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EFFECTS OF SURFACE TENSION ON TRAPPED MODES IN A TWO-LAYER FLUID

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Abstract

We consider a two-layer fluid of finite depth with a free surface and, in particular, the surface tension at the free surface and the interface. The usual assumptions of a linearized theory are considered. The objective of this work is to analyse the effect of surface tension on trapped modes, when a horizontal circular cylinder is submerged in either of the layers of a two-layer fluid. By setting up boundary value problems for both of the layers, we find the frequencies for which trapped waves exist. Then, we numerically analyse the effect of variation of surface tension parameters on the trapped modes, and conclude that realistic changes in surface tension do not have a significant effect on the frequencies of these.

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1. Introduction

If the free surface of a fluid domain in an ocean extends to infinity in some direction, there may be characteristic modes with finite as well as infinite total energy. The former modes retain their energy (neglecting the effect of viscosity), while the latter radiate theirs to infinity. For this reason, modes with finite energy are known as trapped modes; these are the subject of the present paper. A trapped mode persists in some localized region, including the free surface, while decaying rapidly to zero as the free surface extends to infinity. The trapped modes have finite energy if attention is restricted to only a finite length of the submerged body under consideration. In this paper, we will consider a finite length along the axis of the cylinder (see Section 3).

The study of such modes was initiated by Stokes [15] in the context of edge waves. His work was extended by Ursell [18] who gave explicit solutions for a set of edge-wave modes on a plane beach of which the fundamental mode was found by Stokes.

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In a configuration where no trapped modes are possible, periodic motions with periodicity along the direction of the axis of a cylinder of finite length are uniquely defined by the prescribed periodic motion of the boundary, along with a radiation condition at infinity. However, if trapped modes exist, then the radiation condition is no longer sufficient for uniqueness. Earlier proofs of uniqueness were presented by John [5] for single nonbulbous bodies of different types that intersect the free surface, and by Ursell [16] for a submerged horizontal cylinder, lying in an infinitely deep fluid immersed to any depth and for all frequencies. The existence of a trapped mode above a submerged, horizontal circular cylinder in infinitely deep water was first established by Ursell [17]. The most important result emerging out of this work [17], crucial to subsequent investigations, is that the existence of trapped modes depends on the vanishing of a certain infinite determinant. Ursell further proved that zeros of the determinant did exist, if the radius of the cylinder was sufficiently small as compared to the depth of submergence of the cylinder. Afterwards, Jones [6] generalized the result of Ursell [17] for a cylinder of arbitrary but symmetrical cross-section in a finite depth of water. Subsequently, the existence of a new type of trapped mode was uncovered by McIver [10]. She found trapped-mode solutions for the two-dimensional water-wave problem in which there was no longer any finite periodicity in a horizontal direction. The fluid motion was essentially confined to the neighbourhood of a tandem structure giving a free oscillation of finite energy within a fluid of infinite extent. The solution was constructed from two equal-strength wave-sources placed at the free surface, and positioned in such a way that the waves radiating to infinity from each source were cancelled by those from the other, thus giving a local oscillation of the fluid. She showed that there existed families of streamline pairs surrounding the sources that could be interpreted as two surface-piercing structures. These investigations show that it is difficult to analyse the existence of new types of trapped mode which are embedded in a continuous spectrum.

It is reasonable to conclude that only a small amount of progress has been made in the study of the existence of trapped modes in the presence of surface tension. Harter et al. [3, 4] incorporated the effect of surface tension on trapped modes. They showed that its exclusion from the problem was not always justifiable, as its inclusion in the study of a particular submerged body changed the topological nature of the streamline pattern. Motygin and McIver [11] described a criterion that accounted for surface tension, while examining the existence of trapped modes supported by given submerged bodies. The method was applied to pairs of ellipses, and numerical results were used to demonstrate the effects of surface tension.

The results documented above are all related to homogeneous fluids only. A more general class of problem arises in the case of two-layer fluids of different densities having a common interface. An extensive literature survey on the capillary–gravity wave motion in the presence of surface and interfacial tension in a two-layer fluid was given by Mohapatra and Sahoo [12]. The first result concerning trapped modes in a two-layer fluid was obtained by Kuznetsov [8] who examined and proved the existence of trapped modes above a submerged cylinder in the lower layer of infinite depth by means of perturbation methods. Later, Linton and Cadby [9] computed the trapped-mode frequencies for a horizontal circular cylinder submerged either in the upper layer or in the lower layer by the use of a multipole expansion method. The authors of the current paper have also discussed trapped modes in a two-layer fluid by considering a rigid lid [13] and an ice-cover [14] at the upper surface. To the best of our knowledge, none of the work documented above for a two-layer fluid takes into account the effect of surface tension when studying the existence of trapped modes.

In the present problem, we consider the case in which two trapped waves develop at the free surface and the interface of a two-layer fluid of finite depth and include surface tension effects at both the surface and the interface, by introducing a surface-tension parameter for each of them. Third-order boundary conditions are satisfied both at the mean free surface and at the mean interface, which makes the problem more complex.

