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LARGE DEVIATIONS FOR GAUSSIAN STOCHASTIC PROCESSES WITH SAMPLE PATHS IN ORLICZ SPACES

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1. Introduction. Let X be a complete, separable metric space, and $\{\mu_\epsilon:\epsilon \searrow 0\}$ a family of probability measures on the Borel subsets of X. We say that $\{\mu_\epsilon:\epsilon \searrow 0\}$ obeys the large deviation principle (LDP) with a rate function $I(\cdot)$ if there exists a function $I(\cdot)$ from X into $[0, \infty]$ satisfying:

(i) $0 \leqq I(x) \leqq \infty$ for all $x \in X$,

(ii) $I(\cdot)$ is lower semicontinuous,

(iii) for each $l < \infty$ the set $\{x:I(x) \leqq l\}$ is compact set in X,

(iv) for each closed set $C \subset X$

$$\limsup_{\epsilon\to 0} \epsilon \log \mu_\epsilon(C) \leqq -\inf_{x\in C} I(x),$$

(v) for each open set $U \subset X$

$$\liminf_{\epsilon\to 0} \epsilon \log \mu_\epsilon(U) \geqq -\inf_{x\in U} I(x).$$

It is easy to see that if A is a Borel set such that

$$\inf_{x\in A^0} I(x) = \inf_{x\in A} I(x) = \inf_{x\in \bar{A}} I(x)$$

then

$$\lim_{\epsilon\to 0} \epsilon \log \mu_\epsilon(A) = -\inf_{x\in A} I(x)$$

where A^0 and $\bar{A}$ are respectively the interior and the closure of the Borel set A.

Let $\mathbf{R}$ denote the real line and for $N < \infty$ let $(\mathbf{R}^N, \mathbf{B}(\mathbf{R}^N))$ be N-dimensional vector space with Borel σ-algebra $\mathbf{B}(R^N)$. Let μ be a mean-zero, Gaussian measure on $(\mathbf{R}^N, \mathbf{B}(\mathbf{R}^N))$ such that the covariance S is a positive definite matrix. Let $\mu_\epsilon = \mu \circ \epsilon^{-1/2}$ (i.e., $\mu_\epsilon(A) = \mu(\epsilon^{-1/2}A)$ for any measurable set A) then by Cramer's Theorem **[5]** $\{\mu_\epsilon:\epsilon \searrow 0\}$ satisfies the large deviation principle with the rate function

$$I(x) = 2^{-1}\langle x, S^{-1}x\rangle, \quad x \in \mathbf{R}^N.$$

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In the case that μ is a mean-zero Gaussian measure defined on a separable, real Banach space $\{\mu_\epsilon:\epsilon \searrow 0\}$ satisfies the LDP. This follows from a result of Donsker, Varadhan [**6**] in 1976. The above result for locally convex, Hausdorff real vector spaces was discussed by Bahadur, Zabell in 1979 [**1**].

In this paper we are going to prove the large deviation principle for

$$\{\mu_\epsilon:\mu_\epsilon = \mu \circ \epsilon^{-1/2}, \epsilon \searrow 0\}$$

where μ is a mean-zero, non-degenerate Gaussian measure defined on a separable, real Orlicz space L_ϕ which is not necessarily locally-convex vector space. The proof of the main result utilizes Cramer's Theorem for finite dimensional vector spaces and it bases on the series representation of Gaussian random elements with values in L_ϕ spaces as well as on its exponential integrability.

The importance of studying the LDP for the sequence $\{\mu_\epsilon:\epsilon \searrow 0\}$ defined on some Orlicz space L_ϕ lies in the fact that the Orlicz space L_ϕ can be considered as a space of sample paths of measurable stochastic processes. Necessary and sufficient conditions for a measurable Gaussian stochastic process to have almost all its sample paths in some Orlicz space L_ϕ were established in [**2**], in terms of the covariance function of the process.

As an application of the LDP for Orlicz spaces we get an extension of Kallianpur's and Oodaira's (1978), Marlow's (1973) results concerning some asymptotic estimates of the probabilities of high level occupation times for Gaussian stochastic processes with sample paths in Orlicz spaces.

2. Preliminaries. Let $(T, \mathbf{F}, m)$ be an arbitrary σ-finite measure space with σ-algebra $\mathbf{F}$ and a separable measure m. Let S be the space of equivalence classes in measure m of all real valued $\mathbf{F}$ measurable functions. By ϕ let us denote a continuous, non-negative, non-decreasing function defined for $u \geqq 0$ such that:

(i) $\phi(u) = 0$ if and only if $u = 0$,

(ii) satisfies the so-called Δ_2 condition, i.e., there is a positive constant k such that for any u, $\phi(2u) \leqq k\phi(u)$.

For $x \in S$ let us define

$$R_\phi(x) = \int_T \phi(\,|x(t)|\,)m(dt)$$

and let L_ϕ be the set of all $x \in S$ such that $R_\phi(x) < \infty$. The set L_ϕ is a linear space under the usual addition and scalar multiplication. Moreover it becomes a complete, separable metric space under the (usually non-homogeneous) seminorm $\|\cdot\|_\phi$.

$$\|x\|_\phi = \inf\{c:c > 0, R_\phi(c^{-1}x) \leqq c\}.$$

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The space $(L_\phi, \|\cdot\|_\phi)$ is called an Orlicz space. In the case that ϕ is a convex function L_ϕ is a Banach space [**11**]. The best known examples of the Orlicz spaces are $L_p(T, \mathbf{F}, m)$ spaces for $0 \leqq p < \infty$ [**11**].

