ON THE NON-CUTPOINT EXISTENCE THEOREM

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1. Introduction. The theorem of the title asserts that every non-degenerate continuum (that is, every compact connected Hausdorff space containing more than one point) contains at least two non-cutpoints. This is a fundamental result in set-theoretic topology and several standard proofs, each varying from the others to some extent, have been published. (See, for example, [1], [4] and [5]). The author has presented a less standard proof in [3] where the non-cutpoint existence theorem was obtained as a corollary to a result on partially ordered spaces. In this note a refinement of that argument is offered which seems to the author to be the simplest proof extant. To facilitate its exposition, the notion of a weak partially ordered space is introduced and the cutpoint partial order of connected spaces is reviewed.

2. Weak partially ordered spaces. If X is a topological space endowed with a partial order $\leq$ we write

$$L(x) = \{y \in X : y \leq x\},$$

$$M(x) = \{y \in X : x \leq y\}.$$

An element m of X is maximal (minimal) if $M(m) = \{m\}$ ($L(m) = \{m\}$). A subset A of X is said to be increasing (decreasing) if $M(a) \subset A$ for each $a \in A$ ($L(a) \subset A$ for each $a \in A$). The space X is a weak partially ordered space if, for each $x \in X$ such that x is not maximal, there exists a closed set $K(x) \subset M(x)$ such that $K(x) - \{x\}$ is non-empty and increasing. The partial order is said to be weakly continuous if X is a weak partially ordered space. We note that weak continuity is a much weaker condition than upper semicontinuity and related conditions which have been studied in [3]. It permits a generalization of a well-known proposition about the existence of maximal elements in partially ordered spaces which was first enunciated by Wallace [2].

PROPOSITION 1. *A compact weak partially ordered space has a maximal element.*

Proof. Let X be a compact weak partially ordered space. By a standard maximality argument it may be shown that X contains a subset C satisfying the following conditions.

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(1) *The set* C *is simply ordered*.

(2) *If* $x \in C$ *and* x *is not maximal then there exists a closed set* $K(x) \subset M(x)$ *such that* $K(x) - \{x\}$ *is increasing and* $K(x) \cap C - \{x\}$ *is non-empty*.

(3) *The set* C *is maximal with respect to (1) and (2)*.

Since X is compact there exists $z \in \bigcap \{K(x) : x \in C\}$. The maximality of C insures that $z \in C$ and that z is a maximal element of X.

PROPOSITION 2. *Let* X *be a compact weak partially ordered space which is not simply ordered and which satisfies the condition* (S) *if* $x \in X$ *then* L(x) *is simply ordered*. *Then* X *contains at least two distinct maximal elements*.

Proof. Let x and y be elements of X which are not comparable under the partial order. It follows from (S) that the sets M(x) and M(y) are disjoint. Therefore, if K(x) and K(y) are the closed subsets of M(x) and M(y), respectively, whose existence is guaranteed by the weakly continuous partial order, then K(x) and K(y) are disjoint. Now K(x) and K(y) are themselves compact weak partially ordered spaces and so they have maximal elements by Proposition 1. Since $K(x) - \{x\}$ and $K(y) - \{y\}$ are increasing, those elements are also maximal in X.

3. *The cutpoint order*. Let X be a connected Hausdorff space and suppose that e is a cutpoint of X. We define a relation $\leq$ on X by $x \leq y$ if and only if $x = e$ or $x = y$ or x separates e and y. This relation has been called the cutpoint order on X. In this paragraph we summarize a few of its properties. Proofs of Propositions 3 and 6 are implicit in [3] but they are sketched here in order to make the treatment self-contained.

PROPOSITION 3. *The cutpoint order is a partial order*.

Proof. This is straightforward except to show that if $e < x < y < z$ then $x < z$. By definition of the cutpoint order we have

$$X - \{x\} = A \cup B,$$

$$X - \{y\} = C \cup D,$$

where A and B are separated sets, C and D are separated sets, $e \in A \cap C$, $y \in B$ and $z \in D$. Now $\bar{D} = D \cup \{y\}$ is connected, so if $z \in \bar{A}$ then $x \in D$ and hence $y < x$, which is impossible since $\leq$ is asymmetric. Therefore $z \in B$, which is to say that $x < z$.

PROPOSITION 4. *The non-cutpoints of* X *are precisely the maximal points relative to the cutpoint order*.

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Proof. Obvious.

PROPOSITION 5. The cutpoint order is weakly continuous.

Proof. Since $X = M(e)$ is closed and $M(e) - \{e\}$ is increasing we may set $K(e) = X$. If $x \neq e$ and x is not maximal then x is a cutpoint and hence

$$X - \{x\} = E \cup F$$

where E and F are non-empty separated sets and $e \in E$. Now $\bar{F} = F \cup \{x\}$ is closed and $\bar{F}$ is readily seen to be increasing. Therefore we may set $\bar{F} = K(x)$ and the proposition follows.

PROPOSITION 6. If $x \in S$ then $L(x)$ is simply ordered.

Proof. It is sufficient to show that if p and q are members of $L(x) - \{e, x\}$ and $p \not\leq q$ then $q \leq p$. Now by definition p and q are cutpoints separating e and x and therefore

$$X - \{p\} = G \cup H,$$

$$X - \{q\} = I \cup J,$$

where G and H are separated sets, I and J are separated sets, $e \in G \cap I$ and $x \in H \cap J$. Further, since $p \not\leq q$ it follows that $q \in G$ and hence the connected set $\bar{J} = J \cup \{q\}$ contains p. But then $q \leq p$.

PROPOSITION 7. The cutpoint order is not a simple order.

Proof. Since e is a cutpoint of X there exist elements a and b of X such that e separates a and b. It follows that a does not separate e and b and that b does not separate e and a, i.e., a and b are not comparable.

4. The main result. The non-cutpoint existence theorem can now be obtained from the foregoing propositions.

THEOREM. A non-degenerate continuum has at least two non-cutpoints.

Proof. The theorem is obvious if X is cutpoint free so we may assume that X contains a cutpoint, e. We give X the cutpoint order. By Propositions 3 and 5 X is a weak partially ordered space. By Propositions 6 and 7 and the compactness of X the hypotheses of Proposition 2 are satisfied so that X contains at least two distinct maximal elements. The theorem now follows from Proposition 4.

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