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Tree structure of spectra of spectral Moran measures with consecutive digits

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Abstract. Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of integers larger than 1. We will study the harmonic analysis of the equal-weighted Moran measures $\mu_{\{b_n\},\{\mathcal{D}_n\}}$ with $\mathcal{D}_n = \{0, 1, 2, \ldots, q_n - 1\}$, where q_n divides b_n for all $n \geq 1$. In this paper, we first characterize all the maximal orthogonal sets of $L^2(\mu_{\{b_n\},\{\mathcal{D}_n\}})$ via a tree mapping. By this characterization, we give some sufficient conditions for the maximal orthogonal set to be an orthonormal basis.

1 Introduction

Let μ be a compactly supported Borel probability measure on $\mathbb{R}^d$. We say that μ is a *spectral measure* if there exists a countable set Λ of $\mathbb{R}^d$ such that $E(\Lambda) = \{e^{2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\}$ forms an orthonormal basis for $L^2(\mu)$. In this case, Λ is called a *spectrum* of μ. Spectral theory for the Lebesgue measure on sets has been studied extensively since it is initialed by Fuglede in 1974 [16].

There exist probability measures that are not the restriction of the Lebesgue measure to bounded sets, but they admit spectra. In 1998, Jorgensen and Pedersen [18] constructed the first example of a singular, non-atomic spectral measure $\mu_{4,\{0,2\}}$ (i.e., the one-fourth standard Cantor measure) and proved that the set

$$\Lambda = \Big\{ \sum_{k=0}^{n} 4^k d_k : d_k \in \{0,1\}, n \geq 0 \Big\}$$

is a spectrum of $\mu_{4,\{0,2\}}$. Following this discovery, more interesting spectral measures were found and new spectra for $\mu_{4,\{0,2\}}$ were found (see [2, 5, 6, 8, 10, 19, 20]). Subsequently, some singular phenomena different from the spectral theory of Lebesgue measures were discovered. For example, there exists only one spectrum with containing 0 for $L^2[0,1]$, while a given singular spectral measure μ has more than one spectrum which is not obtained by the translations of each other [10, 14]. Another surprising and interesting difference is that the Fourier expansions of functions in $L^2(\mu_{4,\{0,2\}})$ with respect to different spectra may have different convergence properties. Strichartz [22, 23] proved that for any continuous function

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on $\mathbb{R}$, its Fourier expansion with respect to Λ is uniformly convergent; but Dutkay et al. showed in [11] that there exists a continuous function on $\mathbb{R}$ whose Fourier expansion with respect to the spectrum 17Λ is divergent at 0. Based on the above exotic phenomena, a natural question is raised:

For a given spectral measure, can we find all spectra?

It is quite challenging to "characterize" all the spectra (no example is known with this property). Motivated by the above problem, many researchers concentrated their work on investigating self-similar/self-affine/Moran spectral measures and the construction of their spectra from various aspects (see [1, 3, 4, 7–10, 12, 14, 17, 21, 25] and the references therein).

Nowadays, there are a few of literature on this topic to construct new spectra. Most of this literature deals with the issue of self-similar/self-affine measures. Various of new spectra have been constructed for the measure $\mu_{4,\{0,2\}}$ by Dutkay et al. [10], for self-similar measure with consecutive digits by Dai et al. [7], Dai [5], for infinite Bernoulli convolution by Li [20], Fu et al. [14], for certain self-affine measures on $\mathbb{R}^d$ by Deng et al. [9], etc. Besides these results, some answers for spectral Moran measures have been given in [15, 24]. Motivated by these, the main focus of this paper is to study the classification of spectra of the so-called Moran measures with consecutive digits, where $\mu_{4,\{0,2\}}$ and Cantor measures with consecutive digits are special cases.

Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of integers larger than 1 and $\mathcal{D}_n = \{0, 1, 2, \ldots, q_n - 1\}$ for each $n \geq 1$, there exists a Borel probability measure $\mu_{\{b_n\},\{\mathcal{D}_n\}}$, which is defined by the following infinite convolutions of finite measures:

$$\mu_{\{b_n\},\{\mathcal{D}_n\}} = \delta_{b_1^{-1}\mathcal{D}_1} * \delta_{(b_1 b_2)^{-1}\mathcal{D}_2} * \ldots, \tag{1.1}$$

where $rE = \{rx : x \in E\}$, $\delta_E = \frac{1}{\#E}\sum_{e\in E}\delta_e$ ($\#E$ is the cardinality of the finite set E, δ_e is the Dirac measure at the point $e \in \mathbb{R}$) and the convergence is in a weak sense. Note that $\mathcal{D}_n = \{0, 1, 2, \ldots, q_n - 1\}$, we call the measure $\mu_{\{b_n\},\{\mathcal{D}_n\}}$ a Moran measure with consecutive digits. In particular, when $b_n = b$ and $\mathcal{D}_n = \{0, 1, \ldots, q-1\} := \mathcal{D}$ for all $n \in \mathbb{N}$, then the measure $\mu_{\{b_n\},\{\mathcal{D}_n\}}$ is reduced to the *self-similar measure* $\mu_{b,\mathcal{D}}$ (see [13]).

Let us first recall the known results on the spectrality of the measure $\mu_{\{b_n\},\{\mathcal{D}_n\}}$. In the self-similar case, Dai, He, and Lai [7] showed that the measure $\mu_{b,\mathcal{D}}$ is spectral if and only if $q|b$. Recently, An and He [2] proved that if $q_n|b_n$ for all $n \geq 1$, then $\mu_{\{b_n\},\{\mathcal{D}_n\}}$ is a spectral measure. Continuing the above research, the main goal of this paper is to investigate the structure of spectra for the Moran spectral measure $\mu_{\{b_n\},\{\mathcal{D}_n\}}$.

The starting point of our approach is to analyze the precise structure of the zero set of the Fourier transform $\widehat{\mu}_{\{b_n\},\{\mathcal{D}_n\}}$, and then we introduce the maximal orthonormal sets, as candidates for the spectra of the measure $\mu_{\{b_n\},\{\mathcal{D}_n\}}$. We shows that there is one-to-one correspondence between a maximal orthonormal set and a tree mapping (Theorem 1.2). More precisely, we decomposed the maximal orthonormal sets of the measure $\mu_{\{b_n\},\{\mathcal{D}_n\}}$ using $\mathcal{C}_j$-adic expansion and put them into a labeling of the tree.

Definition 1.1 Let $b_n = q_n r_n$ for all $n \geq 1$, we say that τ is a *tree mapping* if it is a map $\tau : \mathcal{D}_{0,*} \to \mathbb{Z}$ which satisfies:

(i) $\tau(0^n) = 0$ for all $n \geq 1$,

(ii) for all $n \geq 1$, $\tau(\sigma_1 \dots \sigma_n) \in (\sigma_n + q_n\mathbb{Z}) \bigcap \{-1, 0, 1, \dots, q_n r_{n+1} - 2\}$,

(iii) for any word $\sigma \in \mathcal{D}_{0,n}(n \geq 1)$, there exists $\sigma' \in \mathcal{D}_{n,\infty}$ such that $\tau\big((\sigma\sigma')|_j\big) = 0$ for all j sufficiently large.

Given a tree mapping τ, we define the following sets:

$$\mathcal{D}(\tau) = \big\{\sigma \in \mathcal{D}_{0,\infty} : \tau(\sigma|_m) = 0 \text{ for all } m \text{ large enough}\big\}$$

and

$$\Lambda(\tau) = \Big\{\sum_{k=1}^{\infty} \tau(\sigma|_k) r_k b_1 \dots b_{k-1} \big| \sigma \in \mathcal{D}(\tau)\Big\}.$$

It will play a special role in the following theorem, which gives a characterization of the maximal orthogonal set of $L^2(\mu_{\{b_n\},\{\mathcal{D}_n\}})$.

Theorem 1.2 *The set Λ with $0 \in \Lambda$ is a maximal orthonormal set of $\mu_{\{b_n\},\{\mathcal{D}_n\}}$ if and only if there exists a tree mapping τ such that $\Lambda = \Lambda(\tau)$.*

While a maximal orthonormal set is not necessarily a spectrum since it may lack of completeness in $L^2(\mu_{\{b_n\},\{\mathcal{D}_n\}})$. So what we need to do is to investigate under what conditions on τ such that the set $\Lambda(\tau)$ to be a spectrum for the measure $\mu_{\{b_n\},\{\mathcal{D}_n\}}$. We first introduce a good path to ensure the above maximal orthogonal set $E\big(\Lambda(\tau)\big)$ to be a complete set in $L^2(\mu_{\{b_n\},\{\mathcal{D}_n\}})$.

