A STIELTJES-VOLTERRA INTEGRAL EQUATION THEORY

D. B. HINTON

Suppose $S = [a, b]$ is a number interval and F is a function from $S \times S$ to a normed algebraic ring N with multiplicative identity I. We consider the problem of finding, for appropriate conditions on F, a function M from $S \times S$ to N such that for all t and x,

$$M(t, x) = I + (\mathrm{L}) \int_x^t d_s\, F(t, s) . M(s, x)$$

where the integral is a Cauchy-left integral.

In §2, a class $\mathfrak{F}$ of functions F is defined in which the above homogeneous equation is uniquely solvable. This gives rise to a mapping $\mathfrak{E}$ on $\mathfrak{F}$ defined by $\mathfrak{E}(F) = M$ where M is the unique solution of the above integral equation. The mapping $\mathfrak{E}$ has the property that $\mathfrak{E}(\mathfrak{E}(F)) = F$, which implies that it is one–one and onto. The non-homogeneous equations

$$Y(t) = G(t) + (\mathrm{L}) \int_a^t d_s\, F(t, s) . Y(s)$$

and

$$Z(t) = G(t) + (\mathrm{L}) \int_a^t Z(s) . d_s\, F(s, t)$$

have solutions that are representable in terms of M and G. Moreover, the solution M of the homogeneous equation has representations in terms of solutions of non-homogeneous equations.

This paper is a continuation of an integral equation theory studied by H. S. Wall **(10)** and developed extensively by J. S. MacNerney **(5–8)**. Wall's theory has been expanded by T. H. Hildebrandt **(3)**, where a Lebesgue–Stieltjes integral is used. A non-linear version of Wall's work has been developed by J. W. Neuberger **(9)**. The principal distinction between the equations here and the other linear theories is that in the above references the function F is a one-place function, i.e., a function M on $S \times S$ is sought such that

$$M(t, x) = I + \int_x^t dF(s) . M(s, x).$$

The connection with the previous work is primarily with the mapping developed by MacNerney in **(7)**. For the application of MacNerney's theory

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to number intervals, the class $\mathfrak{F}$ contains properly the multiplicative functions $\mathfrak{OM}$ and additive functions $\mathfrak{OA}$ (under a trivial embedding) used in **(7)**. An approximation theorem proved in §5 gives as corollaries the existence of a continued sum for members of $\mathfrak{OM}$ and a continued product for members of $\mathfrak{OA}$. This, coupled with properties of $\mathfrak{C}$, gives as a further corollary parts (i) and (ii) of Theorem 3.3 and Theorem 4.3 of **(7)**.

1. Cauchy left and right integrals. Hereafter let a and b denote numbers,

$$c = \min\{a, b\}, \qquad d = \max\{a, b\}, \qquad S = [c, d],$$

and let N be a non-degenerate ring, with additive identity element denoted by 0 and multiplicative identity element denoted by I. Suppose $||\cdot||$ is a norm for N, i.e., a function from N to the non-negative numbers such that for all x and y in N,

(i) $||x|| = 0$ if and only if $x = 0$,

(ii) $||x + y|| \leqslant ||x|| + ||y||$, and

(iii) $||xy|| \leqslant ||x||\,||y||$,

and suppose that N is complete with respect to this norm. The norm of I is then $\geqslant 1$ since $0 < ||I|| = ||I^2|| \leqslant ||I||^2$. A function F from the number interval S to N is quasi-continuous if it has left and right limits at each interior point of S and one-sided limits at each end point of S; and F is of bounded variation on S if there is a number K such that

$$\sum_{i=1}^{n} ||F(s_i) - F(s_{i-1})|| \leqslant K$$

for all sequences $\{s_i\}^n$ increasing from c to d. The least such number K is denoted by $V[F, a, b]$ or $V[F, b, a]$. A function of bounded variation is also quasi-continuous.

A number sequence is monotone if it is either non-decreasing or non-increasing, and a chain $\{s_i\}_0^n$ from the number x to the number y is a monotone sequence such that $s_0 = x$ and $s_n = y$. The chain $\{t_i\}_0^m$ is a refinement of the chain $\{s_i\}_0^n$ if $\{s_i\}_0^n$ is a subsequence of $\{t_i\}_0^m$.

If both F and G are functions from S to N, then

(i) the Cauchy-left integral

$$(\mathrm{L}) \int_a^b dF(x).G(x)$$

denotes an element Y of N with the following property: for each positive number ϵ, there is a chain $s = \{s_i\}_0^n$ from a to b such that if $t = \{t_i\}_0^m$ is a refinement of s, then

$$||Y - (\mathrm{L}) \textstyle\sum_t dF.G|| < \epsilon$$

where $(\mathrm{L})\sum_t dF\cdot G$ denotes the sum

$$\sum_{i=1}^{m} [F(t_i) - F(t_{i-1})]G(t_{i-1}).$$

(ii) the Cauchy-right integral

$$(\mathrm{R}) \int_a^b dF(x).G(x)$$

denotes an element Y of N with the following property: for each positive number ϵ, there is a chain $s = \{s_i\}_0^n$ from a to b such that if $t = \{t_i\}_0^m$ is a refinement of s, then

$$||Y - (\mathrm{R})\textstyle\sum_t dF\cdot G|| < \epsilon$$

where $(\mathrm{R})\sum_t dF.G$ denotes the sum

$$\sum_{i=1}^{m} [F(t_i) - F(t_{i-1})]G(t_i).$$

The Cauchy-left and right integrals

$$(\mathrm{L}) \int_a^b G(x).dF(x) \quad \text{and} \quad (\mathrm{R}) \int_a^b G(x).dF(x)$$

are defined similarly. For each integral, classical arguments show that if the integral exists it is unique.