In Section 2, the problem is formulated for a cylinder placed in either of the layers. The velocity potentials for the progressive waves along with the dispersion relation are determined using all the boundary conditions and the governing equation. Section 3 describes a solution based on the multipole expansion method adopted by Linton and Cadby [9], in which the singular solutions of the modified Helmholtz equation are modified to include all the prescribed boundary conditions. The total potential is expressed as the sum over all the relevant multipoles. Then, by using no-flow conditions on the cylinder surfaces, an infinite system of homogeneous linear equations is obtained whose nontrivial solutions for a truncated system correspond to the trapped modes. Subsequently, the frequencies for which trapped modes exist are numerically found by locating the zeros of the suitably truncated determinant of this infinite system of homogeneous linear equations. Finally, the effect of variation of both of the surface tension parameters on the values and patterns of the trapped modes is examined. Then, we conclude that the exclusion of surface tension in formulating the problems is justifiable. The type of trapped mode that is being investigated here is more straightforward to investigate, because a cut-off value can be introduced and those modes whose frequencies are below that cut-off, can be located. Hence, these discrete trapped-mode frequencies are outside the continuous spectrum. The paper concludes with a brief discussion in Section 4.

2. Mathematical formulation of the problem

Cartesian coordinates are chosen such that the xy-plane coincides with the undisturbed interface between the two fluids. Each fluid is assumed to be of infinite horizontal extent in the plane of the x- and y-directions, while the finite depth is along the z-direction which is considered to be positive vertically upwards. The upper fluid layer ($-\infty < x < \infty$, $-\infty < y < \infty$, $0 < z < d$) is of constant density ρ_I in the presence of surface tension T_1, with $z = d$ as the mean free surface. The lower fluid ($-\infty < x < \infty$, $-\infty < y < \infty$, $-h < z < 0$) is assumed to be of constant density ρ_{II} in the presence of surface tension T_2, with the mean interface at $z = 0$, and the bottom surface is considered to be at $z = -h$ (Figure 1). Assuming that the

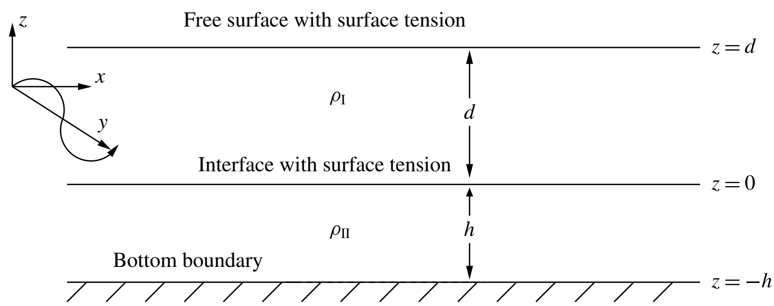

FIGURE 1. Schematic diagram for a two-layer fluid with surface tension at free surface and interface.

fluids are inviscid, incompressible and immiscible, and their motions irrotational, the fluid motion is described by the two velocity potentials $\Phi^j(x, y, z, t)$, $j = I, II$. Let $\eta(x, y, t)$ and $\zeta(x, y, t)$ be the small displacements at the free surface and the interface, respectively. The governing equation for the boundary value problem involving the potential $\Phi^j(x, y, z, t)$, $j = I, II$, is the Laplace's equation

$$\nabla^2\Phi^j = 0 \quad \text{in the respective fluid regions.} \tag{2.1}$$

The linearized kinematic conditions at the mean free surface and the mean interface are, respectively, given by

$$\frac{\partial \eta}{\partial t} = \frac{\partial \Phi^I}{\partial z} \quad \text{on } z = d \quad \text{and} \tag{2.2}$$

$$\frac{\partial \zeta}{\partial t} = \frac{\partial \Phi^I}{\partial z} = \frac{\partial \Phi^{II}}{\partial z} \quad \text{on } z = 0. \tag{2.3}$$

In the presence of surface tension, the general relations connecting surface tension and pressure gradient are given by

$$P_0 - p_I = \frac{T_1}{R} \quad \text{for the free surface, and}$$

$$p_{II} - p_I = \frac{T_2}{R} \quad \text{for the interface,}$$

where $1/R$ is the mean curvature, P_0 is the constant atmospheric pressure and p_j, $j = I, II$, is the hydrodynamic pressure in fluid region j. In Cartesian coordinates, the mean curvature is

$$\frac{1}{R} = \begin{cases} \dfrac{\eta_{xx} + \eta_{yy}}{(1 + \eta_x^2 + \eta_y^2)^{3/2}} & \text{for the free surface, and} \\ \dfrac{\zeta_{xx} + \zeta_{yy}}{(1 + \zeta_x^2 + \zeta_y^2)^{3/2}} & \text{for the interface.} \end{cases} \tag{2.4}$$

According to the linearized theory of water-waves, the hydrodynamic pressure p_j in the corresponding fluid region is given by Bernoulli's equation [1]

$$p_j = -\rho_j g\Big(z + \frac{1}{g}\frac{\partial \Phi^j}{\partial t}\Big) \quad \text{for } j = I, II, \tag{2.5}$$

where g is the acceleration due to gravity.