Theorem 1.3 *Let $\{q_n, b_n\}_{n=1}^{\infty}$ be a sequence of positive integers larger than 1 with $q_n | b_n$ for all $n \in \mathbb{N}$ and $\sup_{n\geq 1}\{b_n\} < \infty$. Let τ be a tree mapping of $\mu_{\{b_n\},\{\mathcal{D}_n\}}$. Suppose there is a constant integer $N > 0$ such that for any word $\sigma \in \mathcal{D}_{0,k}(k \geq 1)$, there exists $\sigma' \in \mathcal{D}_{k,\infty}$ such that $\sigma\sigma' \in \mathcal{D}(\tau)$ and one of the following two conditions is satisfied:*

(i) $\tau\big((\sigma\sigma')|_{k+j}\big) = 0$ *for all* $j \geq 1$;

(ii) $\tau\big((\sigma\sigma')|_{k+j}\big) = 0$ *for all* $1 \leq j \leq n$, $\tau\big((\sigma\sigma')|_{k+n+1}\big) \in q_{k+n+1}\mathbb{Z}\backslash\{0\}$ *and*

$$\max\Big\{j \geq 1 : \tau\big((\sigma\sigma')|_{k+n+j}\big) \neq 0\Big\} \leq N. \tag{1.2}$$

Then $\Lambda(\tau)$ is a spectrum of $\mu_{\{b_n\},\{\mathcal{D}_n\}}$.

We remark that (1.2) shows that the *efficient length* of the word $\sigma\sigma'$ is uniformly bounded, that is to say, σ' is crucial in the proof of the completeness. However, the *length* of $\sigma\sigma'$ i.e., $\max\Big\{j \geq 1 : \tau\big((\sigma\sigma')|_{k+j}\big) \neq 0\Big\}$ may be not uniformly bounded since it also depends on the choice of n in 0^n.

Next, we consider other digits that can be used for the base expansion of the integers in the candidate set Λ, and give a sufficient condition when these will generate spectra for the measure $\mu_{\{b_n\},\{\mathcal{D}_n\}}$.

Definition 1.4 We say that τ is a *generalized tree mapping* if it is a mapping $\tau : \mathcal{D}_{0,*} \to \mathbb{Z}$, which satisfies:

(i) $\tau(\sigma_1\sigma_2 \dots \sigma_n) = 0$ if $\sigma_n = 0$;

(ii) $\tau(\sigma_1\sigma_2 \dots \sigma_n) \in \sigma_n + q_n\mathbb{Z}$ if $\sigma_n \neq 0$.

We say that $\sigma \in \mathcal{D}_{0,\infty}$ is a *useful word* about the generalized tree mapping τ if there exists an integer $\tau^{\diamond}(\sigma) \in \mathbb{Z}$ satisfying

$$\tau^{\diamond}(\sigma) \equiv \sum_{k=1}^{n} \tau(\sigma|_k) r_k B_{k-1} \pmod{r_{n+1}B_n}, \qquad \text{for all } n \geq 1.$$

Given a generalized tree mapping τ, we let

$$\Lambda^{\diamond}(\tau) = \{\tau^{\diamond}(\sigma) | \sigma \text{ is a useful word}\}.$$

Theorem 1.5 *Let $\{q_n, b_n\}_{n=1}^{\infty}$ be a sequence of positive integers larger than 1 with $q_n | b_n$ for all $n \in \mathbb{N}$ and $\sup_{n\geq 1}\{b_n\} < \infty$. Let τ be a generalized tree mapping,*

(i) *$\Lambda^{\diamond}(\tau)$ is a maximal orthogonal set of $\mu_{\{b_n\},\{\mathcal{D}_n\}}$;*

(ii) *if $\sup_{\sigma\in\mathcal{D}_{0,*}} |\tau(\sigma)| < \infty$, then $\Lambda^{\diamond}(\tau)$ is a spectrum of $\mu_{\{b_n\},\{\mathcal{D}_n\}}$.*

The paper is organized as follows. In Section 2, we recall some preliminary results on basic criterion for spectral measures to prove our main results in the sequel, and characterize all the maximal orthonormal sets Λ for $L^2(\mu_{\{b_n\},\{\mathcal{D}_n\}})$. Theorem 1.3 is proved in Section 3. In Section 4, the proof of Theorem 1.5 is given.

2 Preliminaries and maximal orthogonal set

In this section, we first give some necessary definitions and facts that we need in the proof of the main theorem. Subsequently, we characterize all the maximal orthogonal sets Λ for $L^2(\mu_{\{b_n\},\{\mathcal{D}_n\}})$ via a tree mapping (see Definition 1.1).

2.1 A criterion for spectral measures

Let μ be a probability measure with compact support on $\mathbb{R}$. As usual, we define the Fourier transform of the measure μ,

$$\widehat{\mu}(\xi) = \int e^{-2\pi i\xi x} d\mu(x). \tag{2.1}$$

A countable set $\Lambda \subseteq \mathbb{R}$ is called an *orthonormal set/ maximal orthonormal set/spectrum*, respectively, of μ if $E(\Lambda) := \{e^{2\pi i\lambda x} : \lambda \in \Lambda\}$ is an orthonormal set/maximal orthogonal set/orthonormal basis, respectively, for $L^2(\mu)$. It is easy to show that Λ is an orthogonal set of μ if and only if $\widehat{\mu}(\lambda_i - \lambda_j) = 0$ for any $\lambda_i \neq \lambda_j \in \Lambda$, which is equivalent to

$$(\Lambda - \Lambda) \setminus \{0\} \subseteq \mathcal{Z}(\widehat{\mu}),$$

where $\mathcal{Z}(f) := \{\xi : f(\xi) = 0\}$ is the set of the roots of the function $f(\xi)$.

Combining (1.1) and (2.1), we can conclude that

$$\widehat{\mu}_{\{b_n\},\{\mathcal{D}_n\}}(\xi) = \prod_{j=1}^{\infty} M_{\mathcal{D}_j}\Big(\frac{\xi}{b_1 b_2 \dots b_j}\Big),$$

where

$$M_{\mathcal{D}_j}(\xi) = \frac{1}{q_j}\Big(1 + e^{-2\pi i\xi} + \dots + e^{-2\pi i(q_j-1)\xi}\Big).$$

A direct calculation shows that

$$\mathcal{Z}(M_{\mathcal{D}_j}) = \left\{\frac{a}{q_j} : q_j \nmid a,\ a \in \mathbb{Z}\right\},$$

where $q_j \nmid a$ means that q_j does not divide a. Hence, the zero of $\widehat{\mu}_{\{b_n\},\{\mathcal{D}_n\}}$ is given by

$$\mathcal{Z}(\widehat{\mu}_{\{b_n\},\{\mathcal{D}_n\}}) = \bigcup_{j\geq 1}\left\{r_j B_{j-1} a : a \in \mathbb{Z} \text{ and } q_j \nmid a\right\}, \tag{2.2}$$

where $r_j = b_j/q_j$ for all $j \geq 1$ and $B_j = b_0 b_1 \cdots b_j$ with $b_0 = 1$.

To this end, we introduce a fundamental criterion for spectral measures, which is a direct application of Stone–Weierstrass Theorem and Parseval's identity. For any $\xi \in \mathbb{R}$, denote

$$Q(\xi) = \sum_{\lambda\in\Lambda} |\widehat{\mu}(\xi+\lambda)|^2.$$

Theorem 2.1 [18] *Let μ be a Borel probability measure with compact support on $\mathbb{R}$, and let $\Lambda \subseteq \mathbb{R}$ be a countable subset. Then:*

(i) *Λ is an orthonormal set of μ if and only if $Q(\xi) \leq 1$ for any $\xi \in \mathbb{R}$;*
(ii) *Λ is a spectrum of μ if and only if $Q(\xi) \equiv 1$ for any $\xi \in \mathbb{R}$;*
(iii) *$Q(\xi)$ is an entire function in the complex plane if Λ is an orthogonal set of μ.*

2.2 Maximal orthogonal sets

We start with some notations for simplicity. Fix such an integer $n \geq 0$, let

$$\mathcal{D}_{n,k} = \{\sigma_{n+1}\sigma_{n+2}\ldots\sigma_{n+k} : \sigma_j \in \mathcal{D}_j, n+1 \leq j \leq n+k\}$$

be the set of all words with length k. We adopt that $\mathcal{D}_{n,*} = \bigcup_{k=1}^{\infty}\mathcal{D}_{n,k}\bigcup\{\varnothing\}$ is the set of all the finite words beginning with $\mathcal{D}_{n+1}$ and $\mathcal{D}_{n,\infty} = \{\sigma_{n+1}\sigma_{n+2}\ldots : \sigma_k \in \mathcal{D}_k, k \geq n+1\}$ is the set of all the infinite words beginning with $\mathcal{D}_{n+1}$. For any $\sigma = \sigma_{n+1}\sigma_{n+2}\ldots\sigma_{n+k} \in \mathcal{D}_{n,*}$, we use $|\sigma| = k$ to be its length and $\sigma|_j := \sigma_{n+1}\sigma_{n+2}\ldots\sigma_{n+j}(1 \leq j \leq |\sigma|)$. For any $\sigma \in \mathcal{D}_{n,*}$ and $\sigma' \in \mathcal{D}_{n',*}\bigcup\mathcal{D}_{n',\infty}$, the word $\sigma\sigma'$ is their nature conjunction. In particular, $\varnothing I = I$ and $0^\infty = 000\ldots$.