For convenience let us recall some necessary facts concerning probability measures on Orlicz spaces $(L_\phi, \mathbf{B}(L_\phi))$, where $\mathbf{B}(L_\phi)$ denotes Borel σ-algebra of subsets of L_ϕ.

A. For each probability measure μ on $(L_\phi, \mathbf{B}(L_\phi))$ one can construct a measurable stochastic process

$$\xi = \{\xi(t):t \in T\} \quad \text{on } (\Omega, \Sigma, P) = (L_\phi, \mathbf{B}(L_\phi), \mu),$$

with sample paths in L_ϕ such that $\xi(x) = x$ μ a.e.; induced measure μ_ξ is equal to μ, and for every pair (s, u) or real numbers

$$\xi(t, sx \pm uy) = s\xi(t, x) \pm u\xi(t, y) \quad m \times \mu \times \mu \text{ a.e.}$$

Conversely, each jointly measurable stochastic process $\xi(t, \omega)$ defined on $T \times \Omega$ with almost all its sample paths in L_ϕ induces an $L_\phi(T, \mathbf{F}, m)$ valued random element [**2**].

B. An L_ϕ-valued random element ξ (or p.m. μ on $(L_\phi, \mathbf{B}(L_\phi))$) is Gaussian if for any pair of independent copies of ξ, X_1 and X_2 the random elements $X_1 + X_2$ and $X_1 - X_2$ are independent; this is equivalent to: the process ξ with sample paths in L_ϕ is Gaussian if and only if there exists a measurable subset T_0, $m(T_0) = 0$ such that for all finite sets $\{t_1, \dots, t_k\} \subset T \setminus T_0$ the random vectors $\langle \xi(t_1), \dots, \xi(t_k) \rangle$ are Gaussian [**2**].

C. Let $\xi = \{\xi(t):t \in T\}$ be a measurable Gaussian stochastic process and let

$$\theta(t) = E\xi(t),\ K(s, t) = E(\xi(s) - \theta(s))(\xi(t) - \theta(t)).$$

Then for almost every ω, $\xi(\cdot, \omega) \in L_\phi$ if and only if $\theta(t) \in L_\phi$ and $K^{1/2}(t, t) \in L_\phi$. If almost all sample paths of the process ξ belong to the space L_ϕ then the measure μ_ξ induced by ξ on $(L_\phi, \mathbf{B}(L_\phi))$ is Gaussian [**2**].

D. Let μ be a mean-zero, non-degenerate Gaussian measure on $(L_\phi, \mathbf{B}(L_\phi))$ and let $\xi = \{\xi(t):t \in T\}$ be a measurable stochastic process, such as in A, inducing the measure μ. By A there exists a measurable subset T_0, $m(T_0) = 0$ such that for any $t \in T \setminus T_0$

$$\xi(t, x \pm y) = \xi(t, x) \pm \xi(t, y) \quad \mu \times \mu \text{ a.e.}$$

Let

$$H_\mu = \overline{\text{lin}\{\xi(t):t \in T \setminus T_0\}}^{L_2(\mu)}.$$

From [**8**] it follows that the space H_μ does not depend on the version of the stochastic process inducing the measure μ and consists of all quasi-additive measurable functionals (q.m.f.) F defined on $(L_\phi, \mathbf{B}(L_\phi), \mu)$ [**8**], i.e.,

$$H_\mu = \{F{:}F{:}L_\Phi \to \mathbf{R},\ \text{measurable},\ F(x \pm y) = F(x) \pm F(y) \quad \mu \times \mu \text{ a.e.}\}.$$

For each $F \in H_\mu$ let

$$\Lambda F(\,\cdot\,) = \left[\int \xi(\,\cdot\,, x)F(x)\mu(dx)\right] = [\Lambda_\xi F(\,\cdot\,)]$$

where $[\,\cdot\,]$ denotes the class of functions equivalent in m a.e. In [**8**] it was shown that Λ is a one-to-one map which embeds continuously the space H_μ into L_Φ, and this embedding does not depend on the version of the stochastic process inducing the measure μ.

Let $\{E_j\}_{j=1}^\infty$ be a C.O.N.S. in H_μ and $\psi_j(t) = \langle \xi(t), E_j\rangle$, where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the inner product in H_μ, then by [**3**, **8**]

$$\xi(t, x) = \sum_{j=1}^\infty \psi_j(t)E_j(x)$$

μ a.e. in the L_Φ-seminorm, implying that supp $\mu \subseteq \Lambda \bar{H}_\mu^{L_\Phi}$.

For the remainder of this paper we assume that there exists for certain p, $0 < p \leqq 1$, a p-homogeneous F-norm $\|\cdot\|$ equivalent to the original one $\|\cdot\|_\Phi$. This class of spaces contains for example all L_p, $0 < p < \infty$, spaces or by [**11**] spaces L_Φ for which ϕ satisfies additionally the condition

$$\inf_{0<t<\infty} \inf\{c > 0{:}2\phi(ct) \geqq \phi(t)\} > 0.$$

PROPOSITION 2.1. *Let $\{X_n\}_{n=1}^\infty$ be a sequence of mean-zero, L_Φ-valued Gaussian random elements such that $X_n \to 0$ in P (probability) as $n \to \infty$, then*

$$\forall n\ \exists\ \alpha_n\ \forall\ \alpha < \alpha_n \quad E\exp(\alpha^2\|X_n\|^{2/p}) < \infty,$$

and $\alpha_n \to \infty$ as $n \to \infty$.