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Hence, from equations (2.4) and (2.5), the linearized dynamic free-surface boundary condition in the presence of surface tension T_1 at the mean free surface $z = d$ is given by

$$\rho_I\Big(g\eta + \frac{\partial \Phi^I}{\partial t}\Big) - T_1\Big(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}\Big) = 0. \tag{2.6}$$

Further, from equations (2.4) and (2.5), the linearized dynamic condition at the mean interface $z = 0$, in the presence of interfacial surface tension T_2 is given by

$$\rho_{II}\Big(g\zeta + \frac{\partial \Phi^{II}}{\partial t}\Big) - T_2\Big(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2}\Big) = \rho_I\Big(g\eta + \frac{\partial \Phi^I}{\partial t}\Big). \tag{2.7}$$

Here the domain is infinite along the y-direction, and an infinite horizontal circular cylinder with its generator running along the y-direction is to be considered in Section 3 to find the trapped modes. These trapped modes can be recognized only if we restrict our study to some finite length of the axis of the cylinder (for instance, if the cylinder is placed between two vertical walls at $y = 0$ and $y = \pi/l$). Assuming that the fluid motion is harmonic in time and in the y-direction, the velocity potential, free surface and interface elevations can be written in the form given by Mohapatra and Sahoo [12] as

$$\begin{aligned}
\Phi^j(x, y, z, t) &= \mathrm{Re}[\phi^j(x, z)\cos ly\, e^{-i\omega t}],\\
\eta(x, y, t) &= \mathrm{Re}[\bar{\eta}(x)\cos ly\, e^{-i\omega t}],\\
\zeta(x, y, t) &= \mathrm{Re}[\bar{\zeta}(x)\cos ly\, e^{-i\omega t}],
\end{aligned}$$

where Re denotes the real part of the quantity in the bracket, l is the wavenumber along the y-direction and ω is the radian frequency (both l and ω are taken to be real and positive so that the solution stays bounded for all y and t).

In this case, each $\phi^j(x, z)$ for $j = I, II$, satisfies the modified Helmholtz equation $(\nabla^2_{x,z} - l^2)\phi^j = 0$. By combining the kinematic and dynamic boundary conditions (2.2), (2.3), (2.6) and (2.7), the linearized boundary conditions at the mean free surface and mean interface can be written as

$$\begin{aligned}
\frac{\partial \phi^I}{\partial z} - K\phi^I - M_1\frac{\partial}{\partial z}\Big(\frac{\partial^2}{\partial x^2} - l^2\Big)\phi^I &= 0 \quad \text{on } z = d,\\
\frac{\partial \phi^{II}}{\partial z} - K\phi^{II} - \widetilde{M}_2\frac{\partial}{\partial z}\Big(\frac{\partial^2}{\partial x^2} - l^2\Big)\phi^{II} &= \rho\Big(\frac{\partial \phi^I}{\partial z} - K\phi^I\Big) \quad \text{on } z = 0,
\end{aligned} \tag{2.8}$$

where $M_1 = T_1/(\rho_I g)$, $\widetilde{M}_2 = T_2/(\rho_{II} g)$ and $\rho = \rho_I/\rho_{II}$ with $0 < \rho < 1$. An equivalent form of the interface condition (2.8) is given by

$$\frac{\partial \phi^{II}}{\partial z} - K\phi^{II} - M_2\frac{\partial}{\partial z}\Big(\frac{\partial^2}{\partial x^2} - l^2\Big)\phi^{II} = \rho\Big\{\frac{\partial \phi^I}{\partial z} - K\phi^I - M_2\frac{\partial}{\partial z}\Big(\frac{\partial^2}{\partial x^2} - l^2\Big)\phi^I\Big\},$$

where $M_2 = T_2/\{(\rho_{II} - \rho_I)g\}$.

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The impermeable bottom boundary condition is

$$\frac{\partial \phi^{II}}{\partial z} = 0 \quad \text{on } z = -h.$$

Within this framework, progressive waves or incident waves take the form (up to an arbitrary multiplicative constant)

$$\phi^I = \exp(\pm ix\sqrt{u^2 - l^2})\{F_+(u)e^{u(z-d)} + F_-(u)e^{-u(z-d)}\}, \tag{2.9}$$

$$\phi^{II} = \exp(\pm ix\sqrt{u^2 - l^2})\cosh u(z+h)F(u), \tag{2.10}$$

where

$$F(u) = \frac{F_+(u)e^{-ud} - F_-(u)e^{ud}}{\sinh uh},$$

$$F_\pm(u) = (1 + M_1u^2)u \pm K,$$

and u satisfies the dispersion relation

$$\{u(1 + M_2u^2) + K\sigma\}F_+(u)e^{-2u(d+h)} + \{u(1 + M_2u^2) - K\sigma\}F_-(u) - \{u(1 + M_2u^2) - K\}F_+(u)e^{-2ud} - \{u(1 + M_2u^2) - K\}F_+(u)e^{-2ud} = 0,$$

where $\sigma = (1+\rho)/(1-\rho)$.