For simplicity, we set $b_0 = 1$ and $B_k = b_0 b_1 b_2 \ldots b_k$ for all $k \geq 1$. Given a tree mapping τ, recall that

$$\mathcal{D}(\tau) = \left\{\sigma \in \mathcal{D}_{0,\infty} : \tau(\sigma|_m) = 0 \text{ for all } m \text{ large enough}\right\}. \tag{2.3}$$

For any $\sigma \in \mathcal{D}(\tau)$, we let

$$\tau(\sigma) = \sum_{k=1}^{\infty}\tau(\sigma|_k) r_k B_{k-1} \quad \text{and} \quad \Lambda(\tau) = \{\tau(\sigma) | \sigma \in \mathcal{D}(\tau)\}, \tag{2.4}$$

which plays a special role in Theorem 1.2.

Before giving a proof of Theorem 1.2, we give the following lemma.

Lemma 2.2 *Let $\mathcal{C}_n = \{-1, 0, 1, \ldots, q_n r_{n+1} - 2\}$ and $\boldsymbol{C}_n = \mathcal{C}_1 \times \cdots \times \mathcal{C}_n$ for all $n \geq 1$. Then for any $k \in \mathbb{Z}(k \neq 0)$, there exists a unique word $\sigma = \sigma_1\sigma_2\ldots\sigma_n \in \boldsymbol{C}_n$ with $\sigma_n \neq 0$,*

such that

$$k = \sigma_1 + \sigma_2 r_2 q_1 + \cdots + \sigma_n r_n q_1 b_2 \ldots b_{n-1} := \pi(\sigma).$$

Moreover, if $k = 0$, *then* $\sigma = \sigma_1 = 0$.

Proof For any $k \in \mathbb{Z}$ and $|k| < q_1 r_2$, let

$$\sigma = \begin{cases} 1(-1), & k = -(q_1 r_2 - 1), \\ (q_1 r_2 + k)(-1), & -(q_1 r_2 - 1) < k < 0, \\ k, & 0 \le k < q_1 r_2 - 1, \\ -11, & k = q_1 r_2 - 1, \end{cases}$$

then $k = \pi(\sigma)$. When $|k| > q_1 r_2$, then k can be decomposed uniquely as $k = c_1 + k_1 q_1 r_2$, where $c_1 \in \mathcal{C}_1$. If $|k_1| < q_2 r_3$, then k_1 has to be decomposed as in the previous step. Otherwise, we further decompose k_1 in a similar way and we get $|k_n| < q_{n+1} r_{n+2}$ after finite number k steps. The expansion is unique since each decomposition is unique. ∎

Now, we can give the proof of Theorem 1.2.

Proof of Theorem 1.2 Suppose $\Lambda = \Lambda(\tau)$ for some tree mapping τ. We show that it is a maximal orthogonal of $\mu_{\{b_n\},\{\mathcal{D}_n\}}$. To see this, we firstly show the orthogonality of Λ. Pick two distinct $\lambda, \lambda' \in \Lambda$, by the definition of $\Lambda(\tau)$, we can find $\sigma \neq \sigma' \in \mathcal{D}(\tau)$ such that

$$\lambda = \sum_{j=1}^{\infty} \tau(\sigma|_j) r_j B_{j-1} \quad \text{and} \quad \lambda' = \sum_{j=1}^{\infty} \tau(\sigma'|_j) r_j B_{j-1}.$$

Let k be the first index such that $\sigma|_k \neq \sigma'|_k$. Then, for some integer M, we can write

$$\lambda - \lambda' = r_k B_{k-1} \Big(\tau(\sigma|_k) - \tau(\sigma'|_k) + M q_k \Big).$$

By (ii) in Definition 1.1, $\tau(\sigma|_k)$ and $\tau(\sigma'|_k)$ are in distinct residue classes of q_k. This implies $\lambda - \lambda' \in \mathcal{Z}(\widehat{\mu}_{\{b_n\},\{\mathcal{D}_n\}})$ and then $\Lambda(\tau)$ is orthogonal.

With respect to the maximality of the orthogonal set Λ, the proof is by contradiction. Suppose $\theta \in \mathbb{R} \backslash \Lambda$ and θ is orthogonal to all elements in Λ. Since $0 \in \Lambda$, it gets that $\theta = \theta - 0 \in \mathcal{Z}(\widehat{\mu}_{\{b_n\},\{\mathcal{D}_n\}})$. Hence, by (2.2), there exists $k \in \mathbb{Z}$ such that $\theta = r_k B_{k-1} a$, where $a \in \mathbb{Z}$ and q_k does not divide a. By Lemma 2.2, it yields that

$$a = \varepsilon_k + \varepsilon_{k+1} q_k r_{k+1} + \cdots + \varepsilon_{k+l} q_k r_{k+l} b_{k+1} b_{k+2} \ldots b_{k+l-1},$$

where $\varepsilon_j \in \mathcal{C}_j (k \le j \le k+l)$ and q_k does not divide ε_k. Consequently, it can infer that

$$\theta = r_k B_{k-1} a = \varepsilon_k r_k B_{k-1} + \varepsilon_{k+1} r_{k+1} B_k + \cdots + \varepsilon_{k+l} r_{k+l} B_{k+l-1}.$$

Note that there exists unique $\sigma_s (0 \le \sigma_s \le q_s - 1)$ such that

$$\varepsilon_s \equiv \sigma_s (\operatorname{mod} q_s) \quad \forall\, 1 \le s \le k+l.$$

Denote $\varepsilon_s = \sigma_s = 0$ for all $s > k + l$. Since $\theta \notin \Lambda$, we can find the smallest integer α such that $\tau(\sigma_1 \dots \sigma_\alpha) \neq \varepsilon_\alpha$. By (iii) in the definition of τ, we can find

$$\lambda = \sum_{j=1}^{\alpha} \tau(\sigma|_j) r_j B_{j-1} + M B_\alpha$$

for some integer M. Then there exists M' such that

$$\theta - \lambda = r_\alpha B_{\alpha-1}\Big(\varepsilon_\alpha - \tau(\sigma|_\alpha) + M' q_\alpha\Big).$$

By (ii) in the definition of τ, $\tau(\sigma|_\alpha) \equiv \sigma_\alpha \pmod{q_\alpha}$, which is also congruent to ε_α by our construction. This implies $\theta - \lambda$ is not in the zero set of $\widehat{\mu}_{\{b_n\},\{\mathcal{D}_n\}}$ since $q_\alpha \mid \big(\varepsilon_\alpha - \tau(\sigma|_\alpha)\big)$. It contradicts to θ being orthogonal to all $\lambda \in \Lambda$.

Conversely, suppose we are given a maximal orthogonal set Λ of $\mu_{\{b_n\},\{\mathcal{D}_n\}}$ with $0 \in \Lambda$. Then $\Lambda \subset \mathcal{Z}(\widehat{\mu}_{\{b_n\},\{\mathcal{D}_n\}})$. Hence, we expand λ/r_1 ($\lambda \in \Lambda$) with Lemma 2.2 and get

$$\lambda = \sum_{j=1}^{k} \varepsilon_{\lambda,j} r_j B_{j-1} := \sum_{j=1}^{\infty} \varepsilon_{\lambda,j} r_j B_{j-1}, \tag{2.5}$$

where $\varepsilon_{\lambda,j} \in \mathcal{C}_j (1 \le j \le k)$ with $\varepsilon_{\lambda,k} \neq 0$ and $\varepsilon_{\lambda,j} = 0$ for all $j \ge k+1$. Note that all $\varepsilon_{0,n}$ are zero. We first consider $\varepsilon_{\lambda,1}$ the first coefficients of λ's. As $\varepsilon_{\lambda,1}$ can be written uniquely as $i_k + q_1\alpha_k \in \mathcal{C}_1$ for some $i_k \in \mathcal{D}_1 = \{0, 1, \dots, q_1 - 1\}$, we claim that

$$\{\varepsilon_{\lambda,1}\}_{\lambda\in\Lambda} = \{i + q_1\alpha_i : i \in \mathcal{D}_1\} \subseteq \mathcal{C}_1.$$

(Here α_i depends only on i but not on λ, hence the set has q_1 elements.) In fact, the orthogonality of Λ implies that $\{\varepsilon_{\lambda,1}\}_{\lambda\in\Lambda} \subseteq \{i + q_1\alpha_i : i \in \mathcal{D}_1\}$. On the other hand, if $\{\varepsilon_{\lambda,1}\}_{\lambda\in\Lambda} \subsetneqq \{i + q_1\alpha_i : i \in \mathcal{D}_1\}$, then there exists $0 \le i' \le q_1 - 1$ such that $q_1 \nmid (\varepsilon_{\lambda,1} - i')$ for all $\lambda \in \Lambda$, which contradicts the maximality of Λ. This proves the claim.

