BROUWERIAN GEOMETRY

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1. Introduction. From time to time attention has been directed to the study of spaces in which the "distances" are selected from algebraic structures other than the real or complex numbers. For example, Menger **(9)** and Taussky **(11)** have considered spaces in which the distances are taken from a group. More recently, Ellis **(4)**, Blumenthal **(2)**, and Elliott **(3)**, have investigated spaces in which the distances are elements of a lattice. In particular, Ellis has observed that if with each pair of elements a and b of a Boolean algebra B we associate the symmetric difference $a*b = ab' + a'b$, and refer to this element as the distance between a and b, then this association satisfies (1) $a*b = b*a$, (2) $a*b = O$ if and only if $a = b$, (3) $(a*b) + (b*c) > a*c$. Thus "metrized," the Boolean algebra formally satisfies the usual postulates for a metric space. Since the points of the space as well as the distances are elements of the same lattice, Ellis has called B an autometrized Boolean algebra, while Blumenthal refers to it as a Boolean geometry. We shall use the latter term and reserve the former for more general purposes.

If with each pair of elements of an abstract set S we associate an element of a lattice L with an O such that the association satisfies conditions analogous to conditions (1), (2), and (3) for the association of $a*b$ with a and b, we refer to the structure as an L-metrized space. If S and L coincide, L will be said to be autometrized and in this case the association will be termed a metric operation. Elliott has shown that the symmetric difference is the only operation possible in a Boolean algebra which is simultaneously a metric and a group operation.

It is the aim of this paper to study certain aspects of L-metrized spaces and to extend some of the Boolean geometry to what we call Brouwerian geometries. These geometries are introduced and discussed in §2, and lead to numerous characterizations of Boolean algebras. Subgeometries of Brouwerian geometries are also considered. In §3 some general theorems on L-metrized spaces are established, and the implications of the coincidence of lattice and metric betweenness in autometrized spaces are examined. The concluding §4 contains a few results concerning the concept of congruence order together with some indications of possible future investigations.

2. Brouwerian geometry. If with each two elements a and b of a lattice L having a least element O and greatest element I there is associated a smallest

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element x such that $b + x > a$, then x is denoted by $a - b$ and the lattice is called a Brouwerian algebra (or Brouwerian logic). These algebras can readily be shown to be distributive lattices, and have been extensively studied by McKinsey and Tarski **(7)**. The element $a*b = (a - b) + (b - a)$ is called the *symmetric difference* of a and b and in case L is a Boolean algebra clearly coincides with the usual symmetric difference. Brouwerian algebras are natural extensions of Boolean algebras but are considerably wider since they include, for example, all finite distributive lattices, all chains having an O and an I, and distributive lattices with an I in which all descending chains are finite. A Brouwerian algebra is also the dual of a relatively pseudo-complemented lattice **(1)**. Other examples may be found in **(7)**.

We list below a number of elementary properties of the subtraction operation useful throughout this paper. Additional relations may be found in **(7)**.

THEOREM 2.1. *In a Brouwerian algebra the following relations hold:*

(a) $a - b < a$ (b) $a < b$ *if and only if* $a - b = O$
(c) $a - O = a$ (d) $a - a = O$ (e) $a + (b - a) = a + b$
(f) $a < b$ *implies* $a - c < b - c$ (g) $(a - b) - b = a - b$
(h) $(a + b) - c = (a - c) + (b - c)$ (i) $a - bc = (a - b) + (a - c)$
(j) $a < b$ *implies* $c - b < c - a$ (k) $(a - b) + ab = a$

The element $I - x$ is called the *Brouwerian complement* of x and is denoted by $\urcorner x$. Similarly $\urcorner\urcorner x = I - \urcorner x$. We list some of the properties of the Brouwerian complement below. Others may be found in **(7)**.

THEOREM 2.2. *In a Brouwerian algebra, the following relations hold:*

(a) $a < b$ *implies* $\urcorner b < \urcorner a$ (b) $a + \urcorner a = I$
(c) $\urcorner O = I, \urcorner I = O$ (d) $\urcorner\urcorner a < a$
(e) $\urcorner\urcorner\urcorner a = \urcorner a$ (f) $\urcorner(ab) = \urcorner a + \urcorner b$
(g) $\urcorner(a + b) = \urcorner\urcorner(\urcorner a \cdot \urcorner b)$ (h) $\urcorner(a \cdot \urcorner a) = I$

THEOREM 2.3. *The symmetric difference in a Brouwerian algebra is a metric operation.*

Proof. Relations (1) and (2) of the introduction are immediate, so it remains to establish (3), the "triangle inequality." By properties (i), (h), (k), and (e) of Theorem 2.1 we obtain

$$\begin{aligned} abc + (b*c) + (a*c) &= (ab)c + (c - ab) + [(a + b) - c] \\ &= c + [(a + b) - c] = a + b + c. \end{aligned}$$

Then $(a + b + c) - abc < (b*c) + (a*c)$, and by expansion of the left member using (h), (i) and (d), $(a*b) + (a*c) + (b*c) < (b*c) + (a*c)$, and thus $a*b < (b*c) + (a*c)$.