This equation has exactly two positive real roots u_1 and u_2 ($u_1 < u_2$, say). A detailed analysis of the roots of the dispersion relation is given in the Appendix of the article by Bhattacharjee and Sahoo [2] and a similar derivation can be produced for the case when the surface tension parameters are taken into account. For the existence of trapped modes, it is required that

$$\phi^I, \phi^{II}, |\nabla\phi^I|, |\nabla\phi^{II}| \to 0 \quad \text{as } |x| \to \infty,$$

and since we are interested in discrete trapped modes that exist below a cut-off value, we, therefore, consider the cut-off value as l, and hence restrict l to be in the range $l > u_2 > u_1$, which ensures that no wave propagation to infinity takes place at the interface or near the free surface. In the far-field form of the potentials given by equations (2.9) and (2.10), we have $\exp(-\sqrt{l^2 - u^2}|x|)$, which reduces to zero if l is greater than both of the wavenumbers. Therefore, there will be no propagation of waves along the x-direction.

3. Solutions by multipoles

In this section, we discuss the effect of surface tension on trapped waves. The structure considered here is an impermeable, horizontal circular cylinder of radius a, having its axis along $z = f$, with $|f/a| > 1$ such that the cylinder is totally submerged in either of the layers of the two-layer fluid and its generator runs parallel to the y-axis. If $f > 0$, the cylinder is in the upper fluid, whereas for $f < 0$, the cylinder is in the lower fluid. Polar coordinates (r, θ) are defined in the xz-plane centred at $(0, f)$ as

$$x = r\sin\theta \quad \text{and} \quad z = f - r\cos\theta. \tag{3.1}$$

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Kassem [7] elaborated on different types of multipoles describing the velocity potentials, when each layer of a two-layer fluid was of constant depth. Following his method, we construct multipoles which are singular at $(0, f)$ and symmetrical about $x = 0$. The trapped mode potential is then constructed from a linear combination of all possible multipoles. Moreover, application of the body boundary condition results in an infinite system of homogeneous linear equations.

3.1. Cylinder submerged in the upper layer Since the singularity is in the upper fluid,

$$\phi_n^I \sim K_n(lr)\cos(n\theta) \quad \text{as } r = \sqrt{x^2 + (z-f)^2} \to 0;\; n = 1, 2, 3, \dots,$$

because these are the solutions of $(\nabla^2 - l^2)\phi(r, \theta) = 0$ and are singular at $r = 0$. Here $K_n(.)$ denotes an nth order modified Bessel function of the second kind having the integral representation [17]

$$K_n(lr)\cos n\theta = \begin{cases} \displaystyle\int_0^\infty \cosh nu \cos(lx\sinh u) e^{v(z-f)}\, du & \text{for } z < f, \\ \displaystyle(-1)^n \int_0^\infty \cosh nu \cos(lx\sinh u) e^{v(f-z)}\, du & \text{for } z > f, \end{cases}$$

where $v = l\cosh u$.

Using this integral representation, we try the following multipoles which satisfy the modified Helmholtz equation [9]

$$\phi_n^I = K_n(lr)\cos n\theta + \int_0^\infty \cosh nu \cos(lx\sinh u)[A_U(v)e^{vz} + B_U(v)e^{-vz}]\, du, \quad (3.2)$$

$$\phi_n^{II} = \int_0^\infty \cosh nu \cos(lx\sinh u) C_U(v) \cosh v(z+h)\, du. \quad (3.3)$$

With the help of the boundary conditions at the free surface, the interface and the bottom, the coefficients $A_U(v)$, $B_U(v)$ and $C_U(v)$ in equations (3.2) and (3.3) are

$$A_U(v) = \frac{F_+(v)e^{-2vd}}{F_-(v)}[B_U(v) + (-1)^n e^{vf}],$$

$$B_U(v) = \frac{(-1)^n F_+(v)e^{v(f-2d)} + F_-(v)e^{-vf}}{G(v)} \times [v(1 + M_2v^2) - K - e^{-2vh}\{v(1 + M_2v^2) + K\sigma\}],$$

$$C_U(v) = \frac{2K(1-\sigma)B_U(v)}{\{v(1 + M_2v^2) - K\}e^{vh} - \{v(1 + M_2v^2) + K\sigma\}e^{-vh}},$$

where

$$G(v) = \{v(1 + M_2v^2) - K\}F_+(v)e^{-2vd} + \{v(1 + M_2v^2) + K\}F_-(v)e^{-2vh} - \{v(1 + M_2v^2) + K\sigma\}F_+(v)e^{-2v(d+h)} - \{v(1 + M_2v^2) - K\sigma\}F_-(v). \quad (3.4)$$