From the claim, we can define τ on $\mathcal{D}_{0,1}$ by $\tau(\sigma_1) = \sigma_1 + q_1\alpha_{\sigma_1} (\sigma_1 \in \mathcal{D}_1)$ and in particular $\tau(0) = 0$. Similarly, we can show, for each $0 \le i_1 \le q_1 - 1$, that

$$\{\varepsilon_{\lambda,2} : \varepsilon_{\lambda,1} = i_1 + q_1\alpha_{i_1}\}_{\lambda\in\Lambda} = \{i_2 + q_2\alpha_{i_2} : i_2 \in \mathcal{D}_2\} \subseteq \mathcal{C}_2$$

and define $\tau(\sigma_1\sigma_2) = \sigma_2 + q_2\alpha_{\sigma_2} (\sigma_2 \in \mathcal{D}_2)$. Inductively, we can define a mapping τ on $\mathcal{D}_{0,*}$. By the construction of τ, it is easy to see that (i) and (ii) in Definition 1.1 are satisfied. For any $\sigma = \sigma_1\sigma_2 \dots \sigma_m \in \mathcal{D}_{0,*}$, again by the construction of τ, there exist infinitely many λ such that $\varepsilon_{\lambda,t} = \sigma_t + q_t\alpha_{\sigma_t}$ for $1 \le t \le m$. Together with (2.5), fix such a k, if $m \ge k$, then we have $\lambda = \sum_{j=1}^{\infty} \varepsilon_{\lambda,j} r_j B_{j-1} = \tau(\sigma 0^\infty)$; If $m < k$, there exists $\sigma' = \sigma_{m+1}\sigma_{m+2} \dots \sigma_k$ such that $\varepsilon_{\lambda,n} = \tau(\sigma_1 \dots \sigma_m \sigma_{m+1} \dots \sigma_n)$ for all $m + 1 \le n \le k$. Then

$$\lambda = \sum_{j=1}^{\infty} \varepsilon_{\lambda,j} r_j B_{j-1} = \tau(\sigma\sigma' 0^\infty).$$

This implies that (iii) in Definition 1.1 holds. Hence, τ is a tree mapping and $\Lambda \subset \Lambda(\tau)$. Conversely, $\Lambda(\tau) \subset \Lambda$ since Λ is a maximal orthogonal set. And then $\Lambda = \Lambda(\tau)$. ∎

3 The proof of Theorem 1.3

Theorem 1.2 shows that a countable set is a maximal orthonormal set of the measure $\mu_{\{b_n\},\{\mathcal{D}_n\}}$ if and only if it can be labeled as a tree. In this section, we will give a sufficient condition for a tree mapping to generate a spectrum. In the remainder of this paper, we always assume that the measure $\mu_{\{b_n\},\{\mathcal{D}_n\}}$ satisfies $P = \sup_{n\geq 1}\{b_n\} < \infty$. Given a tree mapping τ for $\mu_{\{b_n\},\{\mathcal{D}_n\}}$, we recall that

$$\mathcal{D}(\tau) = \left\{\sigma \in \mathcal{D}_{0,\infty} : \tau(\sigma|_m) = 0 \text{ for all } m \text{ large enough}\right\}$$

and

$$\tau(\sigma) = \sum_{k=1}^{\infty} \tau(\sigma|_k) r_k B_{k-1}, \quad \forall\, \sigma \in \mathcal{D}(\tau),$$

where $B_n = b_0 b_1 \dots b_n$ with $b_0 = 1$. Similarly, for any $\sigma' \in \mathcal{D}_{0,n}$, we define

$$\tau^*(\sigma') = \sum_{k=1}^{n} \tau(\sigma'|_k) r_k B_{k-1}.$$

We list some propositions, which will be useful in the sequel. Recall that

$$\begin{aligned}\mu_{\{b_n\},\{\mathcal{D}_n\}}(\cdot) &= \delta_{b_1^{-1}\mathcal{D}_1} * \delta_{(b_1 b_2)^{-1}\mathcal{D}_2} * \dots \\ &= \mu_n(\cdot) * \mu_{>n}(b_1 b_2 \dots b_n \cdot),\end{aligned}$$

where μ_n is the convolutional product of the first n discrete measures and

$$\mu_{>n} = \delta_{b_{n+1}^{-1}\mathcal{D}_{n+1}} * \delta_{(b_{n+1}b_{n+2})^{-1}\mathcal{D}_{n+2}} * \dots.$$

Proposition 3.1 *Let τ be a tree mapping. Then, for any $\xi \in \mathbb{R}$ and for any $\sigma \in \mathcal{D}(\tau)$, we have*

$$\widehat{\mu}_{\{b_n\},\{\mathcal{D}_n\}}\Big(\xi + \tau(\sigma)\Big) = \lim_{k\to\infty} \widehat{\mu}_k\Big(\xi + \tau^*(\sigma|_k)\Big).$$

Proof Since

$$\tau(\sigma) = \sum_{j=1}^{\infty} \tau(\sigma|_j) r_j B_{j-1} = \tau^*(\sigma|_k) + \sum_{j=k+1}^{\infty} \tau(\sigma|_j) r_j B_{j-1},$$

it follows from the periodicity of $|M_{\mathcal{D}_s}(x)|$ that

$$\widehat{\mu}_k\Big(\xi + \tau(\sigma)\Big) = \prod_{s=1}^{k} M_{\mathcal{D}_s}\Big(\xi + \tau(\sigma)\Big) = \prod_{s=1}^{k} M_{\mathcal{D}_s}\Big(\xi + \tau^*(\sigma|_k)\Big) = \widehat{\mu}_k\Big(\xi + \tau^*(\sigma|_k)\Big).$$

And then the result follows. ∎

Proposition 3.2 *Let τ be a tree mapping. Then, for all $k \in \mathbb{N}$, we have*

$$\mathcal{D}_{0,k} = \{\sigma|_k : \sigma \in \mathcal{D}(\tau)\}.$$

Proof Clearly, $\{\sigma|_k : \sigma \in \mathcal{D}(\tau)\} \subset \mathcal{D}_{0,k}$. Conversely, it follows from (iii) of Definition 1.1 that for any word $\sigma' \in \mathcal{D}_{0,k}$, there exists $\sigma'' \in \mathcal{D}_{k,\infty}$ such that $\sigma'\sigma'' \in \mathcal{D}(\tau)$.

This implies $\mathcal{D}_{0,k} \subseteq \{\sigma|_k : \sigma \in \mathcal{D}(\tau)\}$ since $(\sigma'\sigma'')|_k = \sigma' \in \mathcal{D}_{0,k}$. Hence, we finish the proof. ∎

Proposition 3.3 *Let τ be a tree mapping. Then, for all $k \geq 1$ and for any $\xi \in \mathbb{R}$, we have*

$$\sum_{\sigma \in \mathcal{D}_{0,k}} \left|\widehat{\mu}_k\big(\xi + \tau^*(\sigma)\big)\right|^2 \equiv 1.$$

Proof We claim that $\{\tau^*(\sigma)\}_{\sigma \in \mathcal{D}_{0,k}}$ is a spectrum of $L^2(\mu_k)$. We can then use Theorem 2.1 to conclude our lemma. Since this set has exactly $q_1 q_2 \dots q_k$ elements, we just need to show the mutual orthogonality.

