Glasgow Math. J. **60** (2018) 513–525. © Glasgow Mathematical Journal Trust 2017.
doi:10.1017/S0017089517000258.

NORM OF THE HILBERT MATRIX OPERATOR ON THE WEIGHTED BERGMAN SPACES

BOBAN KARAPETROVIĆ
University of Belgrade, Faculty of Mathematics
Studentski trg 16, Serbia
e-mail: bkarapetrovic@matf.bg.ac.rs

(Received 8 February 2017; revised 9 August 2017; accepted 1 September 2017; first published online 18 October 2017)

Abstract. We find the lower bound for the norm of the Hilbert matrix operator H on the weighted Bergman space $A^{p,\alpha}$

$$\|H\|_{A^{p,\alpha}\to A^{p,\alpha}} \geq \frac{\pi}{\sin\frac{(\alpha+2)\pi}{p}}, \text{ for } 1<\alpha+2<p.$$

We show that if $4 \leq 2(\alpha+2) \leq p$, then $\|H\|_{A^{p,\alpha}\to A^{p,\alpha}} = \frac{\pi}{\sin\frac{(\alpha+2)\pi}{p}}$, while if $2 \leq \alpha+2 < p < 2(\alpha+2)$, upper bound for the norm $\|H\|_{A^{p,\alpha}\to A^{p,\alpha}}$, better then known, is obtained.

2010 *Mathematics Subject Classification.* 47B35, 30H20.

1. Introduction.

1.1. Hardy and Bergman spaces. Let $\mathcal{H}(\mathbb{D})$ be the space of all functions holomorphic in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

For $0 < p \leq \infty$, the Hardy space H^p is the space of all functions $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{H^p} := \|f\|_p = \sup_{0\leq r<1} M_p(r,f) < \infty,$$

where

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p dt\right)^{\frac{1}{p}}, \ 0<p<\infty;$$

$$M_\infty(r,f) = \sup_{0\leq t<2\pi} |f(re^{it})|.$$

The normalized Lebesgue area measure on $\mathbb{D}$ will be denoted by A, i.e.,

$$dA(z) = \frac{1}{\pi}dxdy = \frac{1}{\pi}rdrdt, \ z = x+iy = re^{it}.$$

Recall that for $0 < p < \infty$ and $\alpha > -1$, the (weighted) Bergman space $A^{p,\alpha} = A^{p,\alpha}(\mathbb{D})$ is the space $\mathcal{H}(\mathbb{D}) \cap L^p(\mathbb{D}, dA_\alpha)$, where

$$dA_\alpha(z) = (\alpha+1)\left(1-|z|^2\right)^\alpha dA(z).$$

If $f \in A^{p,\alpha} = \mathcal{H}(\mathbb{D}) \cap L^p(\mathbb{D}, dA_\alpha)$, we write

$$\|f\|_{A^{p,\alpha}} := \|f\|_{p,\alpha} = \left((\alpha+1)\int_{\mathbb{D}} |f(z)|^p \left(1-|z|^2\right)^\alpha dA(z)\right)^{\frac{1}{p}}.$$

Simply, $A^p = A^{p,0}$ are (unweighted) Bergman spaces. We have that

$$\|f\|_{A^p} = \left(\int_{\mathbb{D}} |f(z)|^p dA(z)\right)^{\frac{1}{p}} = \left(2\int_0^1 rM_p^p(r,f)dr\right)^{\frac{1}{p}},$$

and obviously, $H^p \subset A^p$. Actually, it is well known that $H^p \subset A^{2p}$. The functions in the Bergman spaces exhibit a behaviour somewhat similar to that of the Hardy spaces functions, but often a bit more complicated.

For more information related to the Hardy spaces and the Bergman spaces see monographs [**4**, **8**, **11**].

1.2. The Hilbert matrix. The Hilbert matrix is an infinite matrix H whose entries are $a_{n,k} = \frac{1}{n+k+1}$, $n, k \geq 0$. We note that H as an operator on the space ℓ^2 of all square-summable complex sequences was first studied by Magnus [**10**]. It can be also viewed as an operator on spaces of holomorphic functions by its action on their Taylor coefficients. If $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n)z^n$ is a holomorphic function in $\mathbb{D}$, then we define a transformation H by

$$Hf(z) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\widehat{f}(k)}{n+k+1} z^n.$$

It is well known that the Hilbert matrix operator H is a bounded operator from H^p into H^p if and only if $1 < p < \infty$, and H is a bounded operator from A^p into A^p if and only if $2 < p < \infty$ (see [**1**, **2**, **8**]). In [**2**] was first started the study of the Hilbert matrix as an operator on spaces of holomorphic functions. Namely, the boundedness of the Hilbert matrix as an operator on H^p, $1 < p < \infty$, was first proved by Diamantopoulos and Siskakis [**2**]. In [**3**] it was shown that $\|H\|_{H^p \to H^p} = \frac{\pi}{\sin\frac{\pi}{p}}$, for $1 < p < \infty$.

For some recent results and generalizations related to the Hilbert matrix see [**6**, **7**, **9**].