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The multipoles can be expanded in polar coordinates. If we first put $X = -lr$ and $T = \exp[i(\theta + iu)]$ in the well-known generating function for modified Bessel functions (see [19]), then

$$\exp\Big[\frac{1}{2}X(T + T^{-1})\Big] = \sum_{m=0}^{\infty} \frac{1}{2}\epsilon_m (T^m + T^{-m}) I_m(X),$$

where $I_n(.)$ are modified Bessel functions of the first kind of order n, and $\epsilon_0 = 1$, $\epsilon_m = 2$ for $m \geq 1$. Then, by taking the real and imaginary parts, the resulting expressions can be substituted into (3.2), using (3.1), to give

$$\phi_n^I = K_n(lr)\cos n\theta + \sum_{m=0}^{\infty} A_{mn} I_m(lr)\cos m\theta,$$

where

$$A_{mn} = \epsilon_n \int_0^{\infty} \cosh mu \cosh nu[(-1)^n A_U(v)e^{vf} + B_U(v)e^{-vf}]\, du. \tag{3.5}$$

We note that, since $v > l > u_2 > u_1 > K > 0$, there will be no singularity of the integrand on the real axis. Now, the total velocity potential is

$$\phi_{\mathrm{tot}}^I = \sum_{n=0}^{\infty} \alpha_n \phi_n^I. \tag{3.6}$$

Applying the body boundary condition $\partial\phi/\partial r = 0$ on $r = a$ to the infinite series (3.6), we obtain the infinite system of homogeneous linear equations in the unknowns α_n given by

$$\alpha_n + \frac{I_n'(la)}{K_n'(la)} \sum_{m=0}^{\infty} \alpha_m A_{mn} = 0, \quad n = 0, 1, 2, \ldots, \tag{3.7}$$

where $'$ denotes differentiation with respect to r.

For the existence of trapped modes, we require nontrivial solutions to the infinite system of equations (3.7). Hence, for a fixed geometrical configuration along with a fixed density ratio, we need to find the values of Ka or la (assuming Ka to have a fixed value while computing la and vice-versa) for which the truncated determinant vanishes. For the sake of our numerical computation, we truncate the system to a 32×32 system. The convergence of this system has been derived in one of our earlier works [14]. As discussed previously in Section 2, since we are interested in those trapped modes that occur below a cut-off value of la, we consider a fixed value of la and then take into account different values of Ka up to that fixed value. The bisection method is used to find the values of frequency Ka for which the truncated determinant vanishes. Then, corresponding to those values of frequency, the value of the wavenumbers $u_1 a$ and $u_2 a$ can be determined by using the dispersion relation. To compute the determinant, we need to calculate A_{mn} from equation (3.5) for different values of m and n. Since there is no singularity on the real axis of the integrand on the

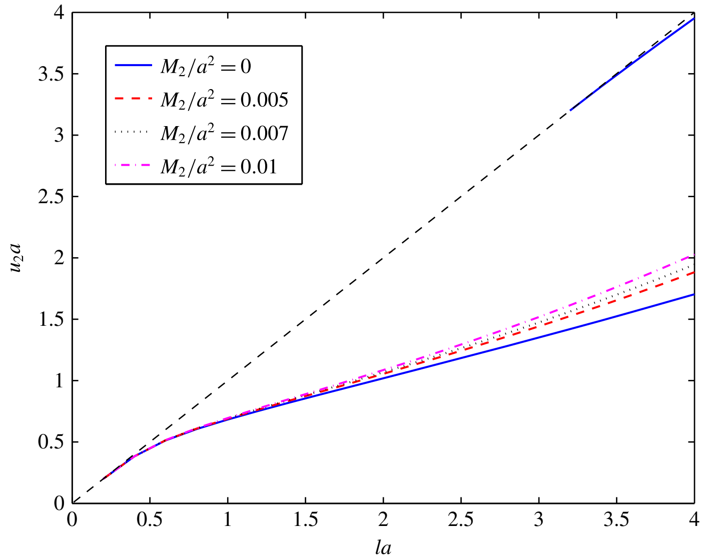

FIGURE 2. Dispersion curves for a cylinder of radius a in the upper layer for different values of M_2/a^2; $\rho = 0.5$, $d/a = 3$, $f/a = 1.01$, $h/a = 6$ and $M_1/a^2 = 0$. (Colour available online.)

right-hand side of equation (3.5), we first break the integral into four convergent integrals and then apply the command NINTEGRATE in Mathematica. Subsequently, we find out the values of A_{mn}. The results presented below are correct to three decimal places. The effects of submergence depth and the depths of each fluid layer have already been covered by the work of Linton and Cadby [9]. We now specifically investigate the effects of variations in both of the surface tension parameters on the dispersion and density plots of trapped modes (if any exist).