DEFINITION 2.1. A Brouwerian algebra autometrized by the symmetric difference is called a *Brouwerian geometry*.

It is often convenient to employ geometrical language and regard a triple of elements a, b, c as the vertices of a triangle with sides $a*b$, $a*c$, and $b*c$. By Theorem 2.3 the sides of any triangle in a Brouwerian geometry satisfy the triangle inequality. The notation $\Delta(a, b, c)$ will be used to designate a triangle with vertices a, b, c.

One reason why a Boolean geometry has so many novel properties is that the symmetric difference in that instance is a group operation. This is not true for Brouwerian geometries, and it is instructive to see how nearly it comes to being one in this case. Thus the group property eliminates the possibility of having isosceles triangles in a Boolean geometry. Brouwerian geometries, on the other hand, may abound with isosceles triangles, as happens, for example, in a Brouwerian chain, where every triangle is isosceles. However:

THEOREM 2.4. *A Brouwerian geometry is a Boolean geometry if and only if it is free of isosceles triangles.*

Proof. The necessity is obvious, as remarked above. If a Brouwerian geometry L is free of isosceles triangles, then $I*O = I*(x \cdot \urcorner x) = \urcorner(x \cdot \urcorner x) = I$ implies $x \cdot \urcorner x = O$. Since $x + \urcorner x = I$, every element has a unique complement and L is a Boolean algebra. Then $a - b = ab'$, $b - a = a'b$ and $a*b = ab' + a'b$, so L is a Boolean geometry.

COROLLARY 2.4.1. *A Brouwerian algebra is a Boolean algebra if and only if symmetric difference is a group operation.*

Proof. It is well known that symmetric difference is a group operation in a Boolean algebra. If it is a group operation in a Brouwerian algebra L, then the associated geometry contains no isosceles triangle and L is a Boolean algebra.

DEFINITION 2.2. Three elements a, b, c of any lattice are said to satisfy the *triangle inequality* if each is under the sum of the other two, and we write $(a, b, c)T$.

THEOREM 2.5. *In a Brouwerian geometry the relation $(a, b, c)T$ is equivalent to each of the relations*

(1) $$a + b = a + c = b + c = a + b + c$$

(2) $$a - b < c, \ b - a < c, \ c - a < b$$

(3) $$a*b < c < a + b$$

(4) $$b - a = c - a, \ a - b = c - b, \ a - c = b - c$$

Proof. If $(a, b, c)T$ then $a < b + c$ implies $a + b + c = b + c$, and (1) follows. Relation (2) is a result of the definition of the subtraction operation, and (3) follows from (2). To prove (4) we have $(a + b) - c = (a + c) - c = a - c = (b + c) - c = b - c$ by (1) above and by (h) of Theorem 2.1. The converse is readily established.

COROLLARY 2.5.1. *If a and b are two elements of a Brouwerian geometry then $(a, b, a*b)T$ and $ab + (a*b) = a + b$.*

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Proof. These relations follow from an examination of $\Delta(O, a, b)$ and by using (1) of Theorem 2.5 and the distributive law.

THEOREM 2.6. *A Brouwerian geometry contains no equilateral triangle.*

Proof. We first establish a few properties of isosceles triangles. If $\Delta(a, b, c)$ is isosceles with $b*c = x, a*b = a*c = y$, then $(a, b, y)T$, $(a, c, y)T$ and $(b, c, x)T$. Now $(x, y, y)T$ implies $x < y$, and by (f) of Theorem 2.1, $x - c < y - c$, $x - b < y - b$, so

$$x = (b-c)+(c-b) = (x-c)+(x-b) < (y-c)+(y-b) = (a-c)+(a-b) < a.$$

Thus the base of an isosceles triangle is "under" the vertex. If $\Delta(a, b, c)$ is equilateral, then $x = y$ and $x < a$, $x < b$ together with $x + ab = a + b$ imply $ab = a + b$ or $a = b$.

DEFINITION 2.3. An ordered set of n pairwise distinct elements $(a_1, a_2, \ldots, a_n)$ of a Brouwerian geometry is called an *equilateral n-circuit* if $a_1*a_2 = a_2*a_3 = \ldots = a_{n-1}*a_n = a_n*a_1$, and $n \geqslant 3$.