1.3. The main results. We are now ready to state the main results of the paper.

THEOREM 1.1. *If $1 < \alpha + 2 < p$, then $\|H\|_{A^{p,\alpha} \to A^{p,\alpha}} \geq \frac{\pi}{\sin\frac{(\alpha+2)\pi}{p}}$.*

In particular, $\|H\|_{A^p \to A^p} \geq \frac{\pi}{\sin\frac{2\pi}{p}}$, for $2 < p < \infty$. This special case was proved in [**3**]. Thus, the lower bound of $\|H\|_{A^{p,\alpha} \to A^{p,\alpha}}$ given in Theorem 1.1 is an extension of the estimate of $\|H\|_{A^p \to A^p}$ and our proof is much simpler than that given in [**3**]. It is based on the use of hypergeometric functions. This method may be also applied to obtain very simple proof that $\|H\|_{H^p \to H^p} \geq \frac{\pi}{\sin\frac{\pi}{p}}$, for $1 < p < \infty$. By a different method this estimate was obtained in [**3**].

THEOREM 1.2. *Let* $\alpha \geq 0$ *and* $p > \alpha + 2$.

(*i*) *If* $p \geq 2(\alpha + 2)$, *then*

$$\|H\|_{A^{p,\alpha}\to A^{p,\alpha}} \leq \frac{\pi}{\sin\frac{(\alpha+2)\pi}{p}};$$

(*ii*) *If* $2\alpha + 3 \leq p < 2(\alpha + 2)$, *then*

$$\|H\|_{A^{p,\alpha}\to A^{p,\alpha}} \leq 2^{\frac{\alpha+1}{p}}\frac{\pi}{\sin\frac{(\alpha+2)\pi}{p}};$$

(*iii*) *If* $\alpha + 2 < p < 2\alpha + 3$, *then*

$$\|H\|_{A^{p,\alpha}\to A^{p,\alpha}} \leq \left(1 + 2^{\frac{2(\alpha+2)}{p}-1}\right)\frac{\pi}{\sin\frac{(\alpha+2)\pi}{p}}.$$

Note that it follows from the main result in [**7**] that H is a bounded operator on $A^{p,\alpha}$ if and only if $1 < \alpha + 2 < p$.

Theorem 1.1 and Theorem 1.2 together give the following result.

COROLLARY 1.3. If $p \geq 2(\alpha + 2)$ and $\alpha \geq 0$, then

$$\|H\|_{A^{p,\alpha}\to A^{p,\alpha}} = \frac{\pi}{\sin\frac{(\alpha+2)\pi}{p}}.$$

For $\alpha = 0$, this was proved in [**3**]. It follows from Theorem 1.2, that if $3 \leq p < 4$, then

$$\|H\|_{A^p\to A^p} \leq 2^{\frac{1}{p}}\frac{\pi}{\sin\frac{2\pi}{p}} \leq \sqrt[3]{2}\frac{\pi}{\sin\frac{2\pi}{p}},$$

and if $2 < p < 3$, then

$$\|H\|_{A^p\to A^p} \leq \left(1 + 2^{\frac{4}{p}-1}\right)\frac{\pi}{\sin\frac{2\pi}{p}} \leq 3\frac{\pi}{\sin\frac{2\pi}{p}}.$$

These two estimates are better than those given in [**3**].

Note that the exact computation of the norm of the Hilbert matrix as an operator on the Bergman space A^p and on the Hardy space H^p is based on the integral representation of H,

$$Hf(z) = \int_0^1 \frac{f(t)}{1 - tz}dt,$$

whenever this integral makes sense for all functions f in the space under consideration (see [**2**]). From the previous representation it also follows, by a change of variables, that the Hilbert matrix operator H can be written as an average of weighted composition operators and this integral representation of H was used in the computation of the norm of the Hilbert matrix as an operator on the Bergman space A^p (see [**1**]). Because of this, the exact computation of the norm of the Hilbert matrix as an operator on A^p is a more difficult problem than its Hardy space counterpart.

We propose the following conjecture.

CONJECTURE. If $1 < \alpha + 2 < p$, then $\|H\|_{A^{p,\alpha} \to A^{p,\alpha}} = \frac{\pi}{\sin \frac{(\alpha+2)\pi}{p}}$.

2. Lower bound for the norm $\|H\|_{A^{p,\alpha} \to A^{p,\alpha}}$.

2.1. An integral representation. As it was noticed in the Introduction, if $1 < \alpha + 2 < p$, then $H : A^{p,\alpha} \to A^{p,\alpha}$ is bounded. A calculation shows that, in this case, if $f \in A^{p,\alpha}$, then

$$Hf(z) = \int_0^1 \frac{f(t)}{1 - tz} dt. \tag{1}$$

Namely, following [7], if $1 < \alpha + 2 < p$, then H is well-defined operator on $A^{p,\alpha}$ and maps this space into itself. Therefore, if f belongs to $A^{p,\alpha}$ and $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$, then we obtain a well-defined holomorphic function Hf on $\mathbb{D}$ and $Hf \in A^{p,\alpha}$. Hence, we find that

$$\begin{aligned} Hf(z) &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{\widehat{f}(k)}{n+k+1} \right) z^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \widehat{f}(k) \int_0^1 t^{n+k} dt \right) z^n \\ &= \int_0^1 \sum_{k=0}^{\infty} \widehat{f}(k) t^k \sum_{n=0}^{\infty} t^n z^n dt \\ &= \int_0^1 \frac{f(t)}{1 - tz} dt, \end{aligned}$$

where the interchange of integrals and sums is easily justified by a geometric series argument.