The Cauchy-left and -right integrals are simply related by

$$(\mathrm{L}) \int_a^b dF(x).G(x) = -(\mathrm{R}) \int_b^a dF(x).G(x)$$

and

$$(\mathrm{L}) \int_a^b G(x).dF(x) = -(\mathrm{R}) \int_b^a G(x).dF(x).$$

If F is a bounded function from a set K to N, then $|F|_K$ denotes the number $\sup\{||F(X)||:x \in K\}$. In particular, a quasi-continuous function from a number interval to N is bounded. If F is a function of bounded variation from S to N and G is a quasi-continuous function from S to N, then $s = \{s_i\}_0^{2n}$ is called an $(F - G - \epsilon)$-chain from a to b if ϵ is a positive number and s is a chain from a to b such that for $i = 0, \ldots, n - 1$,

(i) $||G(x) - G(y)|| \leqslant \epsilon$ if x and y are between s_{2i+1} and s_{2i+2} or equal to s_{2i+1}, and

(ii) $V[F, x, s_{2i+1}] \leqslant \epsilon/2n$ if x is between s_{2i} and s_{2i+1}.

The existence of such chains is easily established by using the following theorem about quasi-continuous functions. The theorem is a consequence of (**4**, Lemma 4.1b).

THEOREM 1.0. *If F is a quasi-continuous function from S to N and ϵ is a positive number, then there is a chain $s = \{s_i\}_0^n$ from a to b such that*

$$||F(x) - F(y)|| \leqslant \epsilon$$

if x and y are between adjacent terms of s.

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To construct an $(F - G - \epsilon)$-chain, first choose a reversible sequence $s_0, s_2, \ldots, s_{2n}$ from a to b so that if x and y are between s_{2i} and s_{2i+2} for some i, then $\|G(x) - G(y)\| \leqslant \epsilon$. Then choose s_{2i+1} between s_{2i} and s_{2i+2} so that condition (ii) will hold. The Cauchy integrals defined above are known to exist if F is of bounded variation and G is quasi-continuous **(2)**. The following approximation theorem will be of use, however.

THEOREM 1.1. *If F is of bounded variation from S to N, G is quasi-continuous from S to N, $s = \{s_i\}_0^{2n}$ is an $(F - G - \epsilon)$-chain from a to b, and $t = \{t_i\}_0^m$ is a refinement of s, then*

$$\|(L)\textstyle\sum_t dF.G - (\mathrm{L})\sum_s dF.G\| \leqslant \epsilon(|G|_S + V[F, a, b]).$$

Remark. It follows that, under the assumptions of Theorem 1.1,

$$\left\|(\mathrm{L}) \int_a^b dF(x).G(x) - (\mathrm{L})\textstyle\sum_s dF.G\right\| \leqslant \epsilon(|G|_S + V[F, a, b]).$$

Since for a chain s from a to b,

$$(\mathrm{L})\textstyle\sum_s dF.G + (\mathrm{R})\sum_s F.dG = F(b)G(b) - F(a)G(a),$$

we have the equality

$$\|(\mathrm{L})\textstyle\sum_t dF.G - (\mathrm{L})\sum_s dF.G\| = \|(\mathrm{R})\sum_t F.dG - (\mathrm{R})\sum_s F.dG\|,$$

and hence an approximation theorem for right integrals.

In addition to the linearity and additive properties of the Cauchy integrals, a few other properties are needed and these are stated below without proof. The use of an integral symbol in the conclusion of a theorem implies the existence of the integral.

THEOREM 1.2. *If F and G are functions from S to N and*

$$(\mathrm{L}) \int_a^b dF(s).G(s) \quad \textit{or} \quad (\mathrm{R}) \int_a^b F(s).dG(s)$$

exists, then

$$(\mathrm{L}) \int_a^b dF(s).G(s) + (\mathrm{R}) \int_a^b F(s).dG(s) = F(b)G(b) - F(a)G(a).$$

THEOREM 1.3. *If F is of bounded variation from S to N, G is quasi-continuous from S to N, f and g are real functions on S such that for $x \leqslant y$,*

$$V[F, x, y] \leqslant f(y) - f(x)$$

and $\|G(x)\| = g(x)$, then

$$\left\|(\mathrm{L}) \int_a^b dF(s).G(s)\right\| \leqslant \left|(\mathrm{L}) \int_a^b df(s)g(s)\right| \leqslant V[f, a, b]|g|_S.$$

The following theorem is a generalization of a well-known inequality in differential equations **(1**, p. 107**)**: it is useful in deriving properties of solutions of the integral equations studied here.

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THEOREM 1.4. *If K is a non-negative number, h is a real function on S non-decreasing in the order from a to b, and m is a real function on S bounded above by the positive number T and such that for each x,*

$$m(x) \leqslant K + (\mathrm{L}) \int_a^x m(s)dh(s), \tag{1.1}$$

then $m(x) \leqslant K \exp[h(x) - h(a)]$ for each x.

Proof. It is sufficient to suppose that $h(a) = 0$. If $a_0, \ldots, a_n$ is a non-decreasing sequence of non-negative numbers and p is a non-negative integer, then

$$\sum_{i=1}^{n} a_{i-1}^p(a_i - a_{i-1}) \leqslant \frac{1}{p+1}\sum_{i=1}^{n} (a_i^p + a_i^{p-1}a_{i-1} + \ldots + a_{i-1}^p)(a_i - a_{i-1})$$
$$= \frac{1}{p+1}\sum_{i=1}^{n} (a_i^{p+1} - a_{i-1}^{p+1}) = \frac{1}{p+1}(a_n^{p+1} - a_0^{p+1}).$$

Hence if x is in S and $s = \{s_i\}_0^n$ is a chain from a to x,

$$(\mathrm{L})\sum_s h^p dh = \sum_{i=1}^{n} h(s_{i-1})^p[h(s_i) - h(s_{i-1})] \leqslant \frac{1}{p+1} h(s_n)^{p+1},$$

from which we conclude that

$$(\mathrm{L}) \int_a^x h(s)^p dh(s) \leqslant \frac{1}{p+1} h(x)^{p+1}. \tag{1.2}$$

By replacing $m(s)$ by T in (1.1), it follows that

$$m(s) \leqslant K + Th(x).$$

Assuming that for some positive integer n,

$$m(x) \leqslant K + Kh(x) + \ldots + [K/(n-1)!]h(x)^{n-1} + [T/n!]h(x)^n,$$

then substitution of the above inequality into the right side of (1.1) for $s = x$ and applying (1.2) yields

$$m(x) \leqslant K + Kh(x) + \ldots + [K/n!]h(x)^n + [T/(n+1)!]h(x)^{n+1}.$$

Hence, $m(x) \leqslant K \exp[h(x)]$.