The preceding theorem shows that there is no equilateral 3-circuit in a Brouwerian geometry. More generally we have:

THEOREM 2.7. *A Brouwerian geometry contains no equilateral n-circuit for n odd.*

Proof. We first show that if a, b, c, d are four elements such that $a*b = b*c = c*d = y$, and $d*a = x$, then $x = y$, so (a, b, c, d) is an equilateral 4-circuit. Let $a*c = u$, $b*d = v$, then $x < u + y$ and by the properties of isosceles triangles $u < y$, $u < b$ and $v < y$, $v < c$, so $x < y$. Since $(u, x, y)T$ and $(v, x, y)T$ we obtain $y = u + y = u + x$, $y = v + y = v + x$ or $y = x + uv$. From $uv < bc$, $y < x + bc$ and $y - bc < x$, or $(y - b) + (y - c) < x$. Since $(b, c, y)T$, $y - b = c - b$, $y - c = b - c$, and $(c - b) + (b - c) < x$ or $y < x$. Thus $x = y$. The theorem is now evident, since the existence of an equilateral n-circuit with n odd would imply the existence of an equilateral $(n - 2)$-circuit and hence that of an equilateral 3-circuit. Examples show that equilateral n-circuits exist for all even n.

THEOREM 2.8. *A Brouwerian geometry is a chain if and only if all triangles are isosceles.*

Proof. If every triangle of a Brouwerian geometry L is isosceles, consider $\Delta(O, a, b)$ with $a \neq b$. Then $a*b = a$ or b. Suppose $a*b = a$. Then $b < a$ and L is a chain. Conversely, if L is a chain, suppose a, b, c, are pairwise distinct with $a < b < c$. Then $b*c = c - b = c$, $a*c = c - a = c$, and every triangle is isosceles.

The preceding theorems have been closely allied to the "uniqueness of solution" property. The following show how much of the associativity of the sym-

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metric difference in a Boolean geometry remains in the symmetric difference operation in a Brouwerian geometry.

DEFINITION 2.4. If x, y, z are the sides of a given triangle, then the triangle with vertices x, y, z is called the *first distance triangle* of the original. Similarly the triangle whose vertices are the sides of the first distance triangle is the *second distance triangle.*

In a Boolean geometry it follows at once from the associativity of the symmetric difference that every first distance triangle has the property of triangular fixity **(4)**. This means that each side is equal to the opposite vertex. This is no longer true for Brouwerian geometries; however, each side of a first distance triangle is under the opposite vertex, since if $x = b*c$, $y = a*c$, $z = a*b$, then $(x, y, z)T$, and $x*y < z$ by (3) of Theorem 2.5. Furthermore:

THEOREM 2.9. *In a Brouwerian geometry every second distance triangle has fixity.*

Proof. Define x, y, z as in the preceding paragraph and let $u = y*z$, $v = x*z$, $w = x*y$, $p = v*w$, $q = u*w$, $r = u*v$. It is sufficient to prove that $p = u$. We have noted that $p < u$, and next show that $u < p$. Since $(x, y, z)T$, $(x, w, y)T$ and $(x, v, z)T$, then by (4) of Theorem 2.5 we have $y - z = x - z = v - z$ and $z - y = x - y = w - y$. By (j) of Theorem 2.1 the inequality $w < z$ implies $v - z < v - w$, and $v < y$ implies $w - y < w - v$. Then $u = (y - z) + (z - y) < (v - w) + (w - v)$, or $u < p$.

THEOREM 2.10. *A Brouwerian geometry is a Boolean geometry if and only if every first distance triangle has fixity.*

Proof. The necessity has been cited earlier. Conversely, a Brouwerian geometry with the fixity property mentioned which is not a Boolean geometry would contain an isosceles triangle, say with sides x, x, y. Then $y = x*x = O$, and the triangle is degenerate.

DEFINITION 2.5. An autometrized space is called *regular* if $a*O = a$ for every element a of the space. The metric operation involved is also called regular. Clearly Brouwerian geometries and Boolean geometries are regular. It may be observed that every lattice admits the regular metric operation $a*b = a + b$, $a \neq b$; $a*a = O$.

DEFINITION 2.6. An autometrized space is called *distributive* if its distance lattice is distributive.

THEOREM 2.11. *In a regular distributive autometrized space* $(a*b) + ab = a + b$.

Proof. Identical with the proof of Corollary 2.5.1.

