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ERGODIC BEHAVIOUR OF EXTREME VALUES

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Abstract

Let $\{X_n, n \geq 1\}$ be independent identically distributed random variables with a common non-degenerate distribution function F. For each $n \geq 1$, denote $M_n = \max\{X_1, \ldots, X_n\}$. Under certain conditions on F, there exist constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that $(M_n - b_n)/a_n \xrightarrow{d} G$. In this paper, we shall show that $\{(M_n - b_n)/a_n\}$ exhibits ergodic behaviour under additional conditions on F.

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1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables with $\mathbb{E}X_1 = 0$ and $\mathbb{E}X_1^2 = 1$. For each $n \geq 1$, set $S_n = \sum_{i=1}^n X_i$. Then for $x \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbf{1}\left(\frac{S_k}{\sqrt{k}} \leq x \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt \quad \text{almost surely.}$$

This is known as the almost sure central limit theorem, and was proved first by Brosamler [1] and Schatte [8]. Recently, the related problems have attracted much attention (see Lacey and Philipp [6], Schatte [9–11]).

It was shown in Cheng *et al.* [2] that the above phenomenon holds also for extreme values (see Lemma 2 in Section 2). In this paper we consider the general problem related to maxima (see below).

We assume throughout that $\{X_n, n \geq 1\}$ are a sequence of independent identically distributed random variables with a common non-degenerate distribution function F.

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For each $n \geq 1$, denote

$$M_n = \max\{X_1, \dots, X_n\}.$$

Suppose there exist constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$(M_n - b_n)/a_n \xrightarrow{d} G \tag{1.1}$$

where G is a non-degenerate distribution function. Then we say G is an extreme value distribution and F is in the domain of attraction of G (notation: $F \in D(G)$).

It is well known that G must be one of the following three types:

$$G(x) = \Phi_\alpha(x) = \begin{cases} \exp\{-x^{-\alpha}\} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0; \end{cases}$$

$$G(x) = \Psi_\alpha(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ \exp\{-(-x)^\alpha\} & \text{if } x < 0; \end{cases}$$

$$G(x) = \Lambda(x) = \exp\{-e^{-x}\} \text{ for } x \in \mathbb{R},$$

where $\alpha > 0$ (see [7, Proposition 0.3]). Furthermore a_n and b_n can be chosen as

$$a_n = \begin{cases} U(n) & \text{if } G(x) = \Phi_\alpha(x), \\ x_F - U(n) & \text{if } G(x) = \Psi_\alpha(x), \\ U(ne) - U(n) & \text{if } G(x) = \Lambda(x), \end{cases} \tag{1.2}$$

and

$$b_n = \begin{cases} 0 & \text{if } G(x) = \Phi_\alpha(x), \\ x_F & \text{if } G(x) = \Psi_\alpha(x), \\ U(n) & \text{if } G(x) = \Lambda(x), \end{cases} \tag{1.3}$$

where $x_F := \sup\{x : F(x) < 1\}$ and $U(x) := \inf\{y : 1/(1 - F(y)) > x\}$.

It is important to have necessary and sufficient conditions for a distribution to belong to the domain of attraction of an extreme value distribution. Some characterisation theorems can be found in [7, Chapter 1]. For example, $F \in D(\Phi_\alpha)$ if and only if

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^{1/\alpha} \quad \text{for all } x > 0$$

(notation: $U \in RV_{1/\alpha}$).

Assume that (1.1) holds. If there exists a positive sequence $\{r_n, n \geq 1\}$ with $\sum_{k=1}^{\infty} r_k = \infty$ such that

$$\lim_{n\to\infty} \frac{1}{\sum_{k=1}^n r_k} \sum_{k=1}^n r_k f\left(\frac{M_k - b_k}{a_k}\right) = \int_{\mathbf{D}} f(x) G(dx) \quad \text{almost surely}$$

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(with $\mathbf{D} := \{x : 0 < G(x) < 1\}$) holds for a class of functions f, then we may say that the sequence $\{(M_n - b_n)/a_n\}$ has ergodic behaviour. It is natural to consider the case where $r_n = n^{-\gamma}$ with $0 < \gamma \leq 1$. Unfortunately, for $\gamma \in (0, 1)$, the above equation is not true even for the indicator function (see Cheng *et al.* [2]). Hence we only consider the case $\gamma = 1$, that is the logarithmic means. In the present paper, the following results are obtained (proofs are given in Section 2).

