DIFFRACTION OF PLANE ELASTIC WAVES BY A CRACK, WITH APPLICATION TO A PROBLEM OF BRITTLE FRACTURE

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Abstract

A crack is assumed to be the union of two smooth plane surfaces of which various parts may be in contact, while the remainder will not. Such a crack in an isotropic elastic solid is an obstacle to the propagation of plane pulses of the scalar and vector velocity potential so that both reflected and diffracted fields will be set up. In spite of the non-linearity which is present because the state of the crack, and hence the conditions to be applied at the surfaces, is a function of the dependent variables, it is possible to separate incident step-function pulses into either those of a tensile or a compressive nature and the associated scattered field may then be calculated. One new feature which arises is that following the arrival of a tensile field which tends to open up the crack there is necessarily a scattered field which causes the crack to close itself with the velocity of free surface waves.

In spite of the non-linearity, cases which involve the superimposition of these step-function results may be considered; it appears that any incident plane field may be considered as such a superposition except for two classes. The first class excluded involves incident plane shear fields which are so skew that they move in the direction of the edge with a phase velocity which is smaller than the dilatation velocity of the elastic medium. This is because the assumption of conical motion which is used in the calculation does not hold. The second class excluded involves any incident tensile field, apart from a single step-function, when it travels along the crack before arriving at the edge. This is excluded because any part of the incident field behind the original front is bound to interact with the change of state of the crack which, as mentioned, travels with the velocity of Rayleigh waves.

The analysis is also extended to throw light on the possible initiation of brittle fracture and the subsequent propagation of a smooth crack in the plane of the existing one. It is shown how the analysis leads to the result

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that since the crack is extended without the application of external forces to the crack itself, it can only travel freely with the velocity of Rayleigh waves.

1. Introduction

On the assumption that the subsequent velocity field is conical, we examine the diffraction of plane waves in an isotropic elastic solid by a semi-infinite plane crack. The edge of the crack being taken for z-axis, we take the half-plane $x > 0$, $y = 0$ to define the position of the crack itself. We shall consider the incidence both of plane shear waves and of plane dilatation waves in the analysis: we take the direction of propagation to make angle β with the edge of the crack, and angles $\pi/2 - \alpha_1$, or $\pi/2 - \alpha_2$ with the y-axis for dilatation waves and shear waves respectively. The elastic solid has Lamé constants λ and μ, and a density ρ, and we refer to the dilatation velocity c_1 and to the shear velocity c_2, where $c_1^2 = (\lambda + 2\mu)/\rho$, $c_2^2 = \mu/\rho$.

It is only when the incident field is of step function time dependence that the assumption of subsequent conical motion is valid. Since the scattered field, together with the incident field, travels steadily in the z-direction with a velocity $c_i \sec \beta_i$ with the appropriate subscript $i = 1$ or 2 according as to whether the incident field is a dilation or a shear wave, then for the former we can take any β_1 in the range $0 < \beta_1 < \pi/2$, and for the latter any β_2 in the range $\arccos (c_2/c_1) < \beta_2 < \pi/2$, and we have a fully supersonic motion in the z-direction. The assumption of conical motion means that the analysis will proceed in very similar fashion to that in a preceding paper [1]. For brevity we shall use the same notation and we shall refer directly to results in that paper; all equations cited are given the prefix A.

The immediate point of interest involves the manner in which a plane crack may initiate the scattering of plane waves when they arrive at the crack. It is possible to consider various models for the behaviour of a crack in a dynamic situation, but in this paper we take the crack to be the union of two *smooth* plane surfaces. There are two possible states to be considered: when the crack is open there are two distinct surfaces at which the stresses must vanish, but when it is closed the normal velocity, displacement, and stress components must be continuous, leaving, in the absence of friction, the tangential stresses to vanish. For an incident field of arbitrary time dependence there is no simple way of deciding whether at a given point the crack is open or shut. However, we may take the case when the incident field represents a displacement of step-function type, and we may consider separately the case when the incident field tends to open the (initially closed) crack from that when it tends to keep the crack closed. When eventually we derive the results for any given incident pulse we can superimpose results in some cases in order to give closed formulae for the scattered stress and displacement fields, regardless of the absence of knowledge of the state of the crack.