THEOREM 2.12. *A Brouwerian algebra is a Boolean algebra if and only if it admits a metric group operation. Furthermore, the operation must be the symmetric difference.*

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Proof. If a Brouwerian algebra is a Boolean algebra, it admits a metric group operation, namely the symmetric difference. As Elliott **(3)** has shown, this is indeed the only metric group operation admitted. Conversely, suppose ϕ is a metric group operation in a Brouwerian algebra L. Since $O\phi O = O$, the zero element of L is the group identity, and $a\phi O = a$. From $\Delta(O, a, b)$, $a - b < a\,\phi\, b$, $b - a < a\,\phi\, b$ by (2) of Theorem 2.5, and so $a*b < a\,\phi\, b$. Now $I\,\phi\, O = I$ and $I\,\phi(a\cdot \neg a) > I*(a\cdot\neg a) = I$, so $a\cdot\neg a = O$. Since $a + \neg a = I$, L is a Boolean algebra, and ϕ is the symmetric difference.

This theorem supersedes Theorem 2.4 and its corollary. Since any finite distributive lattice is a Brouwerian algebra, the above result implies that any finite distributive lattice which admits a metric group operation is a Boolean algebra. Whether this is true for arbitrary distributive lattices is an open question.

DEFINITION 2.7. A *subgeometry* of an autometrized lattice is a subset with the property that the distance between each pair of its points is an element of the subset. For example, every principal ideal in a Boolean geometry is a subgeometry, as is any two elements together with their distance and the O. Thus the subgeometry of a Boolean geometry generated by two elements contains at most four points. Since subgeometries of Boolean geometries are subgroups whose elements are all of order two, it is an easy matter to describe the subgeometries generated by a finite number of elements. For the case of a Brouwerian geometry the problem is more complicated and thus far we have solved it only for the case of two elements.

THEOREM 2.13. *A subgeometry generated by two elements of a Brouwerian geometry contains at most nine elements.*

Proof. Let a and b be the given elements, and define $c = a*b$, $d = a*c$, $e = b*c$, $f = b*d$, $g = a*e$, and $h = f*g$. These eight elements and O will form the desired subgeometry. The calculations of the distances which arise are considerably shortened if we observe that when a and b are interchanged then c is unaltered, d and e are interchanged, f and g are interchanged, while h is unaltered.

We first prove that the points (a, b, e, d) form an equilateral 4-circuit of side c. By reason of the observation of the preceding paragraph and the discussion of equilateral 4-circuits made in Theorem 2.7, it is sufficient to show that $a*d = c$. From $\Delta(a, b, c)$ we obtain the inequalities $d < b$, $e < a$. Since $(a, b, c)T$ and $(a, d, c)T$, then $c = (a - b) + (b - a) = (c - b) + (c - a) = c - ab$, $a*d = (a - d) + (d - a) = (c - d) + (c - a) = c - ad$. Now $ad < ab$ implies $c - ab < c - ad < c$, or $c = a*d$. To complete the calculation of the mutual distances for the five points, a, b, c, d, e observe that $\Delta(c, d, e)$ is the second distance triangle of $\Delta(O, a, b)$ so $c*e = d$, $c*d = e$, by Theorem 2.9.

The distances from f to each of the elements a, b, c, d, e will now be computed, and the distances from g found analogously. It will be shown that $f*d = b$,

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$f*c = c, f*e = e, f*b = d$ and $f*a = a$. First note that $f = b - d < b, g = a - e < a$. Since $(b, f, d)T$, $f*d = (f - d) + (d - f) = (b - d) + (d - f) = f + (d - f) = f + d = b + d = b$. $\Delta(a, b, d)$ being isosceles, $f < a$, and $f < c$. From $\Delta(d, f, c)$ we obtain $b*e < c - f$ or $c < c - f$. Thus $f*c = c$. Since $\Delta(b, d, e)$ is isosceles, $f < e$, and from $\Delta(c, e, f)$ we find $e = c*d < e - f$, or $f*e = e - f = e$. From $\Delta(f, b, e)$ we obtain $d = e*c < b - f$. But $b = f + d$ implies $b - f < d$, so $b*f = d$. Finally from $\Delta(f, a, e)$ we find $a = g*e < a - f$, or $f*a = a$.

The proof of the theorem is completed by computing the distances from h to each of the elements f, a, d, c, obtaining the distances to g, b, e by symmetry. It will be shown that $h*f = g$, $h*a = e$, $h*d = b$, and $h*c = c$. From $\Delta(f, e, g)$ there results $a = h + e, g = a - e < h, h < a$, and from $\Delta(f, d, g)$ the relations $b = h + d, f = b - d < h, h = f + g, h < b$. Now $f < e$ implies $g - e < g - f$, and $g = a - e = g - e$, so $g = g - f$. Then since $(f, g, h)T$, $g - f = h - f = g = h*f$. From $\Delta(a, f, h)$ we find $e = a*g < a - h$, and since $g < h$, $a - h < a - g$. From $(a, g, e)T, a - g = e - g$ so $e < a - h < e - g, h*a = a - h = e$. From $\Delta(f, d, h)$, $b = b*g < h*d$. Since $h*d = (h - d) + (d - h) < b$ then $h*d = b$. Finally from $\Delta(h, c, f)$ we find $c = c*g < c - h$, or $h*c = c$.