THEOREM 1. *Suppose* (1.1) *holds for* $G = \Phi_\alpha$, $F(0-) = 0$, *and* a_n *and* b_n *are defined by* (1.2) *and* (1.3). *Assume* f *is an almost everywhere continuous function which is defined on* $(0, \infty)$. *If there are constants* $B > 0$, $\beta \in (0, \alpha)$ *and* $\tau > 0$ *such that*

$$|f(x)| \leq B(x^\beta + x^{-\tau}) \quad \textit{for all } x > 0, \tag{1.4}$$

then

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} f\left(\frac{M_n - b_n}{a_n}\right) = \int_0^\infty f(x)\Phi_\alpha(dx) \quad \textit{almost surely.} \tag{1.5}$$

REMARK 1. Condition (1.4) ensures that $|\int_0^\infty f(x)\Phi_\alpha(dx)| < \infty$. Indeed, (1.5) still holds if we replace (1.4) by assuming that there are constants $\beta \in (0, \alpha)$ and $\tau \in (0, 1/\alpha)$ such that

$$|f(x)| \leq B(x^\beta + e^{x^{-\tau}}) \quad \text{for all } x > 0.$$

REMARK 2. Assume (1.1) holds for $G = \Phi_\alpha$, and f is an almost sure continuous function f which is defined on $(-\infty, \infty)$. If there are constants $B > 0$ and $\beta \in (0, \alpha)$ such that

$$|f(x)| \leq B(|x| + 1)^\beta \quad \text{for } x \in \mathbb{R},$$

then (1.5) holds.

Note that since

$$\int_0^\infty x^\beta \Phi_\alpha(dx) = \Gamma(1 - \beta/\alpha),$$

we have

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{M_n^\beta}{n a_n^\beta} = \Gamma(1 - \beta/\alpha) \quad \text{almost surely.} \tag{1.6}$$

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THEOREM 2. *Suppose* (1.1) *holds for* $G = \Psi_\alpha$, *and* a_n *and* b_n *are defined by* (1.2) *and* (1.3). *Assume* g *is an almost everywhere continuous function which is defined on* $(-\infty, 0)$. *If there are constants* $B > 0$, $\beta \in (0, \alpha)$ *and* $\tau > 0$ *such that*

$$|g(x)| \le B(|x|^{-\beta} + |x|^{\tau}) \quad \textit{for all } x < 0, \tag{1.7}$$

then

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} g\left(\frac{M_n - b_n}{a_n}\right) = \int_{-\infty}^{0} g(x)\Psi_\alpha(dx) \quad \textit{almost surely.} \tag{1.8}$$

REMARK 3. Assume (1.1) holds for $G = \Psi_\alpha$, and g is an almost everywhere continuous function which is defined on $(-\infty, \infty)$. If there are constants $B > 0$ and $\beta > 0$ such that

$$|g(x)| \le B(|x| + 1)^{\beta} \text{ for } x \in \mathbb{R},$$

then (1.8) holds.

Note that since for any positive integer β

$$\int_{-\infty}^{0} x^{\beta}\Psi_\alpha(dx) = (-1)^{\beta}\Gamma(1 + \beta/\alpha),$$

we have

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{(M_n - x_F)^{\beta}}{n a_n^{\beta}} = (-1)^{\beta}\Gamma(1 + \beta/\alpha) \quad \text{almost surely.} \tag{1.9}$$

THEOREM 3. *Suppose* (1.1) *holds for* $G = \Lambda$, *and* a_n *and* b_n *are defined by* (1.2) *and* (1.3). *Assume* h *is an almost everywhere continuous function which is defined on* $(-\infty, \infty)$. *If there are constants* $B > 0$ *and* $\beta > 0$ *such that*

$$|h(x)| \le B(|x| + 1)^{\beta} \quad \textit{for } x \in \mathbb{R}, \tag{1.10}$$

then

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} h\left(\frac{M_n - b_n}{a_n}\right) = \int_{-\infty}^{\infty} h(x)\Lambda(dx) \quad \textit{almost surely.} \tag{1.11}$$

REMARK 4. Under the conditions of Theorem 3, if β is a positive integer, then

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{(M_n - b_n)^{\beta}}{n a_n^{\beta}} = (-1)^{\beta}\Gamma^{(\beta)}(1) \quad \text{almost surely,} \tag{1.12}$$

where $\Gamma^{(\beta)}(1)$ denotes the β-th derivative of the gamma function at $x = 1$.