Proof. Let $\{\eta_j\}_{j=1}^\infty$ be a sequence of i.i.d. $N(0, 1)$ random variables and

$$G = \overline{\text{lin}\{x\eta_j{:}x \in L_\Phi, j = 1, 2, \dots\}}^{inP}$$

then G is a linear, separable, complete metric space which consists of mean-zero, L_Φ-valued Gaussian random elements. From [**3**] it follows that for each X_n there exists an L_Φ valued random element Y_n such that $Y_n \in G$ and $\mathbf{L}(X_n) = \mathbf{L}(Y_n)$ (where $\mathbf{L}(\,\cdot\,)$ denotes the distribution of a random element). By [**4**] for each $Y \in G$ there exists $\alpha > 0$ such that

$$E\exp(\alpha^2\|Y\|^{2/p}) < \infty.$$

Let $g{:}G \to \mathbf{R}$ be defined as follows

$$g(Y) = \exp\|Y\|^{2/p} - 1,$$

and for $Y \in G$

$$\|Y\|_g = \inf\{c^{-1} > 0 : E(\exp c^2\|Y\|^{2/p} - 1) \leqq c^{-1}\},$$

then $(G, \|\cdot\|_g)$ becomes an Orlicz space in which convergence in $\|\cdot\|_g$ seminorm is equivalent to the convergence in probability.

Let $\{Y_n\} \subset G$ and $Y_n \to 0$ in $\|\cdot\|_g$ as $n \to \infty$ then $Y_n \to 0$ in P as $n \to \infty$, which follows from the fact that $\|Y_n\|_g^{-1} \to \infty$ and the inequality

$$(1)\quad P(\|Y_n\| > \alpha)$$
$$\leqq \exp\{-\alpha^{2/p}(\|Y_n\|_g^{-1} - \epsilon)^2\} \cdot E\exp\{(\|Y_n\|_g^{-1} - \epsilon)^2\|Y_n\|^{2/p}\}$$
$$\leqq \exp\{-\alpha^{2/p}(\|Y_n\|_g^{-1} - \epsilon)^2\} \cdot \{1 + (\|Y_n\|_g^{-1} - \epsilon)^{-1}\},$$

where α and ϵ are arbitrary small positive constants.

The inverse implication follows from the closed graph theorem. Since the convergence in probability is equivalent to the convergence in $\|\cdot\|_g$ seminorm this implies that $\|X_n\|_g^{-1} \to \infty$ as $n \to \infty$ and for any $\alpha < \|X_n\|_g^{-1}$,

$$E\exp \alpha^2\|X_n\|^{2/p} < \infty$$

which finishes the proof.

For future purposes let us recall the following fact:

PROPOSITION 2.2 [13]. *Let X be a complete, separable metric space and $\{\mu_\epsilon\}$ a family of probability measures defined on X which satisfies the large deviation principle with a rate function $I(\cdot)$. Let u be a continuous map from $X \to Y$, where Y is another complete, separable metric space. If we define Q_ϵ on Y by $Q_\epsilon = \mu_\epsilon \circ u^{-1}$, then $\{Q_\epsilon\}$ satisfies the large deviation principle with the rate function $\mathbf{J}(\cdot)$ defined by*

$$\mathbf{J}(y) = \begin{cases} \inf\limits_{x:u(x)=y} I(x) & \text{if } y \in u(X) \\ \infty & \text{if } y \notin u(X). \end{cases}$$

3. Main theorem. The proof of the main theorem will be preceded by several lemmas.

LEMMA 3.1. *Let $\{E_k\}_{k=1}^{\infty}$ be a* C.O.N.S. *in H_μ then $\{E_k(x)\}_{k=1}^{\infty}$ is a sequence of* i.i.d. $N(0, 1)$ *random variables. For each $N < \infty$, let*

$$X_N = \sum_{k=1}^{N} E_k(x)\Lambda E_k$$

and

$$\mu_{N,\epsilon} = \mathbf{L}(\epsilon^{1/2}X_N),$$

then $\{\mu_{N,\epsilon}:\epsilon \searrow 0\}$ *satisfies the large deviation principle with the rate function*

$$I_N(x) = \begin{cases} 2^{-1}\|\Lambda^{-1}x\|_{H_\mu}^2 & \text{if } x \in \Lambda(P_N H_\mu) \\ \infty & \text{if } x \notin \Lambda(P_N H_\mu) \end{cases}$$

where P_N *denotes the orthogonal projection onto* $\text{lin}\{E_k:k = 1, \dots, N\}$.

Proof. Let $N < \infty$ be an arbitrary but fixed and let $\{e_k\}_{k=1}^N$ be a C.O.N.S. in $\mathbf{R}_N$, then

$$Y_N = \sum_{k=1}^{N} E_k(x)e_k$$

is N-dimensional, mean-zero Gaussian random vector. For each $\epsilon > 0$, let

$$\nu_{N,\epsilon} = \mathbf{L}(\epsilon^{1/2}Y_N)$$

then by Cramer's Theorem $\{\nu_{N,\epsilon}:\epsilon \searrow 0\}$ satisfies the large deviation principle with the rate function

$$\mathbf{J}_N(x) = 2^{-1}\langle x, x\rangle \quad \text{for } x \in \mathbf{R}^N.$$