2. The kernel functions. Let $\mathfrak{F}$ be the set of all functions F from $S \times S$ to the complete normed ring N such that (i) $F(x, x) = I$ for all x, (ii) F is quasi-continuous with respect to its first place, and (iii) there is a real non-decreasing function g on S such that $g(c) = 0$ and

$$||F(t, x) - F(t, y)|| \leqslant |g(x) - g(y)| \qquad \text{for all } t, x, \text{ and } y.$$

Such a function g is called a super function for F.

For F in $\mathfrak{F}$ let G_F be the collection of all super functions for F and let g_F be defined on S by:

$$g_F(x) = \inf\{g(x) : g \in G_F\}.$$

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THEOREM 2.1. *If $F \in \mathfrak{F}$, then g_F is a super function for F.*

Proof. For $x \leqslant y$ and g in G_F, $g_F(x) \leqslant g(x) \leqslant g(y)$. Hence

$$g_F(x) \leqslant \inf\{g(y) : g \in G_F\} = g_F(y)$$

and g_F is non-decreasing. For $x \leqslant y$, $g \in G_F$, and $t \in S$,

$$||F(t, y) - F(t, x)|| \leqslant g(y) - g(x) \leqslant g(y) - g_F(x)$$

since $g_F(x) \leqslant g(x)$. Thus

$$||F(t, y) - F(t, x)|| \leqslant \inf\{g(y) : g \in G_F\} - g_F(x) = g_F(y) - g_F(x).$$

THEOREM 2.2. *If $F \in \mathfrak{F}$, then F is bounded.*

Proof. Since a one-place quasi-continuous function on S is bounded, there is a number L such that $||F(t, c)|| \leqslant L$ for all t. Hence

$$||F(t, x)|| \leqslant ||F(t, x) - F(t, c)|| + ||F(t, c)|| \leqslant g_F(d) + L.$$

THEOREM 2.3. (i) *If F in $\mathfrak{F}$ is continuous with respect to its first place and has a continuous super function g, then F is continuous.* (ii) *If F is continuous, then g_F is continuous.*

Indication of proof. (i) The continuity of F at (t, x) follows from the inequality

$$\begin{aligned} ||F(r, s) - F(t, x)|| &\leqslant ||F(r, s) - F(r, x)|| + ||F(r, x) - F(t, x)|| \\ &\leqslant |g(s) - g(x)| + ||F(r, x) - F(t, x)||. \end{aligned}$$

(ii) If g_F has a discontinuity at x, then the function h defined below is a super function for F.

$$h(y) = \begin{cases} g_F(y) & \text{if } y < x, \\ h \text{ is defined by continuity at } x, & \\ g_F(y) - [g_F(x+) - g_F(x-)] & \text{if } x < y. \end{cases}$$

But then $h(d) < g_F(d)$, which contradicts the minimality of g_F.

THEOREM 2.4. *If $F \in \mathfrak{F}$, Q is quasi-continuous from S to N,* X = L *or* X = R, *and P is defined on S by:*

$$P(t) = (\mathrm{X}) \int_a^t d_s\, F(t, s) . Q(s),$$

then P is quasi-continuous. Moreover, if F is continuous with respect to its first place, then P is continuous.

Indication of proof. To prove P has a left limit at the number t, let $\{s_i\}_0^{2n}$ be a $(g_F - Q - \epsilon)$-chain from a to t such that $s_{2n-1} \neq s_{2n}$. For r and y between s_{2n-1} and s_{2n}, define the chains u and v by $u_i = v_i = s_i$ if $0 \leqslant i \leqslant sn - 1$ and

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$u_{2n} = r$ and $v_{2n} = y$. Then both u and v are $(g_F - Q - \epsilon)$-chains; by Theorem 1.1, the sums

$$\sum_{i=1}^{2n} [F(r, u_i) - F(r, u_{i-1})]Q(u_{i-1})$$
$$= \sum_{i=1}^{2n-1} [F(r, s_i) - F(r, s_{i-1})]Q(s_{i-1}) + [I - F(r, s_{2n-1})]Q(s_{2n-1})$$

and

$$\sum_{i=1}^{2n} [F(y, v_i) - F(y, v_{i-1})]Q(v_{i-1})$$
$$= \sum_{i=1}^{2n-1} [F(y, s_i) - F(y, s_{i-1})]Q(s_{i-1}) + [I - F(y, s_{2n-1})]Q(s_{2n-1})$$

are approximations to the integrals

$$(\mathrm{L}) \int_a^r d_s F(r, s) . Q(s) \quad \text{and} \quad (\mathrm{L}) \int_a^y d_s F(y, s) . Q(s)$$

respectively. Since F is quasi-continuous in its first place, each of the differences $||F(r, s_i) - F(y, s_i)||$ can be made as small as desired by choosing r and y sufficiently close to t. Hence, P has a left limit at the number t. The other cases are proved similarly.

THEOREM 2.5. *If $F \in \mathfrak{F}$, Q is of bounded variation from S to N,* $\mathrm{X} = \mathrm{L}$ *or* $\mathrm{X} = \mathrm{R}$, *and P is defined on S by:*

$$P(t) = (\mathrm{X}) \int_a^t dQ(s) . F(s, t),$$

then P is of bounded variation.

Proof. If $a \leqslant x \leqslant y \leqslant b$ or $b \leqslant y \leqslant x \leqslant a$, then

$$P(y) - P(x) = (\mathrm{L}) \int_a^x dQ(s) . [F(s, y) - F(s, x)] + (\mathrm{L}) \int_x^y dQ(s) . F(s, y).$$

Hence, by Theorem 1.3,

$$||(P(y) - P(x)|| \leqslant V[Q, a, x]V[g_F, x, y] + V[Q, x, y]|F|_{S\times S} \leqslant KV[L, x, y]$$

where $K = V[Q, a, b] + |F|_{S\times S}$ and $L(S) = V[Q, a, s] + V[g_F, a, s]$. Hence P is of bounded variation. The proof for $\mathrm{X} = \mathrm{R}$ is similar.