2.2. Hypergeometric functions. To get the lower bound of $\|H\|_{A^{p,\alpha} \to A^{p,\alpha}}$, we will use some classical identities about the Gamma, Beta and Hypergeometric functions (see [5]).

The Beta function is defined by

$$B(s, t) = \int_0^1 x^{s-1} (1 - x)^{t-1} dx = \int_0^{\infty} \frac{x^{s-1}}{(1 + x)^{s+t}} dx,$$

for s, t such that $\operatorname{Re} s > 0$, $\operatorname{Re} t > 0$. The value $B(s, t)$ can be expressed in term of Gamma function as $B(s, t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}$. Moreover, the Gamma function satisfies the functional equation $\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z}$, for non-integral complex numbers z.

As is usual, $F(a, b, c; z)$, $z \in \mathbb{D}$, denotes the hypergeometric function with parameters a, b, c, i.e.,

$$F(a, b, c; z) = \sum_{k=0}^{\infty} \frac{\Gamma(k+a)}{\Gamma(a)} \frac{\Gamma(k+b)}{\Gamma(b)} \frac{\Gamma(c)}{\Gamma(k+c)} \frac{z^k}{k!}.$$

We will use the following integral representation of hypergeometric function

$$F(a, b, c; z) = \frac{1}{B(a, c-a)} \int_0^1 \frac{t^{a-1}(1-t)^{c-a-1}}{(1-tz)^b} dt, \ \operatorname{Re} c > \operatorname{Re} a > 0. \tag{2}$$

2.3. The proof of Theorem 1.1. Let $1 < \gamma < \alpha + 2 < p$ and $f_\gamma(z) = (1-z)^{-\frac{\gamma}{p}}$, $z \in \mathbb{D}$. An easy calculation shows that

$$\|f_\gamma\|_{p,\alpha}^p = F\left(\frac{\gamma}{2}, \frac{\gamma}{2}, \alpha+2; 1\right).$$

By Stirling's formula

$$\frac{\Gamma^2\left(k+\frac{\gamma}{2}\right)}{\Gamma(k+\alpha+2)} \sim \frac{k!}{(k+1)^{\alpha+3-\gamma}}, \ k \to \infty.$$

Thus, $\|f_\gamma\|_{p,\alpha} < \infty$, since $\alpha + 3 - \gamma > 1 \Leftrightarrow \gamma < \alpha + 2$. On the other hand, we have that $\lim_{\gamma \to \alpha+2} \|f_\gamma\|_{p,\alpha} = \infty$. Using (1) and (2) we find that

$$Hf_\gamma(z) = \int_0^1 \frac{dt}{(1-t)^{\frac{\gamma}{p}}(1-tz)} = B\left(1, 1-\frac{\gamma}{p}\right) F\left(1, 1, 2-\frac{\gamma}{p}; z\right).$$

Thus,

$$Hf_\gamma(z) = \Gamma\left(\frac{\gamma}{p}\right) \Gamma\left(1-\frac{\gamma}{p}\right) \sum_{k=0}^{\infty} \frac{\Gamma^2(k+1)}{\Gamma\left(k+2-\frac{\gamma}{p}\right)\Gamma\left(k+\frac{\gamma}{p}\right)} \frac{\Gamma\left(k+\frac{\gamma}{p}\right)}{\Gamma\left(\frac{\gamma}{p}\right)} \frac{z^k}{k!}.$$

Since

$$\frac{\Gamma^2(k+1)}{\Gamma\left(k+2-\frac{\gamma}{p}\right)\Gamma\left(k+\frac{\gamma}{p}\right)} = 1 + O\left(\frac{1}{k+1}\right),$$

we obtain

$$Hf_\gamma(z) = \frac{\pi}{\sin\frac{\pi\gamma}{p}} \left(f_\gamma(z) + g_\gamma(z)\right),$$

where

$$\sup_{1<\gamma<\alpha+2} \|g_\gamma\|_\infty \leq C_{p,\alpha} < \infty,$$

https://doi.org/10.1017/S0017089517000258 Published online by Cambridge University Press

and consequently

$$\sup_{1<\gamma<\alpha+2} \|g_\gamma\|_{p,\alpha} \leq C_{p,\alpha}.$$

Therefore,

$$\|H\|_{A^{p,\alpha}\to A^{p,\alpha}} \geq \frac{\|Hf_\gamma\|_{p,\alpha}}{\|f_\gamma\|_{p,\alpha}} \geq \frac{\pi}{\sin\frac{\pi\gamma}{p}} \frac{\|f_\gamma\|_{p,\alpha} - \|g_\gamma\|_{p,\alpha}}{\|f_\gamma\|_{p,\alpha}}.$$