By u let us denote a map from $\mathbf{R}^N$ into L_ϕ defined as follows

$$u\left(\sum_{k=1}^{N} x_k e_k\right) = \sum_{k=1}^{N} x_k \Lambda E_k,$$

then u is a continuous, one-to-one map from $\mathbf{R}^N$ onto $\Lambda(P_N H_\mu)$, such that $u(Y_N) = X_N$. Let

$$\mu_{N,\epsilon} = \mathbf{L}(\epsilon^{1/2}X_N),$$

since $\mu_{N,\epsilon} = \nu_{N,\epsilon} \circ u^{-1}$ then by Proposition 2.2 $\{\mu_{N,\epsilon}:\epsilon \searrow 0\}$ satisfies the large deviation principle with the rate function

$$I_N(x) = \inf_{y:u(y)=x} \mathbf{J}_N(y).$$

It is easy to see that

$$I_N(x) = \begin{cases} 2^{-1}\|\Lambda^{-1}x\|_{H_\mu}^2 & \text{if } x \in \Lambda(P_N H_\mu) \\ \infty & \text{if } x \notin \Lambda(P_N H_\mu). \end{cases}$$

Remark. For the rest of this paper we will use the following notation

$$I_\mu(x) = \begin{cases} 2^{-1}\|\Lambda^{-1}x\|_{H_\mu}^2 & \text{if } x \in \Lambda H_\mu \\ \infty & \text{if } x \notin \Lambda H_\mu). \end{cases}$$

For A an arbitrary set by A^c we will denote a complementary set and

$$\mathbf{I}_{A,N} = \inf_{x \in A} I_N(x), \quad \mathbf{I}_A = \inf_{x \in A} I_\mu(x),$$

$$A_\delta = \{x \in L_\Phi : \exists y \in A, \|x - y\| \leqq \delta\}.$$

LEMMA 3.2. *Let C be a closed subset of L_Φ such that $C \cap \overline{\Lambda H_\mu}^{L_\Phi} \neq \emptyset$ then for $\delta > 0$ there exists N_δ such that for any $N \geqq M \geqq N_\delta$*

$$\mathbf{I}_{C_\delta} \leqq \mathbf{I}_{C_{\delta,N}} \leqq \mathbf{I}_{C_{\delta,M}} < \infty.$$

Proof. Since $C \cap \overline{\Lambda H_\mu}^{L_\Phi} \neq \emptyset$ then for any $\delta > 0$

$$C_\delta \cap \Lambda H_\mu \neq \emptyset.$$

Let

$$\Lambda F \in C_{\delta/2} \cap \Lambda H_\mu,$$

then there exists N such that

$$\Lambda P_N F \in C_\delta \cap \Lambda H_\mu.$$

Since for any $M < N$

$$\Lambda P_M H_\mu \subseteq \Lambda P_N H_\mu \subseteq \Lambda H_\mu,$$

then for any $\delta > 0$ there exists N_δ such that for $N > M \geqq N_\delta$

$$\emptyset \neq C_\delta \cap \Lambda P_M H_\mu \subseteq C_\delta \cap \Lambda P_N H_\mu \subseteq C_\delta \cap \Lambda H_\mu$$

implying that

$$\mathbf{I}_{C_\delta} \leqq \mathbf{I}_{C_{\delta,N}} \leqq \mathbf{I}_{C_{\delta,M}} < \infty.$$

LEMMA 3.3. (i) *The set $K_r = \{\Lambda F : I_\mu(\Lambda F) \leqq r^2\}$, $0 < r < \infty$ is compact in L_Φ.*

(ii) *$I_\mu(x)$ is a lower-semicontinuous function on ΛH_μ with respect to $\|\cdot\|$-norm convergence, i.e., if $\|\Lambda F_n - \Lambda F\| \to 0$ as $n \to \infty$, F_n, $F \in H_\mu$ then*

$$I_\mu(\Lambda F) \leqq \liminf_{n \to \infty} I_\mu(\Lambda F_n).$$

Proof. First we show that K_r is a compact subset of L_Φ for any $0 < r < \infty$. Let $\{\Lambda F_n\} \subset K_r$ be an arbitrary sequence. By the Banach-Alaoglu Theorem $\{F_n\}$ contains a subsequence $\{F_{n'}\}$ which is weakly convergent to F from $\Lambda^{-1}K_r$. Remark D implies that there exists a measurable subset T_0, $m(T_0) = 0$ such that for any $t \in T \setminus T_0$, $\xi(t) \in H_\mu$ and

$$\Lambda_\xi F_{n'}(t) = \int \xi(t) F_{n'} d\mu \to \int \xi(t) F d\mu = \Lambda_\xi F(t).$$

Since

$$|\Lambda_\xi F_{n'}(t)| \leqq K^{1/2}(t, t)\|F_{n'}\|_{H_\mu} \leqq \sqrt{2}rK^{1/2}(t, t)$$

for m a.e. t, and $K^{1/2}(t, t) \in L_\Phi$ then by the Lebesgue Dominated Convergence Theorem, $\Lambda F_{n'} \to \Lambda F$ in L_Φ, proving that K_r is a compact subset of L_Φ.

Proof of part (ii). Let us denote by $\{F_{n'}\}$ a subsequence such that

$$\liminf_n I_\mu(\Lambda F_n) = \lim_{n'} I_\mu(\Lambda F_{n'}).$$

Since $\|\Lambda F_{n'} - \Lambda F\| \to 0$ as $n' \to \infty$, then there exists a subsequence $\{n''\} \subset \{n'\}$ and a measurable subset T_0, $m(T_0) = 0$ such that for any $t \in T \setminus T_0$, $\xi(t)$ is a q.m.f. and

$$\langle \xi(t), F_{n''}\rangle = \Lambda_\xi F_{n''}(t) \to \Lambda_\xi F(t) = \langle \xi(t), F\rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in H_μ. Let

$$G = \text{lin}\{\xi(t): t \in T \setminus T_0\}$$

then G is a dense subset of H_μ **[8]** and for any $g \in G$

$$\langle g, F_{n''}\rangle \to \langle g, F\rangle \quad \text{as } n'' \to \infty.$$