For any two distinct words $\sigma, \sigma' \in \mathcal{D}_{0,k}$, we have

$$\tau^*(\sigma) = \sum_{j=1}^{k} \tau(\sigma|_j) r_j B_{j-1} \quad \text{and} \quad \tau^*(\sigma') = \sum_{j=1}^{k} \tau(\sigma'|_j) r_j B_{j-1}.$$

Let s $(1 \leq s \leq k)$ be the first index such that $\sigma|_s \neq \sigma'|_s$. Then we can obtain that

$$\tau^*(\sigma) - \tau^*(\sigma') = r_s b : s-1\Big(\tau(\sigma|_s) - \tau(\sigma'|_s) + M q_s\Big)$$

for some integer M. It follows from the periodicity of the exponential function $e^{-2\pi i x}$ that

$$M_{\mathcal{D}_s}\Big(\frac{\tau^*(\sigma) - \tau^*(\sigma')}{B_s}\Big) = M_{\mathcal{D}_s}\Big(\frac{\tau(\sigma|_s) - \tau(\sigma'|_s)}{q_s}\Big) = 0,$$

where the last equation holds because (ii) in the definition of maximal mapping implies that q_s does not divide $\tau(\sigma|_s) - \tau(\sigma'|_s)$. Since

$$\widehat{\mu}_k(\xi) = \prod_{j=1}^{k} \Big|M_{\mathcal{D}_j}\Big(\frac{\xi}{B_j}\Big)\Big|,$$

it follows that $\widehat{\mu}_k\big(\tau^*(\sigma) - \tau^*(\sigma')\big) = 0$. And thus the desired result follows. ∎

We divide the proof of Theorem 1.3 into several lemmas.

Lemma 3.4 *Assume that for any $1 \leq j \leq t$, $c_{k+j} \in \{-1, 0, 1, \dots, q_{k+j} r_{k+j+1} - 2\}$ satisfies*

$$c_{k+1} \in q_{k+1}\mathbb{Z}\backslash\{0\} \quad \textit{and} \quad c_{k+t} \neq 0.$$

Let $|\xi| \leq r_{k+1} - 1/q_k$, $\eta_1 = \xi/b_{k+1}$ and

$$\eta_n = \frac{1}{b_{k+n}}\Big(\eta_{n-1} + \frac{c_{k+n-1}}{q_{k+n-1}}\Big),$$

then one has

$$\frac{1}{P^{n+1}} \leq |\eta_n| \leq \frac{1}{q_{k+n}}\Big(1 - \frac{1}{P^2}\Big),$$

for all $2 \leq n \leq t$, where $P = \sup_{n \geq 1}\{b_n\} < \infty$.

Proof With an easy calculation, it can infer that

$$\eta_n = \frac{\xi + c_{k+1}r_{k+1} + \sum_{j=2}^{n-1} c_{k+j}r_{k+j}b_{k+1}\dots b_{k+j-1}}{b_{k+1}\dots b_{k+n}}, \quad \forall 2 \le n \le t.$$

On the upper bound, it follows $c_{k+j} \in \{-1, 0, 1, \dots, q_{k+j}r_{k+j+1} - 2\}$ for all $1 \le j \le n$, then

$$\begin{aligned}
|\eta_n| &= \left| \frac{\xi + c_{k+1}r_{k+1} + \sum_{j=2}^{n-1} c_{k+j}r_{k+j}b_{k+1}\dots b_{k+j-1}}{b_{k+1}\dots b_{k+n}} \right| \\
&\le \frac{r_{k+1} - 1/q_k + (q_{k+1}r_{k+2} - 2)r_{k+1} + \dots + (q_{k+n-1}r_{k+n} - 2)r_{k+n-1}b_{k+1}\dots b_{k+n-2}}{b_{k+1}b_{k+2}\dots b_{k+n}} \\
&\le \frac{r_{k+n}b_{k+1}\dots b_{k+n-1} - r_{k+n-1}b_{k+1}\dots b_{k+n-2}}{b_{k+1}\dots b_{k+n}} \\
&= \frac{1}{q_{k+n}}\left(1 - \frac{1}{q_{k+n-1}r_{k+n}}\right) \le \frac{1}{q_{k+n}}\left(1 - \frac{1}{P^2}\right), \qquad (3.1)
\end{aligned}$$

where the last inequality holds since $P = \sup_{n\ge1}\{b_n\}$.

On the lower bound, for η_1, a direct calculation shows that

$$|\eta_1| \le \frac{1}{q_{k+1}}\left(1 - \frac{1}{P^2}\right).$$

For $\eta_2 = \frac{1}{b_{k+2}}\left(\eta_1 + \frac{c_{k+1}}{q_{k+1}}\right)$, since

$$c_{k+1} \in q_{k+1}\mathbb{Z}\backslash\{0\} \quad \text{and} \quad c_{k+1} \in \{-1, 0, 1, \dots, q_{k+1}r_{k+2} - 2\},$$

it follows that

$$|\eta_2| \ge \frac{1}{b_{k+2}}\left(1 - \frac{1}{q_{k+1}}\left(1 - \frac{1}{P^2}\right)\right) > \frac{1}{2P} \ge \frac{1}{P^3}.$$

For $\eta_3 = \frac{1}{b_{k+3}}\left(\eta_2 + \frac{c_{k+2}}{q_{k+2}}\right)$, where $c_{k+2} \in \{-1, 0, 1, \dots, q_{k+2}r_{k+3} - 2\}$, one has the following two cases:

(i) if $c_{k+2} = 0$, then $\eta_3 = \frac{1}{b_{k+3}}\eta_2$, and hence

$$|\eta_3| > \frac{1}{P^3 b_{k+3}} \ge \frac{1}{P^4};$$

(ii) if $c_{k+2} \ne 0$, then

$$|\eta_3| > \frac{1}{b_{k+3}}\left(\frac{1}{q_{k+2}} - \frac{1}{q_{k+2}}\left(1 - \frac{1}{P^2}\right)\right) = \frac{1}{q_{k+2}b_{k+3}P^2} \ge \frac{1}{P^4}.$$

Noting that $\frac{1}{P^4} \le \frac{1}{q_{k+3}}\left(1 - \frac{1}{P^2}\right)$. Then we get that

$$\frac{1}{P^4} \le |\eta_3| \le \frac{1}{q_{k+3}}\left(1 - \frac{1}{P^2}\right).$$

Continuing the above procedure several times, we can obtain, for all $2 \le n \le t$, that

$$\frac{1}{P^{n+1}} \le |\eta_n| \le \frac{1}{q_{k+n}}\Big(1 - \frac{1}{P^2}\Big).$$

Hence, the lemma is proved. ∎

Since $\sup_{n\ge1}\{b_n\} < \infty$ and $q_n | b_n$, there are only finitely many $(b_n, \mathcal{D}_n)$, say that $\big(b(1), \mathcal{D}(1)\big)$, $\big(b(2), \mathcal{D}(2)\big), \dots, \big(b(N), \mathcal{D}(N)\big)$ with $D(i) = \{0, 1, \dots, q_{(i)} - 1\}$ for some $N \in \mathbb{N}$. Hence, it is easy to show that

$$c_1 = \min\left\{\big|M_{\mathcal{D}(i)}(\xi)\big| : |\xi| \le \frac{1}{q_{(i)}}\Big(1 - \frac{1}{P^2}\Big)\right\} > 0, \tag{3.2}$$

and

(3.3)

$$c(t) = \min\left\{\Big|M_{\mathcal{D}(i)}\Big(\xi + \frac{j}{q(i)}\Big)\Big| : \begin{array}{l} \xi \in \Big(\Big[\frac{1}{P^{t+1}}, \frac{1}{q_{(i)}}\Big(1 - \frac{1}{P^2}\Big)\Big] \cup \Big[-\frac{1}{q_{(i)}}\Big(1 - \frac{1}{P^2}\Big), -\frac{1}{P^{t+1}}\Big]\Big) \\ \forall j \in \{0, 1, \dots, q(i) - 1\} \end{array}\right\} > 0.$$

The following lemma will be use to prove Lemma 3.6, which is crucial in proof of completeness of the maximal orthogonal set.

Lemma 3.5 *Assume that for any $1 \le j \le t$, $c_{k+j} \in \{-1, 0, 1, \dots, q_{k+j}r_{k+j+1} - 2\}$ satisfies*

$$c_{k+1} \in q_{k+1}\mathbb{Z}\backslash\{0\} \quad \textit{and} \quad c_{k+t} \ne 0.$$

Let $|\xi| \le r_{k+1} - 1/q_k$, and let

$$\eta = \xi + c_{k+1}r_{k+1} + \sum_{j=2}^{t} c_{k+j}r_{k+j}b_{k+1}\dots b_{k+j-1},$$

then there exists a constant $C(t) \in (0,1)$ (only depending on t) such that $|\widehat{\mu}_{>k}(\eta)| \ge C(t)$.