3.1.1. Numerical results Figures 2–4 show the results obtained for trapped modes above a horizontal circular cylinder of radius a, submerged entirely in the upper layer of a two-layer fluid with a free surface, with the inclusion of surface tension at both the free surface and the interface. For all these cases, the depth h/a of the lower layer is taken as 6.0. For the dispersion curve, $u_2 a = la$ is the upper bound for the trapped mode wavenumber $u_2 a$ (Figure 2). With the given set of parameter values, we observed that, when there was no surface tension at the interface, there did exist two trapped modes as seen by Linton and Cadby [9]. However, with the consideration of a nonzero value of nondimensionalized surface tension parameter M_2/a^2 at the interface, there exists only one trapped mode. This mode is little affected by an increase in the parameter M_2/a^2.

Trapped mode wavenumbers are plotted against density ratio in Figures 3–4 for different values of the surface tension parameters M_1/a^2 and M_2/a^2, assuming that one parameter is zero and the other varies. The submergence depth f/a is taken as 1.05 and the depth of the upper layer as 2.10. In Figure 3, for both of the

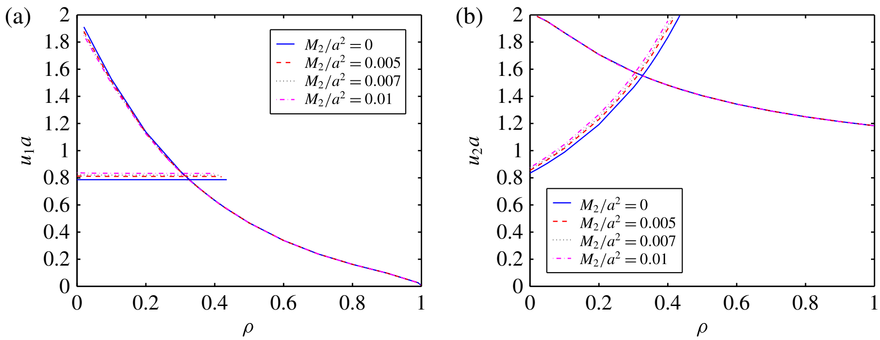

FIGURE 3. Trapped mode wavenumbers plotted against ρ for a cylinder of radius a in the upper fluid layer for different values of free surface tension M_1/a^2; $la = 2$, $d/a = 2.1$, $h/a = 6$, $f/a = 1.05$ and $M_2/a^2 = 0$. (Colour available online.)

wavenumbers, different pairs of curves correspond to four different values of surface tension parameter: $M_1/a^2 = 0, 0.005, 0.007$ and 0.01. It is observed that there are two curves for each of the wavenumbers u_1a and u_2a, corresponding to each value of the surface tension parameter, showing that two modes exist. For the wavenumber u_1a, the first modes (lower ones) correspond to the lines of constant u_1a, which are solutions to the single-layer fluid problem. These first modes appear to cross the second modes, but with closer inspection we can observe near-crossing points or diabolical points. These points have also been observed by Linton and Cadby [9]. If the density ratio increases further, the second modes remain constant and terminate at a certain value of ρ corresponding to different values of the surface tension parameter, whereas the first modes decrease to zero. For the wavenumber u_2a, the first modes increase, and the second modes decrease with an increase in density ratio and they come very close to each other at near crossing points. As ρ increases further, the modes interchange their properties: that is, the second modes increase and then terminate when $u_2a = 2 = la$ is reached, corresponding to a fixed value of ρ for all values of M_1/a^2, while the first mode decreases to some fixed value for each value of M_1/a^2. The second mode for the wavenumbers u_1a and u_2a terminates at some small nonzero value of ρ. The presence of near-crossing points shows that at those specific values of density ratio ρ, the free surface and interfacial modes interchange their properties. More detailed explanation on near-crossing points can be found in the literature (see, for example, [9]). Also, in Figure 4, exactly the same behaviour is observed for different values of the interfacial surface tension parameter M_2/a^2. In contrast to what was found by Linton and Cadby [9], we did not observe that inclusion of surface tension at the free surface and the interface had a very significant effect.

3.2. Cylinder submerged in the lower layer Now consider the problem with the cylinder placed in the lower layer. The multipoles that are singular at $z = f\,(< 0)$ are required to be modified. This can be done by the same method as used previously when the cylinder was placed in the upper layer ($f > 0$). The suitable symmetrical

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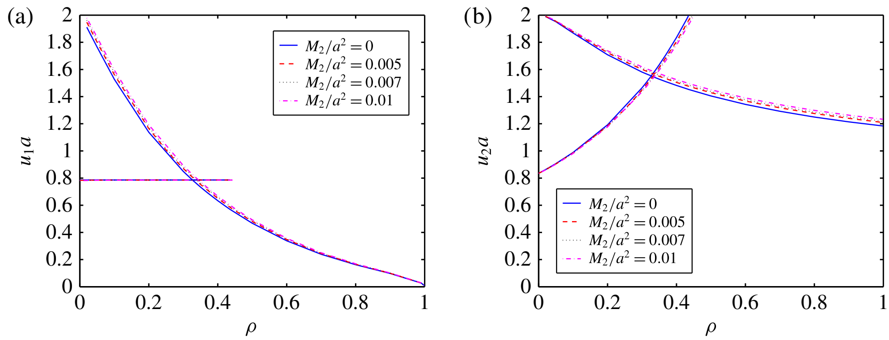