Since

$$\|F_{n''}\|_{H_\mu} = \sup\{\langle g, F_{n''}\rangle : g \in G, \|g\|_{H_\mu} = 1\},$$

then for any $g \in G$, $\|g\|_{H_\mu} = 1$

$$\lim_{n''} \|F_{n''}\|_{H_\mu} \geqq \lim_{n''} \langle g, F_{n''}\rangle = \langle g, F\rangle.$$

This implies that

$$\lim_{n''} \|F_{n''}\|_{H_\mu} \geqq \sup\{\langle g, F\rangle : g \in G, \|g\|_{H_\mu} = 1\} = \|F\|_{H_\mu}$$

proving part (ii) because

$$\liminf_n \|F\|_{H_\mu} = \lim_{n''} \|F_{n''}\|_{H_\mu} \geqq \|F\|_{H_\mu}.$$

LEMMA 3.4. *Let C be a closed subset of L_Φ then $\mathbf{l}_{C_\delta} \nearrow \mathbf{l}_C$ as $\delta \searrow 0$.*

Proof. Clearly $\mathbf{l}_{C_\delta} \leqq \mathbf{l}_C$ and $\mathbf{l}_{C_\delta} \nearrow$ as $\delta \searrow 0$. Suppose that $\mathbf{l}_{C_\delta} \leqq \mathbf{l}' < \mathbf{l}_C$ for all $\delta > 0$. Let n_0 be such that for all $n > n_0$ $\mathbf{l}' + n^{-1} < \mathbf{l}$. Let $\Lambda F_n \in C_{n^{-1}}$ be so that

$$I_\mu(\Lambda F_n) \leqq \mathbf{l}' + n^{-1} < \mathbf{l}.$$

Since $\{\Lambda F: I_\mu(\Lambda F) \leqq \mathbf{l}\}$ is a compact set in L_Φ (Lemma 3.3) then $\{\Lambda F_n\}$ contains a convergent subsequence $\{\Lambda F_{n'}\}$ such that $\Lambda F_{n'} \to \Lambda F$ in L_Φ where $\Lambda F \in C$. By the lower semicontinuity of the function $I_\mu(\cdot)$ (Lemma 3.3)

$$I_\mu(\Lambda F) \leqq \liminf_{n'} I_\mu(\Lambda F_{n'}) \leqq \mathbf{I}'$$

and $\mathbf{I}_C \leqq \mathbf{I}'$ which is contradictory unless $\mathbf{I}_{C_\delta} \nearrow \mathbf{I}_C$ as $\delta \searrow 0$.

LEMMA 3.5. *Let U be an open set such that $U \cap \overline{\Lambda H_\mu}^{L_\phi} \neq \emptyset$ then there exists N_U such that for any $N \geqq N_U$*

$$\mathbf{I}_{U,N} < \infty \quad \textit{and} \quad \mathbf{I}_{U,N} \searrow \mathbf{I}_U \quad \textit{as } N \to \infty.$$

Proof. Since U is an open set, $U \cap \overline{\Lambda H_\mu}^{L_\phi} \neq \emptyset$ then

$$U \cap \Lambda H_\mu \neq \emptyset.$$

Let $\Lambda F \in U \cap \Lambda H_\mu$ where F is a q.m.f. then there exists N_F such that for any $N > N_F$

$$\Lambda P_N F \in U \cap \Lambda H_\mu$$

implying that there exists N_U such that for any $N > N_U$ $\mathbf{I}_{U,N} < \infty$. Since for any $M < N$

$$\Lambda P_M H_\mu \subseteq \Lambda P_N H_\mu \subseteq \Lambda H_\mu$$

then

$$\mathbf{I}_{U,M} \geqq \mathbf{I}_{U,N} \geqq \mathbf{I}_U.$$

Let $\epsilon > 0$ be an arbitrary small but fixed and let F be a q.m.f. such that

$$\Lambda F \in U \cap \Lambda H_\mu, \quad \mathbf{I}_U + \epsilon > I_\mu(\Lambda F).$$

There exists N_F such that for $N > N_F$

$$\Lambda P_N F \in U \cap \Lambda P_N H_\mu \quad \text{and} \quad \mathbf{I}_U + \epsilon > I_\mu(\Lambda P_N F) \geqq \mathbf{I}_{U,N}$$

implying that $\mathbf{I}_{U,N} \searrow \mathbf{I}_U$ as $N \to \infty$.

LEMMA 3.6. *Let U be an open set such that $U \cap \Lambda H_\mu \neq \emptyset$ then there exists $\delta_0 > 0$ such that for any $\delta < \delta_0$*

$$((U^c)_\delta)^c \cap \Lambda H_\mu \neq \emptyset \quad \textit{and} \quad \mathbf{I}_{((U^c)_\delta)^c} \searrow \mathbf{I}_U \quad \textit{as } \delta \searrow 0.$$

Proof. Let us observe that $((U^c)_\delta)^c$ is an increasing sequence of open sets as $\delta \searrow 0$. If there is no $\delta > 0$ such that

$$((U^c)_\delta)^c \cap \Lambda H_\mu \neq \emptyset$$

then

$$\bigcup_\delta ((U^c)_\delta)^c \cap \Lambda H_\mu = \emptyset$$

implying that $\Lambda H_\mu \subseteq U^c$ which is contradictory to the assumption. Since $((U^c)_\delta)^c$ is an increasing sequence of sets then

$$\mathbf{l}_{((U^c)_\delta)^c} \searrow \mathbf{l}' \geqq \mathbf{l}_U \quad \text{as } \delta \searrow 0.$$

Let $\epsilon > 0$ be an arbitrary then there exists

$$\Lambda F \in U = \bigcup_{\delta}((U^c)_\delta)^c$$

and δ_0 such that for each $\delta < \delta_0$

$$\mathbf{l}_U + \epsilon > I_\mu(\Lambda F) > \mathbf{l}_{((U^c)_\delta)^c} \geqq \mathbf{l}'$$

proving that $\mathbf{l}_U = \mathbf{l}'$.