Proof We define $\eta_1 = \frac{\xi}{b_{k+1}}$ and

$$\eta_n = \frac{\xi + c_{k+1}r_{k+1} + \sum_{j=2}^{n-1} c_{k+j}r_{k+j}b_{k+1}\dots b_{k+j-1}}{b_{k+1}\dots b_{k+n}}, \quad \forall 2 \le n \le t.$$

According to the fact that

$$\big|\widehat{\mu}_{>k}(\eta)\big| = \prod_{s=1}^{\infty}\Big|M_{\mathcal{D}_{k+s}}\Big(\frac{\eta}{b_{k+1}\dots b_{k+s}}\Big)\Big|,$$

it follows from the periodicity of $|M_{\mathcal{D}_{k+s}}(x)|$ and $c_{k+1} \in q_{k+1}\mathbb{Z}\backslash\{0\}$ that

$$\big|\widehat{\mu}_{>k}(\eta)\big| = \Big|M_{\mathcal{D}_{k+1}}\big(\eta_1\big)\Big| \cdot \prod_{s=2}^{t}\Big|M_{\mathcal{D}_{k+s}}\Big(\eta_s + \frac{a_{k+s}}{q_{k+s}}\Big)\Big| \cdot \prod_{s=t+1}^{\infty}\Big|M_{\mathcal{D}_{k+s}}\Big(\frac{\eta}{b_{k+1}\dots b_{k+s}}\Big)\Big|,$$

where $a_{k+s} \equiv c_{k+s} \pmod{q_{k+s}}$ with $a_{k+s} \in \{0, 1, \dots, q_{k+s} - 1\}$ for $2 \le s \le t$.

We estimate the products one by one. From the proof of Lemma 3.4, it can infer that

$$|\eta_1| \le \frac{1}{q_{k+1}}\Big(1-\frac{1}{P^2}\Big).$$

And thus we have, together with (3.2), that

$$\Big|M_{\mathcal{D}_{k+1}}\Big(\eta_1\Big)\Big| \ge c_1 > 0. \tag{3.4}$$

For any $2 \le s \le t$, using Lemma 3.4 again, we also have that

$$\frac{1}{P^{s+1}} \le |\eta_s| \le \frac{1}{q_{k+s}}\Big(1-\frac{1}{P^2}\Big).$$

It follows from (3.3) that

$$\Big|M_{\mathcal{D}_{k+s}}\Big(\eta_s+\frac{a_{k+s}}{q_{k+s}}\Big)\Big| \ge c(t) > 0. \tag{3.5}$$

For $s \ge t+1$, we set

$$\gamma = \frac{\eta}{b_{k+1}\dots b_{k+t}}$$

for convenience. Thus, we conclude that

$$\Big|\frac{\gamma}{b_{k+t+1}}\Big| \le \frac{1}{q_{k+t+1}}\Big(1-\frac{1}{P^2}\Big).$$

Using (3.2) again, it gets that

$$\begin{aligned}\prod_{s=t+1}^{\infty}\Big|M_{\mathcal{D}_{k+s}}\Big(\frac{\eta}{b_{k+1}\dots b_{k+s}}\Big)\Big| &= \Big|M_{\mathcal{D}_{k+t+1}}\Big(\frac{\gamma}{b_{k+t+1}}\Big)\Big|\cdot\prod_{m=2}^{\infty}\Big|M_{\mathcal{D}_{k+t+s}}\Big(\frac{\gamma}{b_{k+t+1}\dots b_{k+t+m}}\Big)\Big| \\ &= \Big|M_{\mathcal{D}_{k+t+1}}\Big(\frac{\gamma}{b_{k+t+1}}\Big)\Big|\cdot\prod_{m=2}^{\infty}\Big|\frac{\sin q_{k+t+m}\pi\gamma(b_{k+t+1}\dots b_{k+t+m})^{-1}}{q_{k+t+m}\sin\pi\gamma(b_{k+t+1}\dots b_{k+t+m})^{-1}}\Big)\Big| \\ &\ge c_1\cdot\prod_{m=2}^{\infty}\Big|\frac{\sin q_{k+t+m}\pi\gamma(b_{k+t+1}\dots b_{k+t+m})^{-1}}{q_{k+t+m}\pi\gamma(b_{k+t+1}\dots b_{k+t+m})^{-1}}\Big)\Big|. \end{aligned} \tag{3.6}$$

On the other hand, for any $m \ge 2$, it can be easily checked that

$$\Big|\frac{q_{k+t+m}\pi\gamma}{b_{k+t+1}\dots b_{k+t+m}}\Big| \le \frac{\pi(1-\frac{1}{P^2})}{q_{k+t+1}b_{k+t+2}\dots b_{k+t+m-1}r_{k+t+m}} \le \frac{\pi(1-\frac{1}{P^2})}{2^{m-1}}.$$

Combined this with (3.6), we can obtain from the monotonicity of $|\sin x/x|$ in $[0,\pi]$ and $\sin x/x \ge 1-6^{-1}x^2$ for $x \in [0,\pi/2]$ that

$$\prod_{s=t+1}^{\infty}\Big|M_{\mathcal{D}_{k+s}}\Big(\frac{\eta}{b_{k+1}\dots b_{k+s}}\Big)\Big| \ge c_1\cdot\prod_{m=1}^{\infty}\Big(1-\frac{\pi^2(1-\frac{1}{P^2})^2}{6\cdot 4^m}\Big) := c' > 0. \tag{3.7}$$

Combined with (3.4), (3.5), and (3.7), we get that

$$|\widehat{\mu}_{>k}(\eta)| \ge c_1 c(t)^{t-1}c' > 0.$$

Therefore, the desired result follows from $C(t) = c_1 c(t)^{t-1}c'$. ∎

Lemma 3.6 *For any word $\sigma \in \mathcal{D}_{0,k}(k \geq 1)$, let $\sigma\sigma' \in \mathcal{D}(\tau)$ satisfying (1.2) given as in Theorem 1.3. Let $|\xi| \leq r_{k+1} - 1/q_k$, and let*

$$\eta = \xi + \frac{\sum_{j=1}^{\infty} \tau\big((\sigma\sigma')|_{k+j}\big) r_{k+j} B_{k+j-1}}{B_k}.$$

Then there exists a constant c such that $|\widehat{\mu}_{>k}(\eta)| \geq c$.

Proof We distinguish two cases to prove it.

Case I. When $\sigma' = 0^{\infty}$ with $\tau\big((\sigma\sigma')|_j\big) = 0$ for any $j \geq k+1$, thus $\eta = \xi$ and

$$|\eta| \leq r_{k+1} - \frac{1}{q_k}.$$

It is easy to check that $|\widehat{\mu}_{>k}(\eta)| \geq c' > 0$, where c' is given as in (3.7).

Case II. When $\sigma' = 0^n \delta (n \in \mathbb{N}, \delta \in \mathcal{D}_{k+n,\infty})$ with $\tau\big((\sigma\sigma')|_{k+j}\big) = 0$ $(\forall\, 1 \leq j \leq n)$,

$$\tau\big((\sigma\sigma')|_{k+n+1}\big) \in q_{k+n+1}\mathbb{Z}\backslash\{0\} \quad \text{and} \quad \max\Big\{j : \tau\big((\sigma\sigma')|_{k+n+j}\big) \neq 0, \quad j \geq 1\Big\} \leq N.$$

Together with the periodicity of $|M_{\mathcal{D}_{k+s}}(x)|$, it implies that

$$\begin{aligned} |\widehat{\mu}_{>k}(\eta)| &= \prod_{s=1}^{\infty} \Big| M_{\mathcal{D}_{k+s}}\Big(\frac{\eta}{b_{k+1}\dots b_{k+s}}\Big)\Big| \\ &= \prod_{s=1}^{n} \Big| M_{\mathcal{D}_{k+s}}\Big(\frac{\eta}{b_{k+1}\dots b_{k+s}}\Big)\Big| \cdot \prod_{s=n+1}^{\infty} \Big| M_{\mathcal{D}_{k+s}}\Big(\frac{\eta}{b_{k+1}\dots b_{k+s}}\Big)\Big| \\ &= \prod_{s=1}^{n} \Big| M_{\mathcal{D}_{k+s}}\Big(\frac{\xi}{b_{k+1}\dots b_{k+s}}\Big)\Big| \cdot |\widehat{\mu}_{>k+n}(\gamma)|, \end{aligned} \tag{3.8}$$

where

$$\gamma = \frac{\eta}{b_{k+1}\dots b_{k+n}} = \frac{\xi}{b_{k+1}\dots b_{k+n}} + \tau\big((\sigma\sigma')|_{k+n+1}\big) c_{k+n+1} + \sum_{j=2}^{\infty} \tau\big((\sigma\sigma')|_{k+n+j}\big) r_{k+n+j} b_{k+n+1}\dots b_{k+n+j-1}.$$

Since $\max\Big\{j : \tau\big((\sigma\sigma')|_{k+n+j}\big) \neq 0, \quad j \geq 1\Big\} \leq N$, it follows that γ can be rewritten as

$$\gamma = \frac{\xi}{b_{k+1}\dots b_{k+n}} + \tau\big((\sigma\sigma')|_{k+n+1}\big) c_{k+n+1} + \sum_{j=2}^{N} \tau\big((\sigma\sigma')|_{k+n+j}\big) r_{k+n+j} b_{k+n+1}\dots b_{k+n+j-1}.$$

Using Lemma 3.5, there exists a constant $C(N) \in (0,1)$ (only depending on N) such that

$$|\widehat{\mu}_{>k+n}(\gamma)| > C(N). \tag{3.9}$$

Note that $|\xi| \leq r_{k+1} - 1/q_k$, we also have

$$\prod_{s=1}^{n} \Big| M_{\mathcal{D}_{k+s}}\Big(\frac{\xi}{b_{k+1}\dots b_{k+s}}\Big)\Big| \geq \prod_{s=1}^{\infty} \Big| M_{\mathcal{D}_{k+s}}\Big(\frac{\xi}{b_{k+1}\dots b_{k+s}}\Big)\Big| = |\widehat{\mu}_{>k}(\xi)| \geq c', \tag{3.10}$$

where the last inequality holds from *Case I*.