THEOREM 2.6. *If $P \in \mathfrak{F}$, Q is of bounded variation from S to N, and K is quasi-continuous from S to N, then*

$$(2.0) \quad (\mathrm{L}) \int_a^b dQ(s) . (R) \int_a^s P(s, v) . dK(v)$$
$$= (\mathrm{R}) \int_a^b \left\{ (\mathrm{L}) \int_v^b dQ(s) . P(s, v) \right\} . dK(v).$$

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Proof. First suppose that $a < b$ and e and f are numbers such that $a \leqslant e < f \leqslant b$, K has value T on (e, f), and K has value zero elsewhere.

Let

$$D(v) = -(\mathrm{R}) \int_b^v dQ(s).P(s,v)$$

and

$$\begin{aligned} E(s) &= (\mathrm{R}) \int_a^s P(s,v).dK(v) \\ &= K(s) - P(s,a)K(a) - (\mathrm{L}) \int_a^s d_s\, P(s,v)K(v). \end{aligned}$$

Then by Theorem 2.4 E is quasi-continuous and by Theorem 2.5 D is of bounded variation. A calculation of $E(s)$ gives 0 if $s \leqslant e$, $P(s, e+)T$ if $e < s < f$ and $P(s, e+)T - P(s, f)T$ if $f \leqslant s$. An additional calculation gives

$$(\mathrm{R}) \int_a^b D(v).dK(v) = D(e+)T - D(f)T.$$

Hence,

$$\begin{aligned} &\left\| (\mathrm{L}) \int_a^b dQ(s).E(s) - (\mathrm{R}) \int_a^b D(v).dK(v) \right\| \\ &= \Big\| (\mathrm{L}) \int_e^{e+\delta} dQ(s).E(s) + (\mathrm{L}) \int_{e+\delta}^b dQ(s).[P(s,e+) - P(s,e+\delta)]T \\ &\qquad\qquad - [D(e+) - D(e+\delta)]T \Big\| \end{aligned}$$

for every $\delta > 0$. Considering the approximating sums, we obtain

$$\left\| (\mathrm{L}) \int_e^{e+\delta} dQ(s).E(s) \right\| \leqslant V[Q, e+, e+\delta]|E|_S.$$

By Theorem 1.3, we have

$$\begin{aligned} &\left\| (\mathrm{L}) \int_{e+\delta}^b dQ(s).[P(s,e+) - P(s,e+\delta)]T \right\| \\ &\qquad\qquad \leqslant V[Q,a,b][g_P(e+\delta) - g_P(e+)]||T||. \end{aligned}$$

Thus equation (2.0) holds for K. The proof is similar for $b < a$ and K constant on a segment of $[b, a]$ with value zero elsewhere.

The proof that equation (2.0) holds for a function K that has value zero on S except possibly at one point is similar to the proof above. Since a step function is a finite sum of functions of the above two types, and the integrals of (2.0) are additive with respect to K, we have equation (2.0) for step functions.

An application of Theorem 1.0 proves that if K is quasi-continuous, then it is the uniform limit of a sequence K_1, K_2, ... of step functions. After an integration by parts (i.e., Theorem 1.2), equation (2.0) is seen to be equivalent to the equation

$$(2.1)\quad (\mathrm{L})\int_a^b dQ(s)\,.\,(\mathrm{L})\int_a^s d_v P(s,v)\,.K(v)$$
$$= (\mathrm{L})\int_a^b dQ(s)\,.K(s) + (\mathrm{L})\int_a^b d_v\left\{(\mathrm{L})\int_v^b dQ(s)\,.P(s,v)\right\}.K(v).$$

Application of Theorem 1.3 yields

$$\left\|(\mathrm{L})\int_a^b dQ(s)\,.\,(\mathrm{L})\int_a^s d_v P(s,v)\,.\,[K(u) - K_i(u)]\right\|$$
$$\leqslant V[Q,a,b]V[g_P,a,b]|K - K_i|_S,$$

$$\left\|(\mathrm{L})\int_a^b dQ(s)\,.\,[K(s) - K_i(s)]\right\| \leqslant V[Q,a,b]|K - K_i|_S,$$

and

$$\left\|(\mathrm{L})\int_a^b d_v\left\{(\mathrm{L})\int_v^b dQ(s)\,.P(s,v)\right\}.\,[K(v) - K_i(v)]\right\| \leqslant V[D,a,b]|K - K_i|_S.$$

Hence the uniform convergence of the sequence $K_1, K_2, \ldots$ implies that (2.1) and hence (2.0) holds for K.

3. The mapping $\mathfrak{E}$. Let $F \in \mathfrak{F}$. In this section we construct a function M from $S \times S$ to N such that

$$(3.1)\qquad M(t,x) = I + (\mathrm{L})\int_x^t d_s F(t,s)\,.\,M(s,x) \qquad \text{for all } t \text{ and } x.$$

THEOREM 3.1. *Given $F \in \mathfrak{F}$, there is one and only one function M that is quasi-continuous with respect to its first place and that is a solution of* (3.1). *Moreover, if F is continuous with respect to its first place, then so is M.*

COROLLARY 3.1. *If M is the solution of* (3.1) *which is quasi-continuous in its first place, then*

$$||M(t,x)|| \leqslant ||I|| \exp |g_F(t) - g_F(x)|.$$

Proof of uniqueness. Suppose M_1 and M_2 are solutions of (3.1) and are quasi-continuous in their first places. Let $P = M_1 - M_2$ and $p(t,x) = ||P(t,x)||$. Then

$$P(t,x) = (\mathrm{L})\int_x^t d_s F(t,s)\,.P(s,x)$$

and by Theorem 1.3 (for $x \leqslant t$),

$$p(t,x) \leqslant 0 + (\mathrm{L})\int_x^t dg_F(s)\,.p(s,x);$$

thus $p(t,x) \leqslant 0$ by Theorem 1.4. Similarly, $p(t,x) \leqslant 0$ for $t \leqslant x$. Thus $M_1 = M_2$.