THEOREM 3.7. *Let μ be a mean-zero, non-degenerate Gaussian measure defined on $(L_\Phi, \mathbf{B}(L_\Phi))$ such that there exists a p-homogeneous F-norm $\|\cdot\|$, $0 < p \leqq 1$, equivalent to the original one $\|\cdot\|_\Phi$. Let $\mu_\epsilon = \mu \circ \epsilon^{-1/2}$, then the family $\{\mu_\epsilon : \epsilon \searrow 0\}$ satisfies the large deviation principle with the rate function I_μ, $I_\mu : L_\Phi \to [0, \infty]$, defined as follows*

$$I_\mu(x) = \begin{cases} 2^{-1}\|\Lambda^{-1}x\|^2_{H_\mu} & \text{if } x \in \Lambda H_\mu \\ \infty & \text{if } x \notin \Lambda H_\mu. \end{cases}$$

Proof. Lemma 3.3 implies lower-semicontinuity of the function I_μ and the compactness in L_Φ of the set $\{x : I_\mu(x) \leqq \mathbf{l}\}$ for any $\mathbf{l} > 0$.

Let X be an L_Φ-valued random element, generated by $\xi = \{\xi(t) : t \in T\}$ a measurable stochastic process such as in D, inducing the measure μ, then $\mathbf{L}(X) = \mu$ and by D

$$X_N = \sum_{j=1}^{N} \psi_j E_j \to \sum_{j=1}^{\infty} \psi_j E_j = X$$

μ a.e. in L_Φ. Let $A \in \mathbf{B}(L_\Phi)$ then for $\epsilon > 0$

$$\mu_\epsilon(A) = \mu(\epsilon^{-1/2}A) = P(\epsilon^{1/2}X \in A).$$

Upper bound. We want to prove that for an arbitrary closed set C in L_Φ

$$(2) \quad \limsup_{\epsilon \searrow 0} \epsilon \log \mu_\epsilon(C) \leqq -\inf_{x \in C} I_\mu(x).$$

If $C \cap \overline{\Lambda H_\mu}^{L_\Phi} = \emptyset$ then $\mathbf{l}_C = \infty$ and for any $\epsilon > 0$,

$$\epsilon^{-1/2}C \cap \overline{\Lambda H_\mu}^{L_\Phi} = \emptyset.$$

Since

$$\operatorname{supp} \mu \subseteq \overline{\Lambda H_\mu}^{L_\Phi}$$

then for every $\epsilon > 0$, $\mu_\epsilon(C) = 0$ proving that (2) is true.

Let us assume that

$$C \cap \overline{\Lambda H_\mu}^{L_\phi} = \emptyset$$

and let $\delta > 0$ be an arbitary small but fixed, then

$$P(\epsilon^{1/2}X \in C) \leqq P(\epsilon^{1/2}X_N \in C_\delta) + P(\|\epsilon^{1/2}(X-X_N)\| \geqq \delta).$$

Let $\eta > 0$ be an arbitrary small but fixed. By Proposition 2.1 and Lemma 3.2 there exists N_0 such that for any $N > N_0$

$$(\alpha_N - \eta)^2\delta^{2/p} > \mathbf{I}_{C_\delta,N_\delta} \geqq \mathbf{I}_{C_\delta,N}$$

and

$$P(\|X - X_N\| \geqq \delta\epsilon^{-p/2})$$
$$\leqq \exp\{-(\alpha_N - \eta)^2\delta^{2/p}\epsilon^{-1}\}\, E \exp\{(\alpha_N - \eta)^2\|X - X_N\|^{2/p}\}$$
$$= \exp\{-(\alpha_N - \eta)^2\delta^{2/p}\epsilon^{-1}\}M_N.$$

Let $N > N_0$ be an arbitrary but fixed and $\mathbf{L}(X_N) = \mu_N$ then by Lemma 3.1 $\{\mu_{N,\epsilon}:\epsilon \searrow 0\}$ satisfies the large deviation principle and for sufficiently small $\epsilon > 0$

$$P(\epsilon^{1/2}X_N \in C_\delta) \leqq \exp(-\epsilon^{-1}\mathbf{I}_{C_{\delta,N}} + \epsilon^{-1}\eta).$$

Therefore for sufficiently small $\epsilon > 0$

$$P(\epsilon^{1/2}X \in C) \leqq \exp(-\epsilon^{-1}\mathbf{I}_{C_{\delta,N}} + \epsilon^{-1}\eta)$$
$$+ \exp\{-(\alpha_N - \eta)^2\delta^{2/p}\epsilon^{-1}\}\, M_N$$
$$= \{1 + M_N \exp \epsilon^{-1}[-(\alpha_N - \eta)^2\delta^{2/p} + \mathbf{I}_{C_{\delta,N}} - \eta]\}$$
$$\times \exp(-\epsilon^{-1}\mathbf{I}_{C_{\delta,N}} + \epsilon^{-1}\eta).$$

Then

$$\limsup_{\epsilon\searrow 0} \epsilon \log P(\epsilon^{1/2}X \in C) \leqq -\mathbf{I}_{C_{\delta,N}} + \eta$$

and by Lemma 3.2

$$\limsup_{\epsilon\searrow 0} \epsilon \log P(\epsilon^{1/2}X \in C) \leqq -\mathbf{I}_{C_\delta} + \eta.$$