The literature in the subject of elastodynamic diffraction is rather meagre. Previous work has been done by Maue [2], de Hoop [3] and Miles [4], in the consideration of diffraction by either a perfectly rigid or a perfectly weak half-plane, while a problem involving the sudden opening of a semi-infinite crack was discussed by Maue [5]. Other authors are mentioned in the dissertation of de Hoop. The work of Miles has many similarities to that of the author, who has made use of the present approach in the solution of a number of diffraction problems.

Miles's paper is certainly relevant to the present research, because in considering diffraction of elastic waves by a perfectly weak half plane he is dealing with what I refer to as a permanently open crack. When it comes to the examination of the physical consequences of his solution, we have to realise that there is a tacit requirement that the relative displacement of the upper and lower surfaces of the crack be non-negative. Miles's formula 4.34b for the normal displacement contains a surface wave term of delta function profile which is symmetric about the crack. The sign of the surface displacement is determined according to whether the incident field is tensile or compressive and for one or other of these cases (in fact for the compressive case) this singular displacement does not satisfy the tacit physical requirement. Even though we are dealing with an idealised problem where the strains are infinitesimal and the two crack surfaces, whatever their physical state, have zero separation, the only situation which is covered by the solution of Miles is that where an open crack is being opened further by a tensile field. If we are to deal with the case of an incident compressive field, we have to eliminate contradictions in the solution by ascribing a special behaviour to the crack, and as a simple case we have the idealised crack described above.

As in my earlier paper [1], the start of the problem involves the introduction of the scalar velocity potential ϕ and the vector velocity potential $\boldsymbol{\psi} = A\boldsymbol{i} + C\boldsymbol{k}$. To utilize the notation of part I we keep the quantity 'a' to represent the uniform velocity in the z-direction, with the understanding that $a = c_1 \sec \beta_1$ for a dilatation field, and $a = c_2 \sec \beta_2$ for a shear field. We refer directly to equations in the earlier paper by using the prefix A.

The assumption that the component $(B\boldsymbol{j})$ of $\boldsymbol{\psi}$, which is perpendicular to the crack, vanishes, is not unduly restrictive. It is only made because B must satisfy the condition $\partial B/\partial y = 0$ at the crack, and it can then play no part in the surface coupling effects. This situation arises from the simultaneous vanishing of the x- and z-component of stress at the surface of the crack. A non-zero value for B in the incident field will modify the relative magnitudes of the other components of the shear potential without greatly changing their form.

2. Conditions at the crack

Apart from satisfying the wave equations A.10 and A.11, the three potentials ϕ, A and C must satisfy certain conditions at the crack. Then after introducing polar co-ordinates so that the upper surface is $\theta = 0$ and the lower surface is $\theta = 2\pi$, we have the conditions corresponding to A.22 which are valid when the crack is closed. These are that

$$(1) \qquad aA_s + sC_s = 0, \quad \text{for} \quad \theta = 0, 2\pi,$$

$$(2) \qquad \frac{2}{s}\phi_\theta + \left[s^2\left(\frac{1}{\gamma_2^2} - \frac{1}{a^2}\right) - 2\right] C_s = 0, \quad \text{for} \quad \theta = 0, 2\pi,$$

and

$$(3) \qquad \frac{2}{a}[A_\theta]^0_{2\pi} - \frac{2}{s}[C_\theta]^0_{2\pi} + \left[s^2\left(\frac{1}{\gamma_2^2} - \frac{1}{a^2}\right) - 2\right][\phi_s]^0_{2\pi} = 0,$$

which express respectively the facts that the vector potential is solenoidal, that the tangential stress vanishes on both sides of the crack and that the normal stress component is continuous. The further condition that normal velocity is continuous across the crack takes the form

$$(4) \qquad a[\phi_\theta]^0_{2\pi} + s^2[A_s]^0_{2\pi} - as[C_s]^0_{2\pi} = 0,$$

and we may note immediately that the three conditions 1, 2 and 4 only hold simultaneously if the quantities ϕ_θ, C_s and A_s are separately continuous across the crack.