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Proof of existence. By Theorem 2.4, there is a sequence of functions M_0, $M_1, \ldots$ such that each M_i is quasi-continuous in its first place, and for all t and x in S and $n = 0, 1, \ldots$, $M_0(t, x) = I$ and

$$M_{n+1}(t, x) = I + (\mathrm{L}) \int_x^t d_s\, F(t, s).M_n(s, x).$$

By the definition of M_0 and M_1,

$$||M_1(t, x) - M_0(t, x)|| = ||F(t, t) - F(t, x)|| \leqslant |g_F(t) - g_F(x)|.$$

If n is a positive integer and

$$||M_n(t, x) - M_{n-1}(t, x)|| \leqslant [1/n!]|g_F(t) - g_F(x)|^n,$$

then by Theorem 1.3 and inequality (1.2) respectively (for $x \leqslant t$),

$$\begin{aligned} ||M_{n+1}(t, x) - M_n(t, x)|| &= \left\| (\mathrm{L}) \int_x^t d_s\, F(t, s)[M_n(s, x) - M_{n-1}(s, x)] \right\| \\ &\leqslant [1/n!](\mathrm{L}) \int_x^t dg_F(s)|g_F(s) - g_F(x)|^n \\ &\leqslant [1/(n+1)!]|g_F(t) - g_F(x)|^{n+1}. \end{aligned}$$

Then, for $p > n$,

$$\begin{aligned} (3.2)\quad ||M_p(t, x) - M_n(t, x)|| &\leqslant \sum_{i=n+1}^{p} ||M_i(t, x) - M_{i-1}(t, x)|| \\ &\leqslant \sum_{i=n+1}^{p} [1/i!]|g_F(t) - g_F(x)|^i \\ &\leqslant \sum_{i=n+1}^{p} g_F(d)/i!. \end{aligned}$$

Similarly, the above inequalities follow for $t < x$. Inequality (3.2) ensures that the sequence M_0, $M_1, \ldots$ converges uniformly on $S \times S$ to a function M since N is a complete metric space. Since each M_i is quasi-continuous with respect to its first place and the convergence is uniform, M must be quasi-continuous with respect to its first place. The uniform convergence implies that M is a solution of equation (3.1). If F is continuous with respect to its first place, then by the second part of Theorem 2.4, M is continuous in its first place.

From equation (3.2),

$$\begin{aligned} ||M(t, x)|| &\leqslant ||M_0(t, x)|| + ||M(t, x) - M_0(t, x)|| \\ &\leqslant ||I|| + \sum_{i=1}^{\infty} [1/i!]|g_F(t) - g_F(x)|^i \\ &\leqslant ||I|| \exp |g_F(t) - g_F(x)| \end{aligned}$$

since $1 \leqslant ||I||$. The corollary implies that M is bounded.

THEOREM 3.2. *If $F \in \mathfrak{F}$ and M is the solution of* (3.1) *which is quasi-continuous in its first place, then $M \in \mathfrak{F}$. Moreover, if F is continuous, then M is continuous.*

Proof. To prove M has a super function, let t, x, and y be in S, $x \leqslant y \leqslant t$. Define m on S by:

$$m(v) = ||M(v, y) - M(v, x)||.$$

From equation (3.1), we obtain for $y \leqslant v$

$$M(v, y) - M(v, x) = (\mathrm{L}) \int_y^v d_s F(v, s).[M(s, y) - M(s, x)] - (\mathrm{L}) \int_x^y d_s F(v, s).M(s, x).$$

Hence, by Theorem 1.3,

$$m(v) \leqslant L[g_F(y) - g_F(x)] + (\mathrm{L}) \int_y^v dg_F(s) m(s)$$

where L is a bound for M. Let $K = L \exp[g_F(d)]$. Then applying Theorem 1.4 to the above inequality we obtain $m(v) \leqslant K[g_F(y) - g_F(x)]$ for $y \leqslant v$ and thus for $v = t$. The same result follows if $t \leqslant x$. For $x < t < y$, the result follows by applying the first two cases in conjunction with the triangle inequality. Thus Kg_F is a super function for M.

If F is continuous, Theorem 2.3 implies that g_F is continuous. Hence by Theorem 2.3 and the second part of Theorem 3.1, M is continuous.

If $F \in \mathfrak{F}$, then $\mathfrak{E}(M)$ is defined as the unique member M of $\mathfrak{F}$ which is a solution of equation (3.1). Let I^* denote the member of $\mathfrak{F}$ which has value I everywhere. Then $\mathfrak{E}(I^*) = I^*$.

THEOREM 3.3. *If $\mathfrak{E}(F) = M$, then $\mathfrak{E}(M) = F$.*

Proof. Define Z on $S \times S$ by:

$$Z(t, x) = I - (\mathrm{R}) \int_x^t M(t, s).d_s F(s, x). \tag{3.3}$$

Integrating the right side of (3.3) by parts, we obtain by Theorem 2.4 that Z is quasi-continuous with respect to its first place. By Theorem 2.6,

$$(\mathrm{L}) \int_x^t d_s F(t, s).Z(s, x) = [I - F(t, x)] - (\mathrm{R}) \int_x^t \left\{ (\mathrm{L}) \int_v^t d_s F(t, s).M(s, v) \right\}.d_v F(v, x).$$

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Since M is the solution in $\mathfrak{F}$ of equation (3.1), the above reduces to

$$(L) \int_x^t d_s F(t, s).Z(s, x) = [I - F(t, x)] - (R) \int_x^t [M(t, v) - I].d_v F(v, x)$$

$$= - (R) \int_x^t M(t, v).d_v F(v, x) = Z(t, x) - I.$$

Thus Z is a solution of equation (3.1), and hence $Z = M$. We now integrate the right side of (3.3) by parts obtaining

$$F(t, x) = I + (L) \int_x^t d_s M(t, s).F(s, x)$$

or $F = \mathfrak{G}(M)$.

4. The non-homogeneous equations. Two non-homogeneous equations are solved below. The proofs are similar and only the second is given.