Letting $\gamma \to \alpha + 2$, we get

$$\begin{aligned}
\|H\|_{A^{p,\alpha}\to A^{p,\alpha}} &\geq \lim_{\gamma\to\alpha+2} \left(\frac{\pi}{\sin\frac{\pi\gamma}{p}} \frac{\|f_\gamma\|_{p,\alpha} - \|g_\gamma\|_{p,\alpha}}{\|f_\gamma\|_{p,\alpha}} \right) \\
&= \frac{\pi}{\sin\frac{(\alpha+2)\pi}{p}} \lim_{\gamma\to\alpha+2} \left(1 - \frac{\|g_\gamma\|_{p,\alpha}}{\|f_\gamma\|_{p,\alpha}}\right) \\
&= \frac{\pi}{\sin\frac{(\alpha+2)\pi}{p}},
\end{aligned}$$

because of $\lim_{\gamma\to\alpha+2} \|f_\gamma\|_{p,\alpha} = \infty$ and $\sup_{1<\gamma<\alpha+2} \|g_\gamma\|_{p,\alpha} \leq C_{p,\alpha} < \infty$. This concludes the proof.

2.4. Lower bound for the norm $\|H\|_{H^p\to H^p}$. A similar argument shows that

$$\|H\|_{H^p\to H^p} \geq \frac{\pi}{\sin\frac{\pi}{p}}, \quad \text{for } 1 < p < \infty.$$

To see this, we take $f_\gamma(z) = (1-z)^{-\frac{\gamma}{p}}$, $0 < \gamma < 1 < p$. An easy calculation show that

$$\|f_\gamma\|_{H^p}^p = \sum_{k=0}^{\infty} \frac{\Gamma^2\left(k+\frac{\gamma}{2}\right)}{\Gamma^2\left(\frac{\gamma}{2}\right)} \frac{1}{(k!)^2}.$$

By Stirling's formula

$$\frac{\Gamma^2\left(k+\frac{\gamma}{2}\right)}{(k!)^2} \sim \frac{1}{(k+1)^{2-\gamma}}, \quad k \to \infty.$$

Thus, $\|f_\gamma\|_{H^p} < \infty$ and $\|f_\gamma\|_{H^p} \to \infty$, as $\gamma \to 1$.

On the other hand, by using (1) and (2), we find that

$$Hf_\gamma(z) = \int_0^1 \frac{dt}{(1-t)^{\frac{\gamma}{p}}(1-tz)} = B\left(1, 1-\frac{1}{p}\right) F\left(1, 1, 2-\frac{\gamma}{p}; z\right).$$

Thus,

$$Hf_\gamma(z) = \Gamma\left(\frac{\gamma}{p}\right)\Gamma\left(1-\frac{\gamma}{p}\right) \sum_{k=0}^{\infty} \frac{\Gamma^2(k+1)}{\Gamma\left(k+2-\frac{\gamma}{p}\right)\Gamma\left(k+\frac{\gamma}{p}\right)} \frac{\Gamma\left(k+\frac{\gamma}{p}\right)}{\Gamma\left(\frac{\gamma}{p}\right)} \frac{z^k}{k!}.$$

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Since

$$\frac{\Gamma^2(k+1)}{\Gamma\left(k+2-\frac{\gamma}{p}\right)\Gamma\left(k+\frac{\gamma}{p}\right)} = 1 + O\left(\frac{1}{k+1}\right),$$

we obtain

$$Hf_\gamma(z) = \frac{\pi}{\sin\frac{\pi\gamma}{p}}\left(f_\gamma(z) + g_\gamma(z)\right).$$

Since

$$\sup_{0<\gamma<1}\|g_\gamma\|_{H^p} \le \sup_{0<\gamma<1}\|g_\gamma\|_{H^\infty} \le C < \infty,$$

we get

$$\|H\|_{H^p\to H^p} \ge \frac{\|Hf_\gamma\|_{H^p}}{\|f_\gamma\|_{H^p}} \ge \frac{\pi}{\sin\frac{\pi\gamma}{p}}\frac{\|f_\gamma\|_{H^p} - \|g_\gamma\|_{H^p}}{\|f_\gamma\|_{H^p}}.$$

Letting $\gamma \to 1^-$, we get

$$\begin{aligned}\|H\|_{H^p\to H^p} &\ge \lim_{\gamma\to 1^-}\left(\frac{\pi}{\sin\frac{\pi\gamma}{p}}\frac{\|f_\gamma\|_{H^p} - \|g_\gamma\|_{H^p}}{\|f_\gamma\|_{H^p}}\right)\\ &= \frac{\pi}{\sin\frac{\pi}{p}}\lim_{\gamma\to 1^-}\left(1 - \frac{\|g_\gamma\|_{H^p}}{\|f_\gamma\|_{H^p}}\right)\\ &= \frac{\pi}{\sin\frac{\pi}{p}}.\end{aligned}$$

A different proof of this inequality is given in [**3**].