FIGURE 4. Trapped mode wavenumbers plotted against ρ for a cylinder of radius a in the upper fluid layer for different values of interfacial tension M_2/a^2; $la = 2$, $d/a = 2.1$, $h/a = 6$, $f/a = 1.05$ and $M_1/a^2 = 0$. (Colour available online.)

multipoles are

$$\phi_n^I = \int_0^\infty \cosh nu \cos(lx \sinh u)[A_L(v)e^{vz} + B_L(v)e^{-vz}]\, du,$$

$$\phi_n^{II} = K_n(lr)\cos n\theta + \int_0^\infty \cosh nu \cos(lx \sinh u)[C_L(v)e^{vz} + D_L(v)e^{-vz}]\, du,$$

where the integrals are Cauchy principal value integrals with

$$A_L(v) = \frac{F_+(v)}{F_-(v)} B_L(v) e^{-2vd},$$

$$B_L(v) = \frac{K(1+\sigma)F_-(v)}{G(v)}\{(-1)^{n+1}e^{vf} - e^{-v(f+2h)}\},$$

$$C_L(v) = \frac{B_L(v)}{K(1+\sigma)F_-(v)}[\{v(1+M_2v^2) + K\sigma\}F_+(v)e^{-2vd}\{v(1+M_2v^2) + K\}F_-(v)],$$

$$D_L(v) = (C_L(v) + e^{-vf})e^{-2vh},$$

and where $G(v)$ is given by equation (3.4). Due to the trapped mode condition, there are no singularities on the real axis. Following the previous procedure, the polar expansion of the multipoles is

$$\phi_n^{II} = K_n(lr)\cos n\theta + \sum_{m=0}^{\infty} B_{mn} I_m(lr)\cos m\theta,$$

where

$$B_{mn} = \epsilon_n \int_0^\infty \cosh mu \cosh nu[(-1)^n C_L(v)e^{vf} + D_L(v)e^{-vf}]\, du.$$

By applying the body boundary condition, a system of equations similar to (3.7) is

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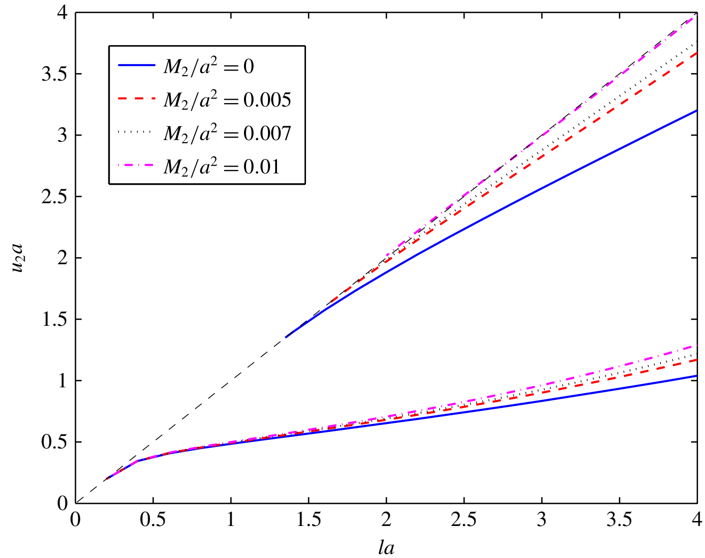

FIGURE 5. Dispersion curves for a cylinder of radius a in the lower layer for different values of M_2/a^2; $\rho = 0.5$, $d/a = 3$, $f/a = -1.01$, $h/a = 6$ and $M_1/a^2 = 0$. (Colour available online.)

obtained for β_n, given by

$$\beta_n + \frac{I_n'(la)}{K_n'(la)} \sum_{m=0}^{\infty} \beta_m B_{mn} = 0, \quad n = 0, 1, 2, \ldots. \tag{3.8}$$

Here, as in the previous case, the zeros of the truncated determinant are conveniently located by varying the frequencies Ka and fixing the other parameters. The results presented next are obtained correct to three decimal places, where a 32×32 system is used after truncating the system obtained from equation (3.8).

3.2.1. Numerical results Figures 5–7 show the plots of the nondimensional trapped mode frequencies for a horizontal circular cylinder of radius a, entirely immersed in the lower layer of the two-layer fluid with the inclusion of surface tension at the free surface and the interface. In all cases, the depth d/a of the upper layer is taken as 3.0, the submergence depth f/a as -1.01 (which means that the cylinder is very close to the interface), and the depth h/a of the lower layer as 6.0. Figure 5 shows the dispersion curves for four different values of the surface tension parameter: $M_2/a^2 = 0$, 0.005, 0.007 and 0.01. For each set of parameter values, there are two curves corresponding to two modes, which are displayed in the graphs of Figure 6. We observe that the trapped mode wavenumber u_2a increases when the surface tension value at the interface increases, the second mode being affected more than the first mode.