THEOREM 4.1. *If $M = \mathfrak{G}(F)$ and G is a quasi-continuous function from S to N, then there is a unique quasi-continuous function Y on S such that*

$$Y(t) = G(t) + (L) \int_a^t d_s F(t, s).Y(s). \tag{4.1}$$

Moreover,

$$Y(t) = G(t) - (L) \int_a^t d_s M(t, s).G(s).$$

THEOREM 4.2. *If $M = \mathfrak{G}(F)$ and G is a function of bounded variation from S to N, then there is a unique function Z of bounded variation on S such that*

$$Z(t) = G(t) + (L) \int_a^t Z(s).d_s F(s, t). \tag{4.2}$$

Moreover,

$$Z(t) = G(t) - (L) \int_a^t G(s).d_s M(s, t).$$

Proof. If both Z_1 and Z_2 are solutions of bounded variation of the above equation and $P = Z_1 - Z_2$, then $P(a) = 0$ and

$$P(t) = (L) \int_a^t P(s).d_s F(s, t).$$

An integration by parts gives, for each t,

$$0 = -(R) \int_a^t dP(s).F(s, t) = (L) \int_t^a dP(s).F(s, t).$$

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Hence by Theorem 2.6, for each x,

$$0 = (\mathrm{R}) \int_x^a \left\{ (\mathrm{L}) \int_t^a dP(s).F(s,t) \right\} .d_t\, M(t,x)$$

$$= (\mathrm{L}) \int_x^a dP(s).(\mathrm{R}) \int_x^s F(s,t).d_t\, M(t,x).$$

Interchanging the order of integration in equation (3.1), we obtain

$$(\mathrm{R}) \int_x^s F(s,t).d_t\, M(t,x) = I - F(s,x)$$

so that

$$0 = (\mathrm{L}) \int_x^a dP(s).[I - F(s,x)] = P(a) - P(x) - (\mathrm{L}) \int_x^a dP(s).F(s,x)$$

and hence $P(x) = 0$.

If Z is defined by:

$$Z(t) = G(t) - (\mathrm{L}) \int_a^t G(s).d_s\, M(s,t)$$

$$= G(a)M(a,t) + (\mathrm{R}) \int_a^t dG(s).M(s,t),$$

then Theorem 2.5 shows Z to be of bounded variation. Interchanging an order of integration, we obtain

$$(\mathrm{L}) \int_a^t Z(s).d_s F(s,t) = G(a)(\mathrm{L}) \int_a^t M(a,s).d_s\, F(s,t)$$

$$+ (\mathrm{L}) \int_t^a dG(v).(\mathrm{R}) \int_t^v M(v,s).d_s\, F(s,t).$$

Utilizing the equations of which M and F are solutions, the right-hand term is reduced to

$$G(a)[M(a,t) - I] + (\mathrm{L}) \int_t^a dG(v).[I - M(v,t)] = Z(t) - G(t).$$

Thus Z is a solution of (4.2).

The following two theorems give representations in $\mathfrak{F}$ of solutions of homogeneous and non-homogeneous equations. The proofs are similar to that of Theorem 4.2.

THEOREM 4.3. *If $M = \mathfrak{S}(F)$, $J = \mathfrak{S}(E)$, and P is the function quasi-continuous in its first place such that*

$$P(t,x) = E(t,x) + (\mathrm{L}) \int_x^t d_s\, F(t,s).P(s,x) \qquad \textit{for all } t \textit{ and } x,$$

then

$$P(t,x) = E(t,x) - \text{(L)} \int_x^t d_s M(t,s).E(s,x)$$

and

$$M(t,x) = J(s,x) - \text{(L)} \int_x^t d_s P(t,s).J(s,x).$$

THEOREM 4.4. *If $M = \mathfrak{E}(F)$, $J = \mathfrak{E}(E)$, and P is the function of bounded variation in its second place such that*

$$P(t,x) = E(t,x) - \text{(R)} \int_x^t P(t,s).d_s F(s,x) \qquad \textit{for all } t \textit{ and } x,$$

then

$$P(t,x) = E(t,x) + \text{(R)} \int_x^t E(t,s).d_s M(s,x)$$

and

$$M(t,x) = J(t,x) + \text{(R)} \int_x^t J(t,s).d_s P(s,x).$$

5. An approximation theorem. The homogeneous equation (3.1) is solvable by a Cauchy polygon process. In this process the integral equation (3.1) is replaced, for a chain $\{r_i\}_0^m$ from x to t, by a system of $m + 1$ equations in $m + 1$ unknowns.

If $F \in \mathfrak{F}$, $\{r_i\}_0^m$ is a chain from the number x to the number t and

$$q_{ij} = F(r_i, r_j) - F(r_i, r_{j-1}) \qquad \text{for } i, j = 1, \ldots, n,$$

then $P_r(F)$ denotes the element z_m in N where $z_0, z_1, \ldots, z_m$ are defined recursively by $z_0 = I$ and

$$z_p = I + q_{p1} z_0 + q_{p2} z_1 + \ldots + q_{pp} z_{p-1} \qquad \text{for } 1 \leqslant p \leqslant m.$$

THEOREM 5.1. *If $M = \mathfrak{E}(F)$, $A(y) = M(y,a)$ for each y, $s = \{s_i\}_0^{2n}$ is a $(g_F - A - \epsilon)$-chain from a to b, $r = \{r_i\}_0^m$ is a refinement of s, and*

$$\epsilon' = 2\epsilon(|A|_S + V[g_F, a, b)],$$

then

$$||P_r(F) - M(b,a)|| \leqslant \epsilon' \prod_{i=1}^{m} [1 + |g_F(r_i) - g_F(r_{i-1})|].$$

Proof. Let q_{ij} and z_i be defined as in the definition of $P_r(F)$ and let $d_i = |g_F(r_i) - g_F(r_{i-1})|$ for each i. Let k be an integer $0 < k \leqslant m$. Let p be the least integer i such that $r_k \leqslant s_i$. The chain s' is defined as $s_0, s_1, \ldots, s_{p-1}, r_k$ if p is even and as $s_0, s_1, \ldots, s_{p-1}, r_k, r_k$ if p is odd. Then s' is a $(g_F - A - \epsilon)$-

chain from a to r_k. The chain $r' = r_0, \ldots, r_k, r_k$ is a refinement of s'. Hence, by Theorem 1.1,

$$\text{(5.1)} \qquad \left\| \text{(L)} \int_a^{r_k} d_y F(r_k, y) . A(y) - \sum_{i=1}^{k} q_{ki} A(r_{i-1}) \right\| \leqslant \epsilon'.$$