THEOREM 2.14. *The subgeometry generated by two comparable elements of a Brouwerian geometry contains at most six elements.*

Proof. Using the notation and result of Theorem 2.13, we suppose $a < b$, and prove that $f = O$, $h = g$, $d = b$. From $(a, b, c)T$, $(b, h, d)T$ and $(b, f, d)T$ there results $b = a + c = h + d = f + d = d + fh$, or $c - fh < d$. Since $c - f = c - h = c$, $c < d$, $h < c < d$ and $h - d = O$. But $f = b - d = h - d$ or $f = O$. Then $g = f*h = h$ and $d = b*f = b$.

COROLLARY 2.14.1. *If $a < b$, the six elements O, a, b, $a*b$, $b*(a*b)$ and $a*[b*(a*b)]$ form a subgeometry of a Brouwerian geometry.*

COROLLARY 2.14.2. *The six elements O, a, I, $\neg a$, $\neg\neg a$, and $a - \neg\neg a$ form a subgeometry of a Brouwerian geometry.*

Examples show that the maxima mentioned in Theorems 2.13 and 2.14 are attained. These theorems are reminiscent of a result due to Kuratowski **(6)**, who showed that the free closure algebra with one generator contains sixteen elements. A natural question to investigate would be whether a set of n elements of a Brouwerian geometry always generates a finite subgeometry. We conjecture that the subgeometry is finite and contains at most $(n + 1)^n$ elements. It is relatively easy to construct examples of Brouwerian geometries in which n elements generate a space containing precisely $(n + 1)^n$ elements.

3. General theorems: Metric betweenness. Blumenthal **(2)** has observed that a Boolean geometry is ptolemaic. A space is termed ptolemaic if the three products of the pairs of opposite distances for each four of its elements satisfy the triangle inequality. A Brouwerian geometry is also ptolemaic, but even more generally we have

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THEOREM 3.1. *Every distributive L-metrized space is ptolemaic.*

Proof. Choose the notation for the six distances determined by four points so that the products of pairs of opposite distances are ax, by, cz, where $(a, b, c)T$, $(a, y, z)T$, $(b, z, x)T$, and $(c, x, y)T$. Then $a < b + c$, $x < b + z$ imply $ax < b + cz$. Also, $a < y + z$, $x < y + c$, imply $ax < y + cz$, so $ax < by + cz$. The remaining inequalities are proved analogously.

COROLLARY 3.1.1. *Every Brouwerian L-metrized space is ptolemaic.*

COROLLARY 3.1.2. *Every Boolean L-metrized space is ptolemaic.*

THEOREM 3.2. *In any L-metrized space, the sum of a pair of sides of a triangle equals the sum of any other pair of sides of the same triangle.*

Proof. The same as for (1) of Theorem 2.5.

THEOREM 3.3. *In an autometrized chain, every triangle is isosceles.*

Proof. This follows from Theorem 3.2 and the fact that any three elements of a chain are pairwise comparable.

DEFINITION 3.1 An autometrized lattice is called *symmetric* if the distance between every two of its elements is equal to the distance between their sum and product. By Theorem 2.1 it is easy to see that Brouwerian geometries (and hence Boolean geometries) are symmetric.

THEOREM 3.4. *An autometrized lattice which is symmetric and contains no isosceles triangles is distributive.*

Proof. A non-distributive lattice must contain one of the special five element sublattices discussed by Birkhoff (**1**, p. 134). It is readily verified that the symmetric property implies in each case the existence of an isosceles triangle.

DEFINITION 3.2. In an L-metrized space, the element b is *metrically between* a and c if $d(a, b) + d(b, c) = d(a, c)$. The points (a, b, c) are said to be linear, and the relation is written $(a, b, c)M$.

It should be observed at once that this is not a betweenness relation in the usual sense, since it fails in many cases to have the so-called special inner point property, viz., $(a, b, c)M$ and $(a, c, b)M$ may both persist without having b and c coincide. However, for ease of locution, it is convenient to adopt the above terminology. It is clear that the relation does have the other basic betweenness property, namely, symmetry in the outer points, i.e. $(a, b, c)M$ if and only if $(c, b, a)M$.