3. Upper bound for the norm $\|H\|_{A^{p,\alpha}\to A^{p,\alpha}}$. If $2 \le \alpha + 2 < p$, then $H : A^{p,\alpha} \to A^{p,\alpha}$ is bounded. Following [**1**, **2**] (see also [**8**]), we have that, if $f \in A^{p,\alpha}$, then

$$Hf(z) = \int_0^1 T_t f(z)dt,$$

where

$$T_t f(z) = \omega_t(z) f(\phi_t(z)),$$

and

$$\omega_t(z) = \frac{1}{1-(1-t)z}, \quad \phi_t(z) = \frac{t}{1-(1-t)z}.$$

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3.1. The proof of Theorem 1.2. First, from the continuous version of Minkowski's inequality, we have

$$\begin{aligned}\|Hf\|_{A^{p,\alpha}} &= \left((\alpha+1)\int_{\mathbb{D}}|Hf(z)|^p\left(1-|z|^2\right)^{\alpha}dA(z)\right)^{\frac{1}{p}}\\ &= (\alpha+1)^{\frac{1}{p}}\left(\int_{\mathbb{D}}\left|\int_0^1 T_tf(z)\left(1-|z|^2\right)^{\frac{\alpha}{p}}dt\right|^p dA(z)\right)^{\frac{1}{p}}\\ &\leq (\alpha+1)^{\frac{1}{p}}\int_0^1\left(\int_{\mathbb{D}}|T_tf(z)|^p\left(1-|z|^2\right)^{\alpha}dA(z)\right)^{\frac{1}{p}}dt\\ &= \int_0^1\|T_tf\|_{A^{p,\alpha}}dt.\end{aligned} \tag{3}$$

Using linear fractional change of variable $w=\phi_t(z)$, $z\in\mathbb{D}$, we obtain

$$\begin{aligned}\|T_tf\|^p_{A^{p,\alpha}} &= (\alpha+1)\int_{\mathbb{D}}|\omega_t(z)|^p|f(\phi_t(z))|^p\left(1-|z|^2\right)^{\alpha}dA(z)\\ &= (\alpha+1)\int_{\phi_t(\mathbb{D})}\left|\omega_t(\phi_t^{-1}(w))\right|^p\frac{|f(w)|^p\left(1-|\phi_t^{-1}(w)|^2\right)^{\alpha}}{\left|\phi_t'(\phi_t^{-1}(w))\right|^2}dA(w)\\ &= \tfrac{t^{2-p}}{(1-t)^2}(\alpha+1)\int_{\phi_t(\mathbb{D})}|w|^{p-4}|f(w)|^p\left(1-\left|\frac{w-t}{(1-t)w}\right|^2\right)^{\alpha}dA(w).\end{aligned}$$

Hence,

$$\|T_tf\|_{A^{p,\alpha}}=\frac{t^{\frac{2}{p}-1}}{(1-t)^{\frac{2}{p}}}\left((\alpha+1)\int_{D_t}|w|^{p-4}|f(w)|^p\left(1-\left|\frac{w-t}{(1-t)w}\right|^2\right)^{\alpha}dA(w)\right)^{\frac{1}{p}},$$

where $D_t=\phi_t(\mathbb{D})$. It is easy to check that $D_t=D\left(\frac{1}{2-t},\frac{1-t}{2-t}\right)$, where $D\left(\frac{1}{2-t},\frac{1-t}{2-t}\right)$ is the Euclidean disc of radius $\frac{1-t}{2-t}$ centered at the point $\frac{1}{2-t}$ in the plane.

On the other hand, we have that

$$\begin{aligned}1-\left|\tfrac{w-t}{(1-t)w}\right|^2 &= \tfrac{(1-t)^2|w|^2-|w-t|^2}{(1-t)^2|w|^2}\\ &= \tfrac{2t\,\mathrm{Re}\,w-t^2-t(2-t)|w|^2}{(1-t)^2|w|^2}\\ &= \tfrac{t}{1-t}\cdot\tfrac{2\,\mathrm{Re}\,w-t-(2-t)|w|^2}{(1-t)|w|^2}.\end{aligned}$$

Therefore,

$$\|T_tf\|_{A^{p,\alpha}}=\frac{t^{\frac{\alpha+2}{p}-1}}{(1-t)^{\frac{\alpha+2}{p}}}\left((\alpha+1)\int_{D_t}|w|^{p-4}|f(w)|^p g_t(w)^{\alpha}dA(w)\right)^{\frac{1}{p}},$$

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where

$$g_t(w) = \frac{2\,\mathrm{Re}\, w - t - (2-t)|w|^2}{(1-t)|w|^2}, \text{ for } w \in D_t.$$