Since $\delta > 0$ and $\eta > 0$ were arbitrarily small then by Lemma 3.4

$$\limsup_{\epsilon\searrow 0} \epsilon \log P(\epsilon^{1/2}X \in C) \leqq -\mathbf{I}_C$$

equivalently

$$\limsup_{\epsilon\searrow 0} \epsilon \log \mu_\epsilon(C) \leqq -\inf_{x\in C} I_\mu(x).$$

Lower bound. We want to prove that for an arbitrary open set U in L_ϕ

(3) $\liminf_{\epsilon \searrow 0} \epsilon \log \mu_\epsilon(U) \geqq -\inf_{x \in U} I_\mu(x).$

If $U \cap \overline{\Lambda H_\mu}^{L_\phi} = \emptyset$ then $\mathbf{I}_U = \infty$ and for any $\epsilon > 0$,

$$\epsilon^{-1/2} U \cap \overline{\Lambda H_\mu}^{L_\phi} = \emptyset.$$

Since supp $\mu \subseteq \overline{\Lambda H_\mu}^{L_\phi}$, then for each ϵ, $\mu_\epsilon(U) = 0$ proving that (3) is true.

Let us assume that

$$U \cap \overline{\Lambda H_\mu}^{L_\phi} \neq \emptyset$$

then $U \cap \Lambda H_\mu \neq \emptyset$ and by Lemma 3.6 there exists δ_0 such that for any $\delta < \delta_0$

$$((U^c)_\delta)^c \cap \Lambda H_\mu \neq \emptyset.$$

Let $\delta < \delta_0$ be arbitrary but fixed then

$$\begin{aligned} P(\epsilon^{1/2} X \in U) &\geqq P(\epsilon^{1/2} X_N \in ((U^c)_\delta)^c, \|\epsilon^{1/2}(X - X_N)\| < \delta) \\ &\geqq P(\epsilon^{1/2} X_N \in ((U^c)_\delta)^c) \\ &\quad + P(\|\epsilon^{1/2}(X - X_N)\| < \delta) - 1 \\ &= P(\epsilon^{1/2} X_N \in ((U^c)_\delta)^c) \\ &\quad - P(\|\epsilon^{1/2}(X - X_N)\| \geqq \delta). \end{aligned}$$

Let $\eta > 0$ be an arbitrary small but fixed. By Proposition 2.1 and Lemma 3.5 there exists N_0 such that for any $N \geqq N_0$

$$(\alpha_N - \eta)^2 \delta^{2/p} - \eta > \mathbf{I}_{((U^c)_\delta)^c, N},$$

$$\mathbf{I}_{((U^c)_\delta)^c} + \eta \geqq \mathbf{I}_{((U^c)_\delta)^c, N},$$

and

$$\begin{aligned} P(\|X - X_N\| \geqq \delta \epsilon^{-p/2}) &\leqq \exp\{-(\alpha_N - \eta)^2 \delta^{2/p} \epsilon^{-1}\} \\ &\quad \times E \exp\{(\alpha_N - \eta)^2 \|X - X_N\|^{2/p}\} \\ &= \exp\{-(\alpha_N - \eta)^2 \delta^{2/p} \epsilon^{-1}\} M_N. \end{aligned}$$

Let $N > N_0$ be arbitrary but fixed and $\mathbf{L}(X_N) = \mu_N$ then by Lemma 3.1 $\{\mu_{N,\epsilon}: \epsilon \searrow 0\}$ satisfies the large deviation principle and for sufficiently small ϵ

$$P(\epsilon^{1/2} X_N \in ((U^c)_\delta)^c) \geqq \exp\{-\epsilon^{-1} \mathbf{I}_{((U^c)_\delta)^c, N} - \epsilon^{-1}\eta\}.$$

Then for sufficiently small ϵ

$$\begin{aligned} &P(\epsilon^{1/2} X \in U) \\ &\geqq \exp\{-\epsilon^{-1} \mathbf{I}_{((U^c)_\delta)^c, N} - \epsilon^{-1}\eta\} - \exp\{-(\alpha_N - \eta)^2 \delta^{2/p} \epsilon^{-1}\} M_N \end{aligned}$$

$$= \{1 - M_N \exp \epsilon^{-1}[-(\alpha_N - \eta)^2\delta^{2/p} + \mathbf{I}_{((U^c)_\delta)^c,N} + \eta]\}$$
$$\times \exp\{-\epsilon^{-1}\mathbf{I}_{((U^c)_\delta)^c,N} - \epsilon^{-1}\eta\}.$$

Hence

$$\liminf_{\epsilon \searrow 0} \epsilon \log P(\epsilon^{1/2}X \in U) \geqq -\mathbf{I}_{((U^c)_\delta)^c,N} - \eta \geqq -\mathbf{I}_{((U^c)_\delta)^c} - 2\eta.$$

Since $\delta > 0$ was an arbitrary small then by Lemma 3.6

$$\liminf_{\epsilon \searrow 0} \epsilon \log P(\epsilon^{1/2}X \in U) \geqq -\mathbf{I}_U - 3\eta$$

and by going with $\eta \searrow 0$ we get

$$\liminf_{\epsilon \searrow 0} \epsilon \log P(\epsilon^{1/2}X \in U) \geqq -\mathbf{I}_U$$

equivalently

$$\liminf_{\epsilon \searrow 0} \epsilon \log \mu_\epsilon(U) \geqq -\inf_{x \in U} I_\mu(x).$$

COROLLARY 3.8 (Extension of Cramer's Theorem to Orlicz spaces L_ϕ). *Let μ be a mean-zero, non-degenerate Gaussian measure defined on $(L_\phi, \mathbf{B}(L_\phi))$ such that there exists a p-homogeneous F-norm $\|\cdot\|$, $0 < p \leqq 1$, equivalent to the original one $\|\cdot\|_\phi$. Let $\{X_i : i \geqq 1\}$ be a sequence of independent, L_ϕ-valued random elements, each with distribution μ. Set*

$$S_n = \sum_{i=1}^{n} X_i$$

and let $\mu_{n^{-1}}$ be the distribution of S_n/n, then $\{\mu_{n^{-1}} : n \nearrow \infty\}$ satisfies the large deviation principle with the rate function I_μ.