THEOREM 3.5. *Three linear points of an L-metrized space fail to have the special inner point property if and only if they are the vertices of an isosceles triangle.*

Proof. If $\Delta(a, b, c)$ has $d(a, b) = d(a, c)$, then $d(b, c) < d(a, b)$ and $(a, b, c)M$, $(a, c, b)M$. Conversely, if these relations hold, then by Theorem 3.2, $d(a, b) = d(a, c)$.

DEFINITION 3.3. If a betweenness relation R has the property that $(a, b, c)R$ and $(a, x, b)R$ imply $(x, b, c)R$, the relation is said to have transitivity t_1. If the same two relations imply $(a, x, c)R$, the relation is said to have transitivity t_2. **(10)**.

THEOREM 3.6. *Metric betweenness has transitivity t_2.*

Proof. If $*$ denotes the metric operation, then $(a, b, c)M$ and $(a, x, b)M$ imply $(a*b) + (b*c) = a*c$ and $(a*x) + (x*b) = a*b$. Then $(a*x) + (x*b) + (b*c) = (a*c)$. Since $(x*b) + (b*c) > (x*c)$, then $a*c > (a*x) + (x*c)$, and by the triangle inequality, $a*c = (a*x) + (x*c)$, or $(a, x, c)M$. In general, metric betweenness fails to have t_1, an obvious situation in a space containing an isosceles triangle. Autometrized lattices, however, do exist in which t_1 fails for four points, no three of which determine an isosceles triangle.

DEFINITION 3.4. An element b of a lattice is *lattice between* a and c, written $(a, b, c)L$, if and only if $ab + bc = b = (a + c)(b + c)$. This relation has transitivity t_1, **(10)**, but does not, in general, have t_2. It is a betweenness relation in the usual sense. We note that in a distributive lattice, $(a, b, c)L$ if and only if $ac < b < a + c$. We shall say that metric and lattice betweenness coincide in an autometrized lattice, provided $(a, b, c)M$ if and only if $(a, b, c)L$.

THEOREM 3.7. *In an autometrized lattice L, metric and lattice betweenness coincide if and only if* (1) *metric betweenness has t_1,* (2) *L is symmetric,* (3) *$a < b < c$ implies $(a, b, c)M$.*

Proof. We employ a theorem due to Pitcher and Smiley **(10**, Theorem 10.1**)**. In verifying the hypotheses of that theorem, we observe that (1) implies that L is free of isosceles triangles and so by Theorem 3.5 that metric betweenness has the special inner point property. It is also readily shown that (2) and (3) imply $(a, a + b, b)M$ and $(a, ab, b)M$. Conversely, if metric and lattice betweenness coincide, (1) and (3) are immediate. The symmetric property is shown as follows: Consider the quadruple $(a, b, ab, a + b)$. Then $(a + b, a, ab)L$ and $(a + b, b, ab)L$ imply $[(a + b)*a] + (a*ab) = [(a + b)*b] + (b*ab) = (a + b)*ab$. Similarly, $(a, a + b, b)L$ and $(a, ab, b)L$ imply $[(a + b)*a] + [(a + b)*b] = (a*ab) + (b*ab) = a*b$. Then $(a + b)*ab = a*b$, as is evident from summing the terms.

COROLLARY 3.7.1. *An autometrized lattice in which metric and lattice betweenness coincide is distributive.*

Proof. By Theorems 3.7 and 3.4.

THEOREM 3.8. (Ellis). *In a Boolean geometry, metric and lattice betweenness coincide.*

Proof. We present a shorter version. If $(a, b, c)L$, then $ac < b < a + c$ and $b' < a' + c'$. Now $(a*b) + (b*c) = ab' + a'b + bc' + b'c = (a + c)b' + b(a' + c') < (a + c)(a' + c') = a*c$, and the triangle inequality yields equality, so that

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$(a, b, c)M$. Conversely, $(a, b, c)M$ implies $(a + c)b' + b(a' + c') = ac' + a'c$. Multiplying by ac and $a'c'$ respectively, we obtain $acb' = O$ and $a'c'b = O$, so that $ac < b < a + c$. Thus $(a, b, c)L$. Theorem 3.8 is also a consequence of Theorem 3.7.

Since a non-Boolean Brouwerian geometry contains an isoceles triangle, it is evident from Theorem 3.5 that metric and lattice betweenness do not, in general, coincide in Brouwerian geometries. Indeed, the following theorems show that the coincidence of metric and lattice betweenness plus regularity serve to characterize Boolean geometries among Brouwerian geometries.