Using

$$\begin{aligned} g_t(w) &\leq \tfrac{2|w|-t-(2-t)|w|^2}{(1-t)|w|^2} \\ &\leq \tfrac{1+|w|^2-t-(2-t)|w|^2}{(1-t)|w|^2} \\ &= \tfrac{1-|w|^2}{|w|^2} \end{aligned}$$

and $\alpha \geq 0$, we find that

$$g_t(w)^\alpha \leq |w|^{-2\alpha} \left(1-|w|^2\right)^\alpha .$$

Hence, we get

$$\|T_t f\|_{A^{p,\alpha}} \leq \frac{t^{\frac{\alpha+2}{p}-1}}{(1-t)^{\frac{\alpha+2}{p}}} \left((\alpha+1)\int_{D_t} |w|^{p-2(\alpha+2)}|f(w)|^p \left(1-|w|^2\right)^\alpha dA(w)\right)^{\frac{1}{p}}. \qquad (4)$$

***Case* (i):** $p \geq 2(\alpha+2)$. Using (4) and $|w|^{p-2(\alpha+2)} \leq 1$, for $w \in D_t \subset \mathbb{D}$, we have that

$$\begin{aligned} \|T_t f\|_{A^{p,\alpha}} &\leq \tfrac{t^{\frac{\alpha+2}{p}-1}}{(1-t)^{\frac{\alpha+2}{p}}} \left((\alpha+1)\int_{D_t} |f(w)|^p \left(1-|w|^2\right)^\alpha dA(w)\right)^{\frac{1}{p}} \\ &\leq \tfrac{t^{\frac{\alpha+2}{p}-1}}{(1-t)^{\frac{\alpha+2}{p}}} \left((\alpha+1)\int_{\mathbb{D}} |f(w)|^p \left(1-|w|^2\right)^\alpha dA(w)\right)^{\frac{1}{p}} \\ &= \tfrac{t^{\frac{\alpha+2}{p}-1}}{(1-t)^{\frac{\alpha+2}{p}}} \|f\|_{A^{p,\alpha}}. \end{aligned}$$

By using (3), we obtain

$$\begin{aligned} \|Hf\|_{A^{p,\alpha}} &\leq \int_0^1 \frac{t^{\frac{\alpha+2}{p}-1}}{(1-t)^{\frac{\alpha+2}{p}}} dt \cdot \|f\|_{A^{p,\alpha}} \\ &= B\left(\tfrac{\alpha+2}{p}, 1-\tfrac{\alpha+2}{p}\right) \|f\|_{A^{p,\alpha}} \\ &= \Gamma\left(\tfrac{\alpha+2}{p}\right)\Gamma\left(1-\tfrac{\alpha+2}{p}\right) \|f\|_{A^{p,\alpha}} \\ &= \tfrac{\pi}{\sin\frac{(\alpha+2)\pi}{p}} \|f\|_{A^{p,\alpha}}. \end{aligned}$$

Hence, in this case, we conclude that

$$\|H\|_{A^{p,\alpha}\to A^{p,\alpha}} \leq \frac{\pi}{\sin\frac{(\alpha+2)\pi}{p}}.$$

***Case* (ii):** $2\alpha+3 \le p < 2(\alpha+2)$. We have that $|w|^{p-2(\alpha+2)} \le \frac{1}{|w|}$, for $w \in D_t \subset \mathbb{D}$. Then, by using (4), we get

$$\|T_t f\|_{A^{p,\alpha}} \le \frac{t^{\frac{\alpha+2}{p}-1}}{(1-t)^{\frac{\alpha+2}{p}}} \left((\alpha+1) \int_{D_t} \frac{1}{|w|} |f(w)|^p \left(1-|w|^2\right)^\alpha dA(w) \right)^{\frac{1}{p}}.$$

Then,

$$\begin{aligned}
\int_{D_t} \frac{1}{|w|} |f(w)|^p \left(1-|w|^2\right)^\alpha dA(w) &\le \int_{\mathbb{D}} \frac{1}{|w|} |f(w)|^p \left(1-|w|^2\right)^\alpha dA(w) \\
&= 2\int_0^1 \left(1-r^2\right)^\alpha M_p^p(r,f)dr \\
&\le 2^{\alpha+1} \int_0^1 (1-r)^\alpha M_p^p(r,f)dr \\
&= 2^{\alpha+2} \int_0^1 r(1-r^2)^\alpha M_p^p(r^2,f)dr \\
&\le 2^{\alpha+2} \int_0^1 r(1-r^2)^\alpha M_p^p(r,f)dr \\
&= 2^{\alpha+1} \int_{\mathbb{D}} |f(w)|^p \left(1-|w|^2\right)^\alpha dA(w).
\end{aligned}$$

Here, we used the fact that $M_p(\cdot, f)$ is an increasing function. We have that

$$\begin{aligned}
\|T_t f\|_{A^{p,\alpha}} &\le 2^{\frac{\alpha+1}{p}} \frac{t^{\frac{\alpha+2}{p}-1}}{(1-t)^{\frac{\alpha+2}{p}}} \left((\alpha+1) \int_{\mathbb{D}} |f(w)|^p \left(1-|w|^2\right)^\alpha dA(w) \right)^{\frac{1}{p}} \\
&= 2^{\frac{\alpha+1}{p}} \frac{t^{\frac{\alpha+2}{p}-1}}{(1-t)^{\frac{\alpha+2}{p}}} \|f\|_{A^{p,\alpha}}.
\end{aligned}$$