Proof. Since $\mathbf{L}(S_n/n) = \mathbf{L}(X_1/\sqrt{n})$ then the proof is an immediate consequence of Theorem 3.7 if we take $\epsilon = n^{-1}$.

COROLLARY 3.9. *Let μ be a mean-zero, non-degenerate Gaussian measure defined on $(L_\phi, \mathbf{B}(L_\phi))$ such that there exists a p-homogeneous F-norm $\|\cdot\|$, $0 < p \leqq 1$, equivalent to the original one $\|\cdot\|_\phi$. Let*

$$a = \inf\{I_\mu(x) : \|x\| \geqq 1\}$$

then $0 < a < \infty$ and

$$\lim_{R \to \infty} R^{-2} \log \mu(\{x : \|R^{-1}x\| > 1\}) = -a.$$

Proof. The proof follows from Theorem 3.7 and Proposition 8 in **[9]**.

As a next application of Theorem 3.7 we get an extension of Kallianpur's and Oodaira's (1978), Marlow's (1973) results concerning

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some asymptotic estimates of the probabilities of high level occupation times for Gaussian stochastic processes with sample paths in Orlicz spaces. Let

$$D_\beta = \{f(t):f(t) \in L_\Phi, m(\{t:f(t) > 1\}) > \beta\}.$$

Then for any $\beta > 0$, D_β is an open set in L_Φ [**9**].

COROLLARY 3.10. *Let $\xi = \{\xi(t):t \in T\}$ be a mean-zero, Gaussian stochastic process with almost all sample paths in an Orlicz space L_Φ such that there exists a p-homogeneous F-norm $\|\cdot\|$, $0 < p \leqq 1$, equivalent to the original one $\|\cdot\|_\Phi$. Let for any $\beta > 0$*

$$a_\beta = \inf\{I_\mu(x):x \in D_\beta\}, \quad \bar{a}_\beta = \inf\{I_\mu(x):x \in \bar{D}_\beta\},$$

then

$$-a_\beta \leqq \liminf_{\alpha\to\infty} \alpha^{-2} \log P(\{\omega:m(\{t:\xi(t, \omega) > \alpha\}) > \beta\})$$

$$\leqq \limsup_{\alpha\to\infty} \alpha^{-2} \log P(\{\omega:m(\{t:\xi(t, \omega) > \alpha\}) > \beta\}) \leqq -\bar{a}_\beta.$$

If T is a metric space with the measure m such that for any open set U, $m(U) > 0$, the covariance function $K(s, t)$ of the process $\xi = \{\xi(t):t \in T\}$ is continuous and for each $\beta > 0$

$$m(\{s:m(\{t:K(s, t) > 0\}) > \beta\}) > 0$$

then $0 < a_\beta < \infty$ and

$$\lim_{\alpha\to\infty} \alpha^{-2} \log P(\{\omega:m(\{t:\xi(t, \omega) > \alpha\}) > \beta\}) = -a_\beta.$$

Proof. This follows from Theorem 3.7 and Theorem 9 in [**9**].

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REFERENCES

1. R. R. Bahadur and S. L. Zabell, *Large deviations of the sample mean in general vector spaces*, Ann. Probab. *7* (1979), 587-621.

2. T. Byczkowski, *Gaussian measures on L_p spaces*, $0 < p < \infty$, Studia Math. *59* (1977), 249-261.

3. ——— *Norm convergent expansion for L_Φ-valued Gaussian random elements*, Studia Math. *64* (1979), 87-95.

4. T. Byczkowski and T. Zak, *On the integrability of Gaussian random vectors*, Lecture Notes in Math. *828* (Springer, New York, 1979), 21-30.

5. H. Cramér, *Sur un nouveau théorème-limite de la théorie des probabilites*, Actualités Sci. Indust. *736* (1938), 5-23.

6. M. D. Donsker and S. R. S. Varadhan, *Asymptotic evaluation for certain Markov processes expectations for large-time* III, Comm. Pure Appl. Math. *29* (1976), 389-461.

7. G. Kallianapur and H. Oodaira, *Freidlin-Wentzell type estimates for abstract Wiener spaces*, Sankhya *40*, Series A (1978), 116-137.

8. A. T. Lawniczak, *Gaussian measures on Orlicz spaces and abstract Wiener spaces*, Lecture Notes in Math. *939* (Springer, New York, 1982), 81-97.

9. ——— *High level occupation times for Gaussian stochastic processes with sample paths in Orlicz spaces*, Can. J. Math. *39* (1987), 239-256.

10. N. Marlow, *High level occupation times for continuous Gaussian processes*, Ann. Probab. *3* (1973), 388-397.

11. S. Rolewicz, *Metric linear spaces* (Polish Scientific Publishers, PWN, 1972).

12. D. W. Stroock, *An introduction to the theory of large deviations* (Springer-Verlag, New York, 1984).

13. S. R. S. Varadhan, *Large deviations and applications*, SIAM (1984).

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