We shall prove by induction that

$$\text{(5.2)} \qquad ||A(r_v) - z_v|| \leqslant \epsilon' \prod_{i=1}^{v} [1 + d_i]$$

for $v = 1, \ldots, m$. First suppose $v = 1$. Then by equation (5.1), for $k = 1$,

$$||A(r_1) - z_1|| = \left\| \text{(L)} \int_a^{r_1} d_y F(r_1, y) . A(y) - q_{11} z_0 \right\| \leqslant \epsilon' \leqslant \epsilon'(1 + d_1).$$

Then if $u - 1$ is a positive integer and equation (5.2) holds for $v \leqslant u - 1$, we have by equation (5.1) for $k = u$,

$$||A(r_u) - z_u|| = \left\| \text{(L)} \int_a^{r_u} d_y F(r_u, y) . A(y) - \sum_{i=1}^{u} q_{ui} z_{i-1} \right\|$$
$$\leqslant \epsilon' + \sum_{i=1}^{u} ||q_{ui}[A(r_{i-1}) - z_{i-1}]||.$$

Using $||q_{ui}|| \leqslant d_i$ and the induction hypothesis, we obtain

$$||A(r_u) - z_u|| \leqslant \epsilon'(1 + d_1) + \epsilon' \sum_{i=2}^{u} d_i \prod_{j=1}^{i-1} (1 + d_j) = \epsilon' \prod_{i=1}^{u} (1 + d_i).$$

Thus (5.2) holds for $v = m$. Since $A(r_m) = M(b, a)$ and $z_m = P_r(F)$, this completes the proof.

In (7), MacNerney defines a function V from $S \times S$ to N to be $\mathfrak{O}$-additive if and only if

$$V(x, y) + V(y, z) = V(x, z) \quad \text{and} \quad V(z, y) + V(y, x) = V(z, x)$$

for all $x \leqslant y \leqslant z$. For such a function V there is a unique pair (V_1, V_2) **(7, p. 149)** with the following properties:

(i) V_1 and V_2 are from S to N and satisfy $V_1(a) = V_2(a) = 0$.

$$\text{(ii)} \qquad V(x, y) = \begin{cases} V_1(y) - V_1(x) & \text{if } x \leqslant y, \\ V_2(y) - V_2(x) & \text{if } y \leqslant x. \end{cases}$$

The set $\mathfrak{O}\mathfrak{A}$ is the set of all $\mathfrak{O}$-additive functions V such that, if

$$(x, y) \in S \times S,$$

there is a number m such that

$$\sum_{i=1}^{n} ||V(s_i, s_{i-1})|| \leqslant m$$

for every chain $\{s_i\}^n$ from x to y. This condition on V is equivalent to both V_1 and V_2 being of bounded variation. Thus if $V \in \mathfrak{D}\mathfrak{A}$, then $I^* - V \in \mathfrak{F}$. Moreover, if $F \in \mathfrak{F}$ and $V = I^* - F$ is $\mathfrak{D}$-additive, then $V \in \mathfrak{D}\mathfrak{A}$ since

$$||V(x, y)|| = ||I - F(x, y)|| = ||F(x, x) - F(x, y)|| \leqslant |g_F(x) - g_F(y)|.$$

MacNerney defines a function M from $S \times S$ to N to be $\mathfrak{D}$-multiplicative if and only if

$$M(x, y)M(y, z) = M(x, z) \quad \text{and} \quad M(z, y)M(y, x) = M(z, x)$$

for all $x \leqslant y \leqslant z$.

The set $\mathfrak{D}\mathfrak{M}$ is the set of all $\mathfrak{D}$-multiplicative functions M such that, if $(x, y) \in S \times S$, there is a number m such that

$$\sum_{i=1}^{n} ||M(s_i, s_{i-1}) - I|| \leqslant m$$

for every chain $\{s_i\}_0^n$ from x to y. Such an M satisfies $M(x, x) = I$ for all x; cf. **(7**, p. 152**)**.

Let $\mathfrak{D}\mathfrak{M}^+$ denote the set of all $\mathfrak{D}$-multiplicative functions μ from $S \times S$ to the ring of real numbers such that $\mu(t, x) \geqslant 1$ for all t and x. Then $M \in \mathfrak{D}\mathfrak{M}$ if and only if there exists a $\mu \in \mathfrak{D}\mathfrak{M}^+$ such that $||M(t, x) - I|| \leqslant \mu(t, x) - 1$ for all t and x; cf. **(7**, Lemma 3.2**)**.

If $\mu \in \mathfrak{D}\mathfrak{M}^+$, then μ is bounded since for $x \leqslant y$,

$$\mu(y, x) = \frac{\mu(d, y)\mu(y, x)\mu(x, c)}{\mu(d, y)\mu(x, c)} = \frac{\mu(d, c)}{\mu(d, y)\mu(x, c)} \leqslant \mu(d, c),$$

and for $y < x$ we have $\mu(y, x) \leqslant \mu(c, d)$. Let

$$h(x) = \mu(c, x) - \mu(c, c) - \mu(d, x) + \mu(d, c)$$

for all x. Then h is non-decreasing since for $x \leqslant y$,

$$\mu(c, y) - \mu(c, x) = \mu(x, x)[\mu(x, y) - 1] \geqslant 0$$

and

$$\mu(d, y) - \mu(d, x) = \mu(d, y)[1 - \mu(y, x)] \leqslant 0.$$

The function h is a super function for μ. For the case $x \leqslant y \leqslant t$, this follows from

$$0 \leqslant \mu(t, x) - \mu(t, y) = [\mu(d, x) - \mu(d, y)]/\mu(d, t)$$
$$\leqslant \mu(d, x) - \mu(d, y) \leqslant h(y) - h(x).$$

The other cases are similar. The function μ has the additional property that $\mu(x, x) = 1$ since

$$1 \leqslant \mu(x, x) = \mu(x, x)^2.$$

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From (7, Lemma 3.2), it follows that if $M \in \mathfrak{O}\mathfrak{M}$, then M is bounded. If L is a bound for M and $x \leqslant y \leqslant t$, then

$$\begin{aligned}||M(t,x) - M(t,y)|| = ||M(t,y)[M(y,x) - I]|| &\leqslant L[\mu(y,x) - 1] \\ &= L(\mu(y,x) - \mu(y,y)] \leqslant L[h(y) - h(x)]\end{aligned}$$

and

$$\begin{aligned}||M(x,t) - M(y,t)|| = ||[M(x,y) - I]M(y,t)|| &\leqslant L[\mu(x,y) - 1] \\ &\leqslant L[h(y) - h(x)].\end{aligned}$$

The same inequalities follow for $t \leqslant x$ and $x < t < y$. Thus $M \in \mathfrak{F}$.