When trapped-mode wavenumbers are plotted against density ratio for different values of surface tension parameter M_1/a^2 at the free surface, we observe that the

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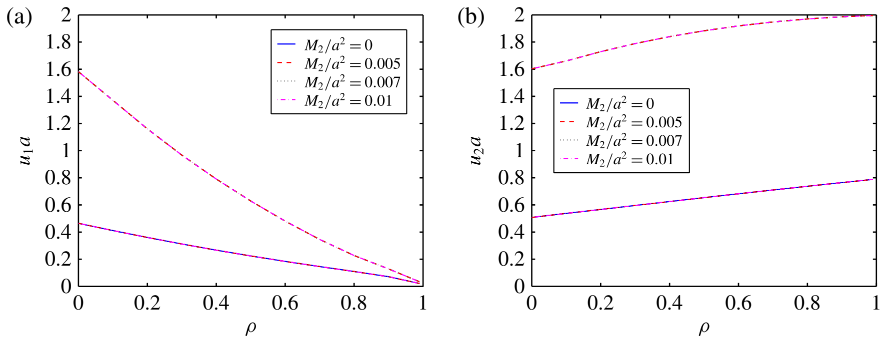

FIGURE 6. Trapped mode wavenumbers plotted against ρ for a cylinder of radius a in the lower fluid layer for different values of M_1/a^2; $la = 2$, $d/a = 3.0$, $h/a = 6.0$, $f/a = -1.01$ and $M_2/a^2 = 0$. (Colour available online.)

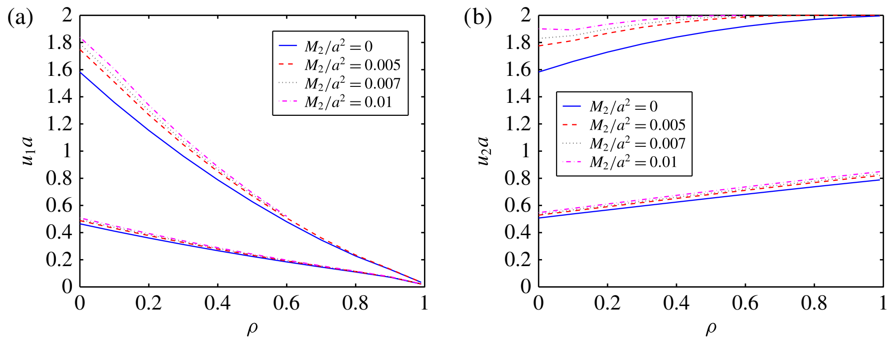

FIGURE 7. Trapped mode wavenumbers plotted against ρ for a cylinder of radius a in the lower fluid layer for different values of M_2/a^2; $la = 2$, $d/a = 3.0$, $h/a = 6.0$, $f/a = -1.01$ and $M_1/a^2 = 0$. (Colour available online.)

modes are unaffected by the variation. But, with the variation of surface tension parameter M_2/a^2, as seen from Figure 7, trapped modes for both of the wavenumbers do get affected. With an increase in M_2/a^2, the second trapped mode for both of the wavenumbers u_1a and u_2a shows a bigger increase as compared to the first mode.

4. Conclusion

We examine the effect on the trapped modes supported by a horizontal circular cylinder placed entirely in one of the layers of a two-layer fluid, when surface tension at the free surface and the interface is incorporated. The dispersion curves are analysed for different values of the interfacial surface tension when the cylinder is placed in each of the layers. In both cases, the second mode gets affected more than the first mode. When the cylinder is placed in the upper layer, the second mode does not exist when an increase in interfacial surface tension takes place. When the cylinder is placed

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in the lower layer, the value of the wavenumbers for the second mode decreases, corresponding to an increase in the same surface tension parameter. We also plot trapped mode wavenumbers against density ratio for different values of free surface and interfacial surface tension when the cylinder is placed in either of the layers (see Figures 3, 4, 6 and 7). In both cases, we observe that, by varying both surface tension parameters, the pattern or value of the wavenumbers does not change in a significant manner.

Hence, it is justified to ignore the effect of surface tension from the free surface or the interface as the current authors did for the problems in some earlier works [13, 14]. Its inclusion gives rise to a third-order boundary condition, and hence makes the computation time-consuming. Even after its inclusion, no significant change is observed on the pattern of trapped modes and values of the frequency.

Therefore, we conclude that inclusion of surface tension at the free surface and (or) the interfaces, as dictated by the problem, brings no significant change to the trapped modes. Although the problem carried out in this article is for a two-layer fluid flow with a free surface, similar observation is expected when the free surface is replaced by a rigid lid or an ice-cover.

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