delta}$ is unary and $\delta = \kappa^c \in T_C$. Let $\mathfrak{L}'$ be the list

$$u^1, \ldots, u^{m-1}, s$$

so that $N(\mathfrak{L}') < N(\mathfrak{L})$ and, by our inductive assumption, there are directly independent $a_1, \ldots, a_k \in B$ with

$$u^j_B(a_1, \ldots, a_k) = \alpha d_j, \qquad s_B(a_1, \ldots, a_k) = \alpha d$$

for some $d_1, \ldots, d_{m-1}, d \in B$. We then observe that

$$u^m_B(a_1, \ldots, a_k) = \delta s_B(a_1, \ldots, a_k) = c = \alpha b$$

for some b, by our assumption that B has (CI).

Next, we consider the case where $u^m = \bar{\beta}s(y_1, \ldots, y_k)$, where $\beta \in T$. Now T is left reversible, so that $\beta\alpha\delta = \alpha\gamma$ for some $\delta, \gamma \in T$. Using our inductive assumption for the element $\alpha\delta \in T$, we can find directly independent $a_1, \ldots, a_k \in B$ with

$$u^j_B(a_1, \ldots, a_k) = (\alpha\delta)d_j = \alpha(\delta d_j) \quad \text{and} \quad s_B(a_1, \ldots, a_k) = (\alpha\delta)d$$

for some $d_1, \ldots, d_{m-1}, d \in B$. Then

$$u^m_B(a_1, \ldots, a_k) = \beta s_B(a_1, \ldots, a_k) = \beta(\alpha\delta d) = (\beta\alpha\delta)d = (\alpha\gamma)d = \alpha(\gamma d).$$

Our final case is straightforward. We assume that

$$u^m(y_1, \ldots, y_k) = t(s^1(y_1, \ldots, y_k), \ldots, s^\ell(y_1, \ldots, y_k)),$$

where $\ell \geqslant 2$ and t is a basic ℓ-ary operation. Let $\mathfrak{L}'$ be the list

$$u^1, \ldots, u^{m-1}, s_1, \ldots, s^\ell,$$

so that $N(\mathfrak{L}') < N(\mathfrak{L})$ and our inductive assumption provides us with directly independent elements $a_1, \ldots, a_k \in B$ such that

$$u^i_B(a_1, \ldots, a_k) = \alpha d_i, \quad s^j_B(a_1, \ldots, a_k) = \alpha b_j,$$

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for $1 \leqslant i \leqslant m-1$ and $1 \leqslant j \leqslant \ell$. But B has the distributivity condition, so that

$$\begin{aligned} u_B^m(a_1,\dots,a_k) &= t_B(s_B^1(a_1,\dots,a_k),\dots,s_B^\ell(a_1,\dots,a_k)) \\ &= t_B(\alpha b_1,\dots,\alpha b_\ell) \\ &= \alpha t_B(b_1,\dots,b_\ell), \end{aligned}$$

thus completing our proof that the condition of Proposition 6.2 holds. By that result, End B is straight in End A as required. □

Our final result follows immediately from the comments at the end of § 4.

Corollary 6.4. *Let B be a free T-act on a finite set X with $|X| \geqslant 2$, where T is a cancellative monoid such that finitely generated left ideals are principal, so that B is a finite-rank stable basis algebra. Let A be the free G-set on X, so that A is an independence algebra and B is a reduct of A and* End B *is a left order in* End A*. Then* End B *is straight in* End A *if and only if T is left reversible.*

To see that not all cancellative monoids in which the principal left ideals are linearly ordered are left reversible, we consider the $\mathcal{R}$-class of a certain Bruck–Reilly monoid. It is clear that, for any monoid S, R_1, that is, the $\mathcal{R}$-class of the identity, is a right cancellative monoid.

Example 6.5. Let G be the free group on the set $X = \{x_1, x_2, \dots\}$ and let θ be the endomorphism of G determined by $x_i\theta = x_{i+1}$. Then for $\mathrm{BR}(G,\theta)$ the monoid R_1 is cancellative with principal left ideals linearly ordered, but is not left reversible.

Proof. Notice that θ is certainly one–one. Hence, if

$$(0,g,n)(0,h,m) = (0,g,n)(0,k,\ell),$$

we calculate that

$$(0, gh\theta^n, m+n) = (0, gk\theta^n, \ell+n)$$

and so $m = \ell$ and (as θ^n is one–one) $h = k$, $(0,h,m) = (0,k,\ell)$ and R_1 is cancellative as required.

Suppose now that $(0,w,n), (0,v,n+k) \in R_1$, where $k \geqslant 0$. Then

$$(0,v,n+k) = (0, v(w\theta^k)^{-1}, k)(0,w,n),$$

so that

$$R_1(0,v,n+k) \subseteq R_1(0,w,n)$$

and the principal left ideals of T are linearly ordered.

Finally, R_1 is not left reversible, for if

$$(0,x_1,1)(0,g,n) = (0,\varepsilon,1)(0,h,m),$$

where ε is the identity of G, then we would obtain $x_1 = (h\theta)(g\theta)^{-1} \in \operatorname{Im}\theta$, a contradiction. □

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