Combined with (3.8), (3.9), and (3.10), we get that

$$|\widehat{\mu}_{>k}(\eta)| \geq C(N)c' > 0.$$

Let $c = C(N)c'$, we can obtain the desired result. ∎

Now, we have all ingredients for the proof of Theorem 1.3.

Proof of Theorem 1.3. Since τ is a tree mapping, it follows that $\Lambda(\tau)$ is a maximal orthogonal set. By Theorem 2.1, we have

$$\begin{aligned} Q(\xi) &= \sum_{I\in\mathcal{D}(\tau)} \left|\widehat{\mu}_{\{b_n\},\{\mathcal{D}_n\}}\Big(\xi + \tau(I)\Big)\right|^2 \\ &= \sum_{n=0}^{\infty} \sum_{I\in\mathcal{D}(\tau), l_\tau(I)=n} \left|\widehat{\mu}_{\{b_n\},\{\mathcal{D}_n\}}\Big(\xi + \tau(I)\Big)\right|^2 \leq 1, \quad \forall\ \xi \in \mathbb{R}. \end{aligned}$$

Here, $l_\tau(I)$ is the smallest integer n such that $\tau(I|_k) = 0$ for all $k \geq n$. Moreover, fixed $\xi \in (0,1)$, for any $\varepsilon > 0$, there exists an integer N_ε such that

$$\sum_{I\in\mathcal{D}(\tau), l_\tau(I)\geq N_\varepsilon} \left|\widehat{\mu}_{\{b_n\},\{\mathcal{D}_n\}}\Big(\xi + \tau(I)\Big)\right|^2 < \varepsilon. \tag{3.11}$$

From Theorem 2.1, we only need to show that $Q(\xi) \geq 1$ for all $\xi \in (0,1)$. Together with Proposition 3.1, it can infer that

$$\begin{aligned} Q(\xi) &\geq \sum_{I\in\mathcal{D}(\tau), l_\tau(I)<N_\varepsilon} \left|\widehat{\mu}_{\{b_n\},\{\mathcal{D}_n\}}\Big(\xi + \tau(I)\Big)\right|^2 \\ &= \sum_{I\in\mathcal{D}(\tau), l_\tau(I)<N_\varepsilon} \lim_{k\to\infty} \left|\widehat{\mu}_k\Big(\xi + \tau^*(I|_k)\Big)\right|^2 \\ &= \lim_{k\to\infty} \sum_{I'\in\mathcal{D}(\tau), l_\tau(I)<N_\varepsilon} \left|\widehat{\mu}_k\Big(\xi + \tau^*(I|_k)\Big)\right|^2. \end{aligned}$$

Using the condition of Theorem 1.3, it follows that

$$\begin{aligned} Q(\xi) &\geq \lim_{k\to\infty} \sum_{\sigma\in\mathcal{D}_{0,k}, l_\tau(\sigma\sigma')<N_\varepsilon} \left|\widehat{\mu}_k\Big(\xi + \tau^*\big((\sigma\sigma')|_k\big)\Big)\right|^2 \\ &= \lim_{k\to\infty} \sum_{\sigma\in\mathcal{D}_{0,k}, l_\tau(\sigma\sigma')<N_\varepsilon} \left|\widehat{\mu}_k\Big(\xi + \tau^*(\sigma)\Big)\right|^2, \end{aligned} \tag{3.12}$$

where σ' are given as in the theorem. It can easily be checked that

$$\begin{aligned} \left|\widehat{\mu}_{\{b_n\},\{\mathcal{D}_n\}}\Big(\xi + \tau(\sigma\sigma')\Big)\right| = \left|\widehat{\mu}_k\Big(\xi + \tau^*(\sigma)\Big)\right| \cdot \Bigg|\widehat{\mu}_{>k} \\ \Bigg(\frac{\xi + \tau^*(\sigma)}{B_k} + \frac{\sum_{j=1}^{\infty} \tau\big((\sigma\sigma')|_{k+j}\big) r_{k+j} B_{k+j-1}}{B_k}\Bigg)\Bigg|, \end{aligned}$$

and

$$\left|\frac{\xi+\tau^*(\sigma)}{B_k}\right| = \left|\frac{\xi+\sum_{j=1}^{k}\tau(\sigma|_j)r_jB_{j-1}}{B_k}\right| \le r_{k+1}-\frac{1}{q_k}.$$

Therefore, combining with Lemma 3.6, we get

$$\left|\widehat{\mu}_{\{b_n\},\{\mathcal{D}_n\}}\Big(\xi+\tau(\sigma\sigma')\Big)\right| \ge c\left|\widehat{\mu}_k\Big(\xi+\tau^*(\sigma)\Big)\right|,$$

where c is given in Lemma 3.6. Together with (3.11), it yields that

$$\sum_{\sigma\in\mathcal{D}_{0,k},l_\tau(\sigma\sigma')\ge N_\varepsilon}\left|\widehat{\mu}_k\Big(\xi+\tau^*(\sigma)\Big)\right|^2 \le \frac{1}{c^2}\sum_{\sigma\in\mathcal{D}_{0,k},l_\tau(\sigma\sigma')\ge N_\varepsilon}\left|\widehat{\mu}_{\{b_n\},\{\mathcal{D}_n\}}\Big(\xi+\tau^*(\sigma)\Big)\right|^2 < \frac{\varepsilon}{c^2}.$$

Applying Propositions 3.2 and 3.3, we infer that

$$\sum_{\sigma\in\mathcal{D}_{0,k},l_\tau(\sigma\sigma')<N_\varepsilon}\left|\widehat{\mu}_k\Big(\xi+\tau^*(\sigma)\Big)\right|^2 + \sum_{\sigma\in\mathcal{D}_{0,k},l_\tau(\sigma\sigma')\ge N_\varepsilon}\left|\widehat{\mu}_k\Big(\xi+\tau^*(\sigma)\Big)\right|^2 = 1.$$

Then using (3.12), we obtain that

$$Q(\xi) \ge \lim_{k\to\infty}\Big(1-\sum_{\sigma\in\mathcal{D}_{0,k},l_\tau(\sigma\sigma')\ge N_\varepsilon}\left|\widehat{\mu}_k\Big(\xi+\tau^*(\sigma)\Big)\right|^2\Big) \ge 1-\frac{\varepsilon}{c^2}.$$

Hence, $Q(\xi)\ge 1$ and this completes the proof of Theorem 1.3. ■

4 The proof of Theorem 1.5

The aim of this section is to prove Theorem 1.5. Recall that $\sigma\in\mathcal{D}_{0,\infty}$ is a *useful word* about the generalized tree mapping τ if there exists an integer $\tau^\diamond(\sigma)\in\mathbb{Z}$ satisfying

$$\tau^\diamond(\sigma) \equiv \sum_{k=1}^{n}\tau(\sigma|_k)r_kB_{k-1} \pmod{r_{n+1}B_n}, \qquad \text{for all } n\ge 1.$$

It is easy to see that if $\sigma\in\mathcal{D}_{0,\infty}$ is a useful word, then the associated $\tau^\diamond(\sigma)$ is unique. Given a generalized tree mapping τ, we let

$$\mathcal{D}^\diamond(\tau) = \Big\{\sigma\in\mathcal{D}_{0,\infty} : \sigma \text{ is a useful word}\Big\} \quad\text{and}\quad \Lambda^\diamond(\tau) = \{\tau^\diamond(\sigma)|\sigma\in\mathcal{D}^\diamond(\tau)\}.$$