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REMARK 5. According to [7, Proposition 2.1], if (1.1) holds, then under additional conditions on the left tail of F, we have

$$\lim_{n\to\infty} \mathbb{E}\left(\frac{M_n - b_n}{a_n}\right)^{\beta} = \begin{cases} \Gamma(1-\beta/\alpha) & \text{if } G(x) = \Phi_\alpha(x), \\ (-1)^{\beta}\Gamma(1+\beta/\alpha) & \text{if } G(x) = \Psi_\alpha(x), \\ (-1)^{\beta}\Gamma^{(\beta)}(1) & \text{if } G(x) = \Lambda(x), \end{cases} \tag{1.13}$$

where a_n and b_n are defined by (1.2) and (1.3), and in the last two equations, β should be positive integer. Thus by (1.6), (1.9) and (1.12) we have

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{(M_n - b_n)^{\beta}}{n\mathbb{E}(M_n - b_n)^{\beta}} = 1 \quad \text{almost surely.}$$

REMARK 6. It is obvious that (1.1) holds for constants a'_n and b'_n which satisfy

$$\begin{cases} a'_n/a_n \to 1 \\ (b'_n - b_n)/a_n \to 0 \end{cases}$$

as $n \to \infty$, where a_n and b_n are defined by (1.2) and (1.3) (see [7, Proposition 0.2]). Moreover, Remark 2, Remark 3 and Theorem 3 hold for above constants a'_n and b'_n.

2. Proofs

For every measurable function l let

$$\mathbf{S}(l) = \{x : l \text{ is continuous at } x\}.$$

The proofs of our theorems are mainly based on the following lemmas. The proof of Lemma 1 below is very standard and we omit it.

LEMMA 1. *Assume* $\{Z, Z_n, n \geq 1\}$ *is a sequence of random variables with distribution functions* $\{G, G_n, n \geq 1\}$. *Assume* $\{Z_n\}$ *converges in distribution to* Z, *that is*

$$\lim_{n\to\infty} G_n(x) = G(x), \quad \text{for } x \in \mathbf{S}(G). \tag{2.1}$$

If l *is a real-valued almost everywhere continuous function with respect to* G, *that is* $\Pr(Z \in \mathbf{S}(l)) = 1$ *and* $\{l(Z), l(Z_n), n \geq 1\}$ *is uniformly integrable (for definition of uniformly integrable, see* [3, page 93]), *then*

$$\lim_{n\to\infty} \mathbb{E}l(Z_n) = \mathbb{E}l(Z). \tag{2.2}$$

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LEMMA 2. *Assume* (1.1) *holds. Then*

$$\Pr\left\{\lim_{n\to\infty}\sup_{\mathbf{D}}\left|\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathbf{1}\left(\frac{M_k-b_k}{a_k}\le x\right)-G(x)\right|=0\right\}=1 \tag{2.3}$$

where $\mathbf{1}(A)$ *denotes the indicator function of set* A, *and* a_n *and* b_n *are defined by* (1.2) *and* (1.3).

PROOF. See Cheng *et al.* [2]. □

Next we are going to prove our theorems. Set

$$\Omega_1=\left\{\omega:\lim_{N\to\infty}\sup_{x\in\mathbf{D}}\left|\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\mathbf{1}_{(-\infty,x]}\left(\frac{M_n-b_n}{a_n}\right)-G(x)\right|=0\right\}. \tag{2.4}$$

From Lemma 2 we know

$$\Pr(\Omega_1)=1. \tag{2.5}$$

Assume $\{W_j, j\ge 1\}$ is a sequence of independent random variables with common distribution function Φ_1. It is easily seen that $\{U(1/(1-\Phi_1(W_j))), j\ge 1\}$ is a sequence of independent random variables which have the same distributions as $\{X_j, j\ge 1\}$. For the sake of simplicity, we assume that $X_j = U\big(1/(1-\Phi_1(W_j))\big)$, for $j\ge 1$. Using the well-known inequalities for regular variation and Π-variation (see Geluk and de Haan [4, Proposition 1.7.5 and Proposition 1.19.4]), we may concentrate on dealing with $\{W_j, j\ge 1\}$ (see (2.9) and (2.13) below).

For $1\le m\le n$, set $W(n,m)=\max_{n-m+1\le j\le n} W_j$. Obviously, $W_{n,m}/m$ has distribution function Φ_1, and $M_n = U(1/(1-\Phi_1(W(n,n))))$ for $n\ge 1$. We also have

$$W(n,n)\to\infty \quad \text{almost surely as } n\to\infty. \tag{2.6}$$

PROOF (of Theorem 1). Put $\delta=(\beta/\alpha+1)/2$ and $d^2=(\alpha+\beta)/(2\beta)$. Then $d>1$ and $\delta\in(0,1)$. Throughout the proof we use C to denote a positive constant, and we let $O(1)$ refer to almost surely.