By using (3), we find that

$$\|Hf\|_{A^{p,\alpha}} \le 2^{\frac{\alpha+1}{p}} \frac{\pi}{\sin\frac{(\alpha+2)\pi}{p}} \|f\|_{A^{p,\alpha}}.$$

Therefore,

$$\|H\|_{A^{p,\alpha}\to A^{p,\alpha}} \le 2^{\frac{\alpha+1}{p}} \frac{\pi}{\sin\frac{(\alpha+2)\pi}{p}}.$$

***Case* (iii):** $\alpha+2 < p < 2\alpha+3$. It is easy to check that $|w| \ge \frac{t}{2-t}$, for $w \in D_t$. Then, in this case, we find that $|w|^{p-2(\alpha+2)} \le \left(\frac{2-t}{t}\right)^{2(\alpha+2)-p}$, for $w \in D_t \subset \mathbb{D}$. Now, by using

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(4), we obtain

$$\|T_t f\|_{A^{p,\alpha}} \leq \frac{t^{\frac{\alpha+2}{p}-1}}{(1-t)^{\frac{\alpha+2}{p}}} \left(\frac{2-t}{t}\right)^{\frac{2(\alpha+2)-p}{p}} \left((\alpha+1)\int_{D_t} |f(w)|^p (1-|w|^2)^\alpha dA(w)\right)^{\frac{1}{p}}$$

$$\leq \frac{(2-t)^{\frac{2(\alpha+2)}{p}-1}}{t^{\frac{\alpha+2}{p}}(1-t)^{\frac{\alpha+2}{p}}} \cdot \|f\|_{A^{p,\alpha}}.$$

On the other hand, we also have

$$\frac{(2-t)^{\frac{2(\alpha+2)}{p}-1}}{t^{\frac{\alpha+2}{p}}(1-t)^{\frac{\alpha+2}{p}}} = \frac{(t+2(1-t))^{\frac{2(\alpha+2)}{p}-1}}{t^{\frac{\alpha+2}{p}}(1-t)^{\frac{\alpha+2}{p}}}$$

$$\leq \frac{t^{\frac{2(\alpha+2)}{p}-1}+2^{\frac{2(\alpha+2)}{p}-1}(1-t)^{\frac{2(\alpha+2)}{p}-1}}{t^{\frac{\alpha+2}{p}}(1-t)^{\frac{\alpha+2}{p}}}$$

$$= \frac{t^{\frac{\alpha+2}{p}-1}}{(1-t)^{\frac{\alpha+2}{p}}} + 2^{\frac{2(\alpha+2)}{p}-1}\frac{(1-t)^{\frac{\alpha+2}{p}-1}}{t^{\frac{\alpha+2}{p}}}.$$

Here, we used the fact that $(x+y)^\beta \leq x^\beta + y^\beta$, if $x, y \geq 0$ and $\beta \in (0, 1)$. Therefore,

$$\|T_t f\|_{A^{p,\alpha}} \leq \left[\frac{t^{\frac{\alpha+2}{p}-1}}{(1-t)^{\frac{\alpha+2}{p}}} + 2^{\frac{2(\alpha+2)}{p}-1}\frac{(1-t)^{\frac{\alpha+2}{p}-1}}{t^{\frac{\alpha+2}{p}}}\right]\|f\|_{A^{p,\alpha}},$$

and by using (3), we find

$$\|Hf\|_{A^{p,\alpha}} \leq \left(1+2^{\frac{2(\alpha+2)}{p}-1}\right)\frac{\pi}{\sin\frac{(\alpha+2)\pi}{p}}\|f\|_{A^{p,\alpha}},$$

because,

$$\int_0^1 \frac{t^{\frac{\alpha+2}{p}-1}}{(1-t)^{\frac{\alpha+2}{p}}}dt = \int_0^1 \frac{(1-t)^{\frac{\alpha+2}{p}-1}}{t^{\frac{\alpha+2}{p}}}dt = \frac{\pi}{\sin\frac{(\alpha+2)\pi}{p}}.$$

Hence, we conclude that

$$\|H\|_{A^{p,\alpha}\to A^{p,\alpha}} \leq \left(1+2^{\frac{2(\alpha+2)}{p}-1}\right)\frac{\pi}{\sin\frac{(\alpha+2)\pi}{p}}.$$

This finishes the proof.

3.2. Upper bound for the norm $\|H\|_{A^p\to A^p}$ when $2 < p < 3$. It follows from the Theorem 1.2, for $\alpha = 0$, that if $3 \leq p < 4$, then $\|H\|_{A^p\to A^p} \leq \sqrt[3]{2}\frac{\pi}{\sin\frac{2\pi}{p}}$, and if $2 < p < 3$, then $\|H\|_{A^p\to A^p} \leq 3\frac{\pi}{\sin\frac{2\pi}{p}}$. These two estimates are better then those given in [**3**]. In the following proposition, we show that, if $2 < p < 3$, then $\|H\|_{A^p\to A^p} \leq (1+2^{\frac{1}{p}})\frac{\pi}{\sin\frac{2\pi}{p}}$. Therefore, if $2 < p < 3$, then we have that $\|H\|_{A^p\to A^p} \leq (1+\sqrt{2})\frac{\pi}{\sin\frac{2\pi}{p}}$.