Proof of Theorem 1.5. (i) We first show that $\Lambda^\diamond(\tau)$ is an orthogonal set of $\mu_{\{b_n\},\{\mathcal{D}_n\}}$. For any two distinct words $\sigma,\sigma'\in\mathcal{D}^\diamond(\tau)$, we let n be the first index such that $\sigma|_n\ne\sigma'|_n$. Since

$$\tau^\diamond(\sigma)\equiv\sum_{k=1}^{n}\tau(\sigma|_k)r_kB_{k-1} \pmod{r_{n+1}B_n} \quad\text{and}\quad \tau^\diamond(\sigma')\equiv\sum_{k=1}^{n}\tau(\sigma'|_k)r_kB_{k-1} \pmod{r_{n+1}B_n},$$

it follows that

$$\tau^\diamond(\sigma)-\tau^\diamond(\sigma') = r_nB_{n-1}\Big(\tau(\sigma|_n)-\tau(\sigma'|_n)+q_nM\Big)$$

for some integer M. Using the definition of the generalized tree mapping, we know that

$$q_n \nmid \Big(\tau(\sigma|_n) - \tau(\sigma'|_n)\Big).$$

Together with (2.2), we also have that

$$\tau^\diamond(\sigma) - \tau^\diamond(\sigma') \in \mathcal{Z}(\widehat{\mu}_{\{b_n\},\{\mathcal{D}_n\}}).$$

Hence, $\Lambda^\diamond(\tau)$ is an orthogonal set of $\mu_{\{b_n\},\{\mathcal{D}_n\}}$.

Next, we show the maximality of $\Lambda^\diamond(\tau)$ by contradiction. Suppose that there exists a $\theta \in \mathbb{R}\backslash\Lambda^\diamond(\tau)$ such that θ is orthogonal to all elements in $\Lambda^\diamond(\tau)$. Note that $0 \in \Lambda^\diamond(\tau)$, we have that $\theta \in \mathcal{Z}(\widehat{\mu}_{\{b_n\},\{\mathcal{D}_n\}})$. Hence, by (2.2), there exists $k \in \mathbb{Z}$ such that $\theta = r_k B_{k-1} a$, where $a \in \mathbb{Z}$ and q_k does not divide a. By Lemma 2.2, it yields that

$$a = \varepsilon_k + \varepsilon_{k+1} q_k r_{k+1} + \cdots + \varepsilon_{k+l} q_k r_{k+l} b_{k+1} b_{k+2} \ldots b_{k+l-1},$$

where $\varepsilon_j \in \mathcal{C}_j = \{-1, 0, 1, \ldots, q_j r_{j+1} - 2\} (k \le j \le k+l)$ and q_k does not divide ε_k. Consequently, it can infer that

$$\theta = r_k B_{k-1} a = \varepsilon_k r_k B_{k-1} + \varepsilon_{k+1} r_{k+1} B_k + \cdots + \varepsilon_{k+l} r_{k+l} B_{k+l-1}, \tag{4.1}$$

where q_k does not divide ε_k. Note that there exists unique σ_j $(0 \le \sigma_j \le q_j - 1)$ such that

$$\varepsilon_j \equiv \sigma_j \pmod{q_j} \quad \forall\, k \le j \le k+l.$$

Denote $\sigma_j = 0$ for all $1 \le j < k$ and let $\sigma = \sigma_1 \sigma_2 \ldots \sigma_{k+l}$. By the definition of useful words, it can be easily checked that $\sigma 0^\infty$ is a useful word.

Since $\theta \notin \Lambda^\diamond(\tau)$, we can find the smallest integer α such that $\tau(\sigma_1 \ldots \sigma_\alpha) \ne \varepsilon_\alpha$. Hence,

$$\theta - \tau^\diamond(\sigma 0^\infty) = r_\alpha B_{\alpha-1}\Big(\varepsilon_\alpha - \tau\big((\sigma 0^\infty)|_\alpha\big) + M q_\alpha\Big),$$

for some integer M. Using the definition of the generalized tree mapping again, we know that $(\sigma 0^\infty)|_\alpha \equiv \sigma_\alpha \pmod{q_\alpha}$, which is also congruent to ε_α by our construction. It implies that

$$q_\alpha \mid \Big(\varepsilon_\alpha - \tau\big((\sigma 0^\infty)|_\alpha\big)\Big),$$

and thus $\theta - \tau^\diamond(\sigma 0^\infty) \notin \widehat{\mu}_{\{b_n\},\{\mathcal{D}_n\}}$. This contradicts the assumption and the maximality of $\Lambda^\diamond(\tau)$ follows.

(ii) Together with the conclusion (i) and Theorem 1.2, there exists a tree mapping τ_1 such that $\Lambda^\diamond(\tau) = \Lambda(\tau_1)$. For any $\sigma' \in \mathcal{D}_{0,k}$ $(k \ge 1)$, by the definition of the tree mapping τ_1, there is an infinite word $\sigma'' \in \mathcal{D}_{k,\infty}$ such that $\sigma'\sigma'' \in \mathcal{D}(\tau_1)$. Since $\Lambda^\diamond(\tau) = \Lambda(\tau_1)$, it follows that there exists $\sigma \in \mathcal{D}^\diamond(\tau)$ such that $\tau^\diamond(\sigma) = \tau_1(\sigma'\sigma'')$. Note that $\sigma|_k 0^\infty$ is a useful word, we can find $\sigma''' \in \mathcal{D}(\tau_1)$ such that $\tau^\diamond(\sigma|_k 0^\infty) = \tau_1(\sigma''')$. Hence, one has that

$$\tau_1(\sigma'\sigma'') \equiv \tau^\diamond(\sigma) \equiv \tau^\diamond(\sigma|_k 0^\infty) \equiv \tau_1(\sigma''') \pmod{r_{n+1} B_n} \quad \text{for any } 1 \le n \le k. \tag{4.2}$$

A direct calculation shows that

$$|\ \tau_1^*\big((\sigma'\sigma'')|_n\big) - \tau_1^*(\sigma'''|_n)\ | \le \sum_{j=1}^{n}(q_j r_{j+1} - 2) r_j B_{j-1} < r_{n+1} B_n.$$

Together with (4.2), it implies that $\tau_1^*\big((\sigma'\sigma'')|_n\big) = \tau_1^*(\sigma'''|_n)$ for all $1 \le n \le k$ and thus we have that

$$\sigma'''|_k = (\sigma'\sigma'')|_k = \sigma'.$$

Hence, we rewrite σ''' as $\sigma''' = \sigma'\delta$ with $\delta \in \mathcal{D}_{k,\infty}$. Since $\sup_{\sigma\in\mathcal{D}_{0,*}} |\tau(\sigma)| < \infty$, there exists a constant M such that $|\tau(\sigma)| < 2^M$ for all $\sigma \in \mathcal{D}_{0,*}$. Then

$$|\tau_1(\sigma'\delta)| = |\tau^\diamond(\sigma|_k 0^\infty)| \le 2^M \sum_{j=1}^{k} r_j B_{j-1} \le 2^{M+1} r_k B_{k-1}. \tag{4.3}$$

Repeating the procedure of (4.1), we also have that

$$2^{M+1} r_k B_{k-1} = \vartheta_k r_k B_{k-1} + \vartheta_{k+1} r_{k+1} B_k + \cdots + \vartheta_{k+\ell} r_{k+\ell} B_{k+\ell-1}, \tag{4.4}$$

where $\vartheta_j \in \mathcal{C}_j$ for all $k \le j \le k+\ell$. Then it is easy to check that

$$\begin{aligned} 2^{M+1} r_k B_{k-1} &= |\vartheta_k r_k B_{k-1} + \vartheta_{k+1} r_{k+1} B_k + \cdots + \vartheta_{k+\ell} r_{k+\ell} B_{k+\ell-1}| \\ &\ge r_{k+\ell} B_{k+\ell-1} - \sum_{j=k}^{k+\ell-1} (q_j r_{j+1} - 2) r_j B_{j-1} \\ &\ge r_{k+\ell-1} B_{k+\ell-2}. \end{aligned}$$

Consequently, it can infer that

$$2^{M+1} \ge r_{k+\ell-1} q_k b_{k+1} b_{k+2} \ldots b_{k+\ell-2} \ge 2^{\ell-1},$$

and thus $\ell \le M+2$.

We let m be the largest order of the expansion of $\tau(\sigma'\delta)$ with respect to the base $\{\mathcal{C}_j\}_{j=1}^{\infty}$. Using (4.3) and (4.4), it can obtain that

$$m \le k + \ell \le k + M + 2.$$

Therefore, it follows from the definition of m that

$$\max\Big\{ j \ge 1 : \tau_1\big((\sigma'\delta)|_{k+j}\big) \ne 0 \Big\} \le M + 2.$$

Hence, the conclusion (ii) follows form Theorem 1.3. ∎

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