If $M \in \mathfrak{F}$ is $\mathfrak{O}$-multiplicative, then $M \in \mathfrak{O}\mathfrak{M}$ since

$$||I - M(x,y)|| = ||M(x,x) - M(x,y)|| \leqslant |g_M(x) - g_M(y)|.$$

If x and y are in S, $\{s_i\}_0^n$ is a chain from x to y, $V \in \mathfrak{O}\mathfrak{A}$ and $M \in \mathfrak{O}\mathfrak{M}$, then we compute inductively

$$P_s(I^* - V) = [I + V(s_n, s_{n-1})] \ldots [I + V(s_1, s_0)]$$

and

$$P_s(M) = I - [M(s_1, s_0) - I] - \ldots - [M(s_n, s_{n-1}) - I].$$

MacNerney defines the continued sum ${}_x\sum^t [h]$ and the continued product ${}_x\prod^t[h]$ for a function h from $S \times S$ to N as the σ-limit over chains $\{s_i\}_0^n$ from x to t of sums

$$\sum_{i=1}^{n} h(s_i, s_{i-1}) = h(s_0, s_1) + \ldots + h(s_{n-1}, s_n)$$

and products

$$\prod_{i=1}^{n} h(s_i, s_{i-1}) = h(s_0, s_1) \ldots h(s_{n-1}, s_n),$$

respectively. Standard arguments show that the continued sum ${}_x\sum^t[h]$ is $\mathfrak{O}$-additive and the continued product ${}_x\prod^t[h]$ is $\mathfrak{O}$-multiplicative.

As a consequence of Theorem 5.1 and the above remarks we have the following two corollaries.

Corollary 5.1. *If $V \in \mathfrak{O}\mathfrak{A}$ and $M = \mathfrak{E}(I^* - V)$, then*

$$M(t,x) = {}_t\prod^x[I^* + V] \qquad \textit{for each } t \textit{ and } x.$$

Moreover, $M \in \mathfrak{O}\mathfrak{M}$.

Corollary 5.2. *If $M \in \mathfrak{O}\mathfrak{M}$ and $I^* - V = \mathfrak{E}(M)$, then*

$$V(t,x) = {}_t\sum^x[M - I^*] \qquad \textit{for each } t \textit{ and } x.$$

Moreover, $V \in \mathfrak{O}\mathfrak{A}$.

The mapping $\mathfrak{E}^*$ from $\mathfrak{O}\mathfrak{A}$ into $\mathfrak{O}\mathfrak{M}$ is defined by:

$$\mathfrak{E}^*(V) = \mathfrak{E}(I^* - V).$$

Corollaries 5.1 and 5.2 and Theorem 3.3 yield Corollary 5.3, which is identical with the parts (i) and (ii) of Theorem 3.3 and with Theorem 4.3 of **(7)**.

COROLLARY 5.3. *The mapping $\mathfrak{E}^*$ is reversible from $\mathfrak{O}\mathfrak{A}$ onto $\mathfrak{O}\mathfrak{M}$ and each of the following is a necessary and sufficient condition that $M = \mathfrak{E}^*(V)$:*

(i) $M(t, x) = {}_t\Pi^x[I^* + V]$ *for each t and x.*

(ii) $V(t, x) = {}_t\Sigma^x[M - I^*]$ *for each t and x.*

(iii) *For each t and x,*

$$M(t, x) = I - (\mathrm{L}) \int_x^t dV_1(s).M(s, x) \qquad \text{if } t \leqslant x$$

and

$$M(t, x) = I - (\mathrm{L}) \int_x^t dV_2(s).M(s, x) \qquad \text{if } x \leqslant t.$$

(iv) *For each t and x,*

$$M(t, x) = I - (\mathrm{R}) \int_x^t M(t, s).dV_1(s) \qquad \text{if } t \leqslant x$$

and

$$M(t, x) = I - (\mathrm{R}) \int_x^t M(t, s).dV_2(s) \qquad \text{if } x \leqslant t.$$

REFERENCES

1. R. Bellman, *Stability theory of differential equations* (New York, 1953).
2. R. F. Deniston, *Existence of Stieltjes integrals.* I, Nederl. Akad. Wetensch. Indag. Math., *11*, (1949), 385–393.
3. T. H. Hildebrandt, *On systems of linear differentio-Stieltjes integral equations*, Illinois J. Math., *3* (1959), 352–373.
4. R. Lane, *The integral of a function with respect to a function*, Proc. Amer. Math. Soc., *5* (1954), 59–66.
5. J. S. MacNerney, *Stieltjes integrals in linear spaces*, Ann. of Math., *61* (1955), 354–367.
6. ——— *Continuous products in linear spaces*, J. Elisha Mitchell Sci. Soc., *71* (1955), 185–200.
7. ——— *Integral equations and semigroups*, Illinois J. Math., *7* (1963) 148–173.
8. ——— *A linear initial value problem*, Bull. Amer. Math. Soc., *69* (1963), 314–329.
9. J. W. Neuberger, *Continuous products and non-linear integral equations*, Pacific J. Math., *10* (1960), 1385–1392.
10. H. S. Wall, *Concerning harmonic matrices*, Arch. Math., *5* (1954), 160–167.

The University of Tennessee and
The University of Georgia