We write

$$\Omega_2=\left\{\omega:\limsup_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\left|f\left(\frac{M_n}{a_n}\right)\right|^d<\infty\right\}.$$

First we show that

$$\Pr(\Omega_2)=1. \tag{2.7}$$

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Write $S_N = (\log N)^{-1} \sum_{n=1}^{N} n^{-1} |f(M_n/a_n)|^d$. Then (1.4) implies

$$\begin{aligned} S_N &\le C\left(\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\left(\frac{M_n}{a_n}\right)^{-d\tau} + \frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\left(\frac{M_n}{a_n}\right)^{d\beta}\right) \\ &= C\left(\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\left(\frac{U(1/(1-\Phi_1(W(n,n))))}{U(n)}\right)^{-d\tau}\right. \\ &\quad \left. + \frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\left(\frac{U(1/(1-\Phi_1(W(n,n))))}{U(n)}\right)^{d\beta}\right) \\ &:= C(S_N^{(1)} + S_N^{(2)}). \end{aligned}$$

Since $U \in RV_{1/\alpha}$, Potter-bound inequality (see Geluk and de Haan [4, Proposition 1.7.5]) implies that there exists $t_0 > 0$ such that

$$\frac{U(tx)}{U(t)} \le 2x^{d/\alpha}$$

for all $t > t_0$ and $x \ge 1$. Since $U(x)$ is non-decreasing, we have

$$\frac{U(1/(1-\Phi_1(W(n,n))))}{U(n)} \le 1 + \frac{2}{(n(1-\Phi_1(W(n,n))))^{d/\alpha}}$$

for all $n \ge t_0$. Hence

$$\begin{aligned} S_N^{(2)} &= O(1) + C\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}(n(1-\Phi_1(W(n,n))))^{-d^2\beta/\alpha} \\ &= O(1) + C\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}(n(1-\Phi_1(W(n,n))))^{-\delta} \\ &= O(1) + \frac{O(1)}{\log N}\sum_{n=1}^{N}\frac{1}{n}\left(\frac{W(n,n)}{n}\right)^{\delta}, \end{aligned} \tag{2.8}$$

since $1 - \Phi_1(W(n,n)) \sim (W(n,n))^{-1}$ holds almost surely from (2.6).

Note that for each $N \ge 2$, there exists $m \ge 2$ such that $2^{m-1} \le N < 2^m$, and

$$\begin{aligned} &\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\left(\frac{W(n,n)}{n}\right)^{\delta} \\ &\le \frac{1}{(m-1)\log 2}\sum_{n=1}^{2^m}\frac{1}{n}\left(\frac{W(n,n)}{n}\right)^{\delta} \\ &\le \frac{1}{(m-1)\log 2}\sum_{j=1}^{m}\sum_{n=2^{j-1}}^{2^j}\frac{1}{n}\left(\frac{W(n,n)}{n}\right)^{\delta} \end{aligned} \tag{2.9}$$

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$$\le \frac{2^{\delta+1}}{m-1}\sum_{j=1}^{m}\left(\frac{W(2^j,2^j)}{2^j}\right)^{\delta}$$

$$\le \frac{2^{\delta+1}}{m-1}\sum_{j=1}^{m}\frac{1}{2^{j\delta}}\left(\sum_{i=1}^{j}(W(2^i,2^{i-1}))^{\delta}+(W(1,1))^{\delta}\right)$$

$$= \frac{2^{\delta+1}}{m-1}\left(\sum_{i=1}^{m}(W(2^i,2^{i-1})^{\delta}+(W(1,1))^{\delta}\right)\sum_{j=i}^{m}\frac{1}{2^{j\delta}}$$

$$\le \frac{2^{\delta}}{2^{\delta}-1}\frac{1}{m-1}\left(\sum_{i=1}^{m}\left(\frac{W(2^i,2^{i-1})}{2^{i-1}}\right)^{\delta}+(W(1,1))^{\delta}\right).$$

Since $\{(W(2^i,2^{i-1})2^{1-i})^{\delta}, i\ge 1\}$ is a sequence of identical and independent random variables with finite means $\mathbb{E}\,W_1^{\delta}$, by the strong law of large numbers we have

$$\text{(2.10)}\qquad \frac{1}{m-1}\sum_{i=1}^{m}\left(\frac{W(2^i,2^{i-1})}{2^{i-1}}\right)^{\delta}\to \mathbb{E}\,W_1^{\delta}\quad\text{almost surely.}$$