THEOREM 3.9. *An autometrized lattice with an I is a Boolean geometry if and only if it is regular, and metric and lattice betweenness coincide.*

Proof. The necessity is implied by Theorem 3.8. To show the sufficiency, let $a*I = x$, $I*x = y$, and consider the points (O, I, a, x, ax). Then $x + y = I$, $a + x = I$, and $(a, I, x)L$ imply $x + y = a*x = I$. By the symmetric property $(a*x) = (a + x)*(ax) = (I*ax) = I$. Since $I*O = I$, the absence of isosceles triangles implies $ax = O$. Thus the lattice is complemented, hence, being distributive, is a Boolean algebra. Since it is, *a fortiori*, a Brouwerian algebra, consider points (O, a, b, I), for which $a*I = a'$, $b*I = b'$. Then $(a*b) + ab = a + b$ implies $a*b < a + b$. Since $a*b < a' + b'$, it follows that $a*b < ab' + a'b$. Moreover, $(a + b) - ab < a*b$, or $(a - b) + (b - a) = ab' + a'b < a*b$. Thus $a*b$ is the symmetric difference.

COROLLARY 3.9.1. *A lattice with an I is a Boolean algebra if and only if it admits a metric group operation under which metric and lattice betweenness coincide.*

THEOREM 3.10. *A lattice with an I is a Boolean algebra if and only if it admits a metrization such that the space is of constant width and metric and lattice betweenness coincide.*

Proof. The necessity is obvious. To prove the sufficiency, consider the four elements (O, I, a, b). Then $(a*O) + (a*I) = (b*O) + (b*I) = (I*O)$. Since $a*b < (a*I) + (b*I)$, then $a*b < I*O$, and $I*O$ is the maximum distance which occurs. Since the space is of constant width, corresponding to any element a there is an element b such that $a*b = I*O$. Then $(a + b)*(ab) = I*O$, and $(a + b, ab, O)L$ implies $(a + b)*O = I*O$. Since no triangle can be isosceles, $a + b = I$, and $I*ab = I*O$ implies $ab = O$, and the lattice is a Boolean algebra.

COROLLARY 3.10.1. *A finite autometrized lattice in which metric and lattice betweenness coincide is a Boolean algebra.*

Proof. By Theorem 3.7 the space contains no isosceles triangle and must have constant width I.

Ellis has observed in (4) and (5) that the group of motions of a Boolean geometry is simply transitive. This means that for any two points a, b of the geometry, there is a motion f such that $b = f(a)$. A motion is a 1-1 mapping of

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the space onto itself which preserves distance. It is clear at once that the group of motions of a Brouwerian geometry is not simply transitive, and in fact,

THEOREM 3.11. *A Brouwerian geometry is a Boolean geometry if and only if its group of motions is simply transitive.*

Proof. If a Brouwerian geometry L has a simply transitive group of motions, consider the motion carrying I into O. If $x \to y$, $I*x = \neg x = O*y$, and $x \to \neg x$. Then, $O \to I$, $x \cdot \neg x \to \neg(x \cdot \neg x) = I$, or $x \cdot \neg x = O$, and L is a Boolean geometry. The converse is evident from the remark preceding Theorem 3.11.

4. Congruence order. A space S is said to have congruence order k relative to a class of spaces M containing S, provided any space of M is congruent (isometric) to a subset of S whenever each k of its points are, and k is the smallest number with this property. This concept is due to Menger (**8**, p. 116) who proved, for example, that the congruence order of E_n relative to the class of metric spaces is $n + 3$. In (**4**) it was shown that the congruence order of a Boolean geometry relative to the class of autometrized spaces is three. We have not as yet established the congruence order of Brouwerian geometries, but the following theorems bear on this general question.

THEOREM 4.1. *If the distance function of an autometrized space is a group operation, the congruence order of the space, relative to the class of L-metrized spaces is three.*

Proof. Let L denote the autometrized space whose distance function $*$ is a group operation, and suppose S is any L-metrized space with the property that every three of its points can be congruently embedded in L. Consider any fixed element a of S, x an arbitrary element, and let $d(a, x) = u$, where $u \in L$. If $\bar{a}$ is any point of L, then there exists uniquely a point $\bar{x} \in L$ such that $\bar{a}*\bar{x} = u$. The mapping $x \to \bar{x}$ of S onto a subset of L is one to one, since if $y \in S$, $y \neq x$, and $d(a, y) = u$, then the isosceles triangle with vertices a, x, y, is congruent, by hypothesis, to an isosceles triangle in L, a contradiction. In particular, if $x = a$, then $u = O$, and $\bar{x} = \bar{a}$, so that $a \to \bar{a}$. We next prove that distances are preserved. Let $y \in S$ and $d(a, y) = v$. Then $\bar{a}*\bar{y} = v$. Suppose $d(x, y) = w$, $w \in L$. Then a triangle exists in L with sides u, v, w. However, if two sides of a triangle in L are respectively equal to two sides of another triangle in L, then by the group property, the third sides are equal. Thus, $\bar{x}*\bar{y} = w$. The example of a three point space with distances all equal to the same non-zero element of L shows that the congruence order is not two, since L contains no equilateral triangle.