PROPOSITION 3.1. Let $2 < p < 3$. Then $\|H\|_{A^p\to A^p} \leq (1+2^{\frac{1}{p}})\frac{\pi}{\sin\frac{2\pi}{p}}$.

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Proof. It follows from the Theorem 1.2, for $\alpha = 0$, that if $f \in A^p$, then

$$\|T_t f\|_{A^p} = \frac{t^{\frac{2}{p}-1}}{(1-t)^{\frac{2}{p}}} \left(\int_{D_t} |w|^{p-4} |f(w)|^p dA(w) \right)^{\frac{1}{p}}.$$

We have that $|w|^{p-4} \le \frac{1}{|w|^2}$, for $w \in D_t$ and $D_t \subset E_t \subset \mathbb{D}$, where $E_t = \{w \in \mathbb{C} : \frac{t}{2-t} < |w| < 1\}$. Hence, we obtain

$$\|T_t f\|_{A^p} \le \frac{t^{\frac{2}{p}-1}}{(1-t)^{\frac{2}{p}}} \left(\int_{E_t} \frac{1}{|w|^2} |f(w)|^p dA(w) \right)^{\frac{1}{p}}.$$

On the other hand, we find that

$$\int_{E_t} \frac{1}{|w|^2} |f(w)|^p dA(w) = 2 \int_{\frac{t}{2-t}}^{1} \frac{1}{r^2} \cdot r M_p^p(r, f) dr.$$

Since function $r \mapsto \frac{1}{r^2}$ is decreasing and function $r \mapsto r M_p^p(r, f)$ is increasing, by using Chebyshev's inequality, we get

$$\begin{aligned} \int_{E_t} \frac{1}{|w|^2} |f(w)|^p dA(w) &\le \frac{2}{1-\frac{t}{2-t}} \int_{\frac{t}{2-t}}^{1} \frac{1}{r^2} dr \int_{\frac{t}{2-t}}^{1} r M_p^p(r, f) dr \\ &= \frac{2-t}{t} \cdot 2 \int_{\frac{t}{2-t}}^{1} r M_p^p(r, f) dr \\ &\le \frac{2-t}{t} \cdot 2 \int_0^1 r M_p^p(r, f) dr \\ &= \frac{2-t}{t} \|f\|_{A^p}^p. \end{aligned}$$

Therefore, we have that

$$\begin{aligned} \|T_t f\|_{A^p} &\le \frac{t^{\frac{2}{p}-1}}{(1-t)^{\frac{2}{p}}} \cdot \frac{(2-t)^{\frac{1}{p}}}{t^{\frac{1}{p}}} \|f\|_{A^p} \\ &= \frac{t^{\frac{2}{p}-1}}{(1-t)^{\frac{2}{p}}} \left(1 + 2\frac{1-t}{t}\right)^{\frac{1}{p}} \|f\|_{A^p} \\ &\le \frac{t^{\frac{2}{p}-1}}{(1-t)^{\frac{2}{p}}} \left(1 + 2^{\frac{1}{p}} \frac{(1-t)^{\frac{1}{p}}}{t^{\frac{1}{p}}}\right) \|f\|_{A^p} \\ &= \left(\frac{t^{\frac{2}{p}-1}}{(1-t)^{\frac{2}{p}}} + 2^{\frac{1}{p}} \frac{t^{\frac{1}{p}-1}}{(1-t)^{\frac{1}{p}}} \right) \|f\|_{A^p}. \end{aligned}$$

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Then, by using (3), we obtain

$$\|Hf\|_{A^p} \leq \left(\frac{\pi}{\sin\frac{2\pi}{p}} + 2^{\frac{1}{p}}\frac{\pi}{\sin\frac{\pi}{p}}\right)\|f\|_{A^p}.$$

We have $\sin\frac{2\pi}{p} = 2\sin\frac{\pi}{p}\cos\frac{\pi}{p} \leq \sin\frac{\pi}{p}$, because $2 < p < 3$. Hence, we find that $\frac{\pi}{\sin\frac{2\pi}{p}} \geq \frac{\pi}{\sin\frac{\pi}{p}}$. Now, we get

$$\|Hf\|_{A^p} \leq \left(1 + 2^{\frac{1}{p}}\right)\frac{\pi}{\sin\frac{2\pi}{p}}\|f\|_{A^p},$$

and finally,

$$\|H\|_{A^p\to A^p} \leq \left(1 + 2^{\frac{1}{p}}\right)\frac{\pi}{\sin\frac{2\pi}{p}},$$

which is what we wanted to prove. □

ACKNOWLEDGEMENTS. The author is grateful to his advisor Professor Miroljub Jevtić who pointed out the method which is given in the proof of Theorem 1.1. Boban Karapetrović is supported by NTR Serbia, Project ON174032.

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