Therefore, by (2.8), (2.9) and (2.10) we have $S_N^{(2)}=O(1)$. In order to prove (2.7), we only need to show that

$$\text{(2.11)}\qquad S_N^{(1)}=O(1).$$

Using Potter-bound inequality, for some $t_1>0$ and $C>0$

$$\text{(2.12)}\qquad \frac{U(tx)}{U(t)}\ge Cx^{d/\alpha}$$

holds for all $t>t_1$, $tx>t_1$ and $x\le 1$. From (2.6), $1-\Phi_1(W(n,n))\to 0$ almost surely. Hence,

$$\Pr\left(1-\Phi_1(W(n,n))\ge\frac{1}{t_1},\ \text{infinitely often}\right)=0.$$

It is easy to check from (2.12) that

$$S_N^{(1)}=\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\left(\frac{U(1/(1-\Phi_1(W(n,n))))}{U(n)}\right)^{-d\tau}\mathbf{1}\left(\frac{1}{1-\Phi_1(W(n,n))}\le t_1\right)$$
$$+\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\left(\frac{U(1/(1-\Phi_1(W(n,n))))}{U(n)}\right)^{-d\tau}\mathbf{1}\left(\frac{1}{1-\Phi_1(W(n,n))}>t_1\right)$$
$$=O(1)+\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}(n(1-\Phi_1(W(n,n)))^{d^2\tau/\alpha}$$

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$$= O(1) + \frac{O(1)}{\log N} \sum_{n=1}^{N} \frac{1}{n} \left(\frac{n}{W(n,n)} \right)^{d^2\tau/\alpha}.$$

For $N \geq 2$, $2^{m-1} \leq N < 2^m$, it may easily be proved that

$$\text{(2.13)} \qquad \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \left(\frac{n}{W(n,n)} \right)^{d^2\tau/\alpha} \leq \frac{C}{m-1} \sum_{j=1}^{m} \left(\frac{2^{j-1}}{W(2^j, 2^{j-1})} \right)^{d^2\tau/\alpha},$$

which is bounded almost surely by the strong law of large numbers since $\{(n/W(n,n))^{d^2\tau/\alpha}\}$ is a sequence of identical and independent random variables with finite means $\mathbb{E}(W_1)^{-d^2\tau/\alpha}$. Thus, (2.11) is proved. This completes the proof of (2.7).

Set $\Omega = \Omega_1 \cap \Omega_2$. From (2.5) and (2.7) we have $\Pr(\Omega) = 1$. Put $K(N) = \sum_{n=1}^{N} 1/n$. Fix $\omega \in \Omega$ and write

$$F_N(x) = \frac{1}{K(N)} \sum_{n=1}^{N} \frac{1}{n} \mathbf{1}_{(-\infty, x]} \left(\frac{M_n}{a_n} \right), \quad x \in \mathbb{R}.$$

Then $\{F_N\}$ is a sequence of distribution functions. Let Z_N have distribution F_N and Z has distribution Φ_α. Since $K(N)/\log N \to 1$ as $N \to \infty$, we have

$$\lim_{N\to\infty} \sup_x |F_N(x) - \Phi_\alpha(x)| = 0.$$

Note that

$$\frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} f\left(\frac{M_n}{a_n} \right) = \frac{K(N)}{\log N} \int_0^\infty f(x) dF_N = \frac{K(N)}{\log N} \mathbb{E} f(Z_N).$$

By the definition of Ω we know that $\{f(Z), f(Z_N), N \geq 1\}$ is uniformly integrable. Thus by Lemma 1

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} f\left(\frac{M_n}{a_n} \right) = \int_0^\infty f(x) \Phi_\alpha(dx).$$

This proves (1.5). □

Proof (of Theorem 2). Put $Y_j = 1/(x_F - X_j)$ for $j \geq 1$. Then $\max_{1 \leq j \leq n} Y_j = 1/(x_F - M_n)$ and

$$\frac{\max_{1\leq j\leq n} Y_j}{a_n^{-1}} \xrightarrow{d} \Phi_\alpha.$$

Put $f(x) = g(-x^{-1})$ for $x > 0$. Then (1.4) is satisfied because of (1.7). Using Theorem 1 we have (1.8). □

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PROOF (of Theorem 3). Note that (1.1) implies

$$\lim_{t\to\infty} \frac{U(tx) - U(t)}{U(te) - U(t)} = \log x \quad \text{for all } x > 0$$

(see de Haan [5, Theorem 2.4.1]), using the known inequality for Π-function (see Geluk and de Haan [4, Proposition 1.19.4]), for every $\epsilon > 0$, there exist $C > 0$ and $t_2 > 0$ such that

$$\left| \frac{U(tx) - U(t)}{U(te) - U(t)} \right| \leq C(x^{\epsilon} + x^{-\epsilon})$$

for all $t \geq t_2$ and $tx \geq t_2$. Following the lines of proof of Theorem 1, we have (1.11). □

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