COROLLARY 4.1.1 (Ellis). *The congruence order of a Boolean geometry relative to the class of L-metrized spaces is three.*

THEOREM 4.2. *A Brouwerian geometry is a Boolean geometry if and only if it has congruence order three relative to the class of L-metrized spaces.*

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Proof. As a result of corollary 4.1.1, we need only show that a Brouwerian geometry L with congruence order three is a Boolean geometry. Suppose $x \cdot \urcorner x \neq O$ and consider the L-metrized space consisting of the four distinct elements a, b, c, d, with $a*c = b*d = x \cdot \urcorner x$, and the remaining distances equal to I. Each three points are congruently embeddable on the points $O, I, x \cdot \urcorner x$ of the Brouwerian geometry, and by hypothesis, the entire space is so embeddable. This configuration, however, is impossible in a Brouwerian geometry, for if a, b, c, d map respectively into a_1, b_1, c_1, d_1, of L, then $x \cdot \urcorner x < a_1, x \cdot \urcorner x < c_1$, by properties of isosceles triangles, and $a_1 c_1 + x \cdot \urcorner x = a_1 + c_1$ implies $a_1 c_1 = a_1 + c_1$, or $a_1 = c_1$. Thus $x \cdot \urcorner x = O$, a contradiction, and L is a Boolean geometry.

THEOREM 4.3. *A Brouwerian chain has congruence order four relative to the class of L-metrized spaces.*

Proof. We recall that any chain with O, I is a Brouwerian algebra and $a \dot{-} b = a$ if $a > b$. Let C be such a chain metrized by the symmetric difference, i.e. a Brouwerian chain. Then (1) every triangle is isosceles, (2) if $a > b > c > d$, then $a*b = a*c = a*d = a$, $b*c = b*d = b$, $c*d = c$, (3) opposite sides of a quadruple cannot be equal, since a Brouwerian geometry contains no equilateral triangle.

Let S be any L-metrized space with the property that every four points of S are congruently embeddable in C. If S contains a point O' which is not the vertex of an isosceles triangle, map O' into the O of C, and every point x' of S into its distance x from O'. This establishes a one to one mapping $x' \rightarrow x$ of S onto a subset of C which is also a congruence, as we now prove. If x' and y' are two elements of S whose distances from O' are x and y, then $d(x', y')$ equals x or y, according as $x > y$ or $y > x$, since by hypothesis, the three points O', x', y', are congruently embeddable in C.

If S contains no point O' as described above, then every point is the vertex of an isosceles triangle. However, two isosceles triangles with the same vertex must have their legs equal, otherwise a quadruple including the vertex is determined, which maps into a quadruple in C not satisfying (3) above. We therefore map each point x' of S into x, the leg of an isosceles triangle with vertex x'. This is a one to one mapping of S onto a subset of C, since if x' and y' are distinct points of S, each the vertex of an isosceles triangle of leg x, then a quadruple including x' and y' is determined having opposite sides equal to x, a contradiction, since this implies a similar configuration in C, violating (3).

We next show that the mapping is a congruence. Consider two distinct points x', y' of S and their correspondents x, y under the mapping. We may assume $x > y$, so that $x*y = x$. Let $d(x', u') = x$, $d(y', u') = z$. Then $z \neq x$ by (3), so that $d(x', y') = x$ or z. If $d(x', y') = z$, then $z = y$ by definition of the mapping and $x < y$, a contradiction. Thus $d(x', y') = x = x*y$. We have shown that the congruence order is at most four.

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To show that the congruence order is not less than four, consider a four point L-metrized space S with two opposite distances equal to a, and the remaining distances equal to b where $a < b$ and a, b are distinct elements of C. Each three points of S are congruently embeddable in C, but the entire space is not, since (3) is violated. This completes the proof of the theorem. Examples show that a Brouwerian geometry may have congruence order four without being a chain.

This paper leaves unanswered three principal problems which we are continuing to investigate, namely:

(1) How many elements are contained in a subgeometry of a Brouwerian geometry generated by more than two elements?

(2) Is an autometrized lattice with an I in which metric and lattice betweenness coincide necessarily a Boolean algebra?

(3) What is the congruence order of Brouwerian geometries relative to the class of L-metrized spaces?

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