If the crack is open, then the conditions 1 and 2 hold on each surface, but instead of 3 and 4 we find the condition that

$$(5) \qquad \frac{2}{a}A_\theta - \frac{2}{s}C_\theta + \left[s^2\left(\frac{1}{\gamma_2^2} - \frac{1}{a^2}\right) - 2\right]\phi_s = 0,$$

for $\theta = 0, 2\pi$.

For the closed crack the conditions will simplify if we split the potentials into parts which are symmetric and anti-symmetric about the crack. The symmetric parts of A and C, and the anti-symmetric part of ϕ then satisfy the simple conditions 1, 2 and 5 on each side of the crack together with the condition of continuous ϕ_θ, C_s and A_s. The symmetric part of ϕ needs only to satisfy the condition of vanishing normal derivative, while the anti-symmetric parts of A and C, already linked through equation 1 must each satisfy the condition of vanishing tangential derivative at the crack.

3. Pulse reflection and transmission coefficients

We may consider first the *steady* problem of a pulse system travelling along an infinite plane crack. If the incident pulse is compressive, there will be

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both reflected and transmitted pulses of shear and dilatation type. If the incident field is tensile, tending to open the crack, there will be no transmitted field, and only reflected shear and dilatation pulses will be set up.

We shall consider combinations of shear and dilatation waves which have the same velocity both in the z- and the x-direction, so that

$$a = c_1 \sec \beta_1 = c_2 \sec \beta_2; \quad \gamma_1 \sec \alpha_1 = \gamma_2 \sec \alpha_2,$$

with

$$\gamma_1 = c_1 \operatorname{cosec} \beta_1, \quad \gamma_2 = c_2 \operatorname{cosec} \beta_2.$$

Then for an incident tensile field given as an arbitrary combination of a shear field ψ, given (with $L = \gamma_1/a = \cot \beta_1$) as a multiple of the unit step-function U by the equation

$$\psi = -(\boldsymbol{k} - L \sec \alpha_1 \boldsymbol{i})\, U\left(t - \frac{z}{a} - \frac{x \cos \alpha_2 + y \sin \alpha_2}{\gamma_2}\right), \tag{6}$$

and a dilatation field ϕ given by the equation

$$\phi = -U\left(t - \frac{z}{a} - \frac{x \cos \alpha_1 + y \sin \alpha_1}{\gamma_1}\right),$$

we can derive the amplitude of the reflected waves, and hence the four basic reflection coefficients $R^0_{mn} (m, n = 1, 2)$. The notation here is that the first subscript m refers to the type of incident pulse, while the second subscript n refers to the type of scattered pulse. The subscript 1 is still associated with dilatation fields.

The result of applying the conditions of vanishing stress to this incident field leads to the values

$$\tag{9} \begin{cases} R^0_{11} = R^0_{22} = \{4 \tan \alpha_1 \tan \alpha_2 (1 + L^2 \sec^2 \alpha_1) - [2 - \sec^2 \alpha_2 (1 - L^2 M^2)]^2\}/D, \\ R^0_{12} = 4 \tan \alpha_1 [2 - \sec^2 \alpha_2 (1 - L^2 M^2)]/D(1 + L^2 \sec^2 \alpha_1), \\ R^0_{21} = -4 \tan \alpha_2 [2 - \sec^2 \alpha_2 (1 - L^2 M^2)](1 + L^2 \sec^2 \alpha_1)/D, \\ \text{and} \\ D = [2 - \sec^2 \alpha_2 (1 - L^2 M^2)]^2 + 4 \tan \alpha_1 \tan \alpha_2 (1 + L^2 \sec^2 \alpha_1), \end{cases}$$

where

$$M = \gamma_2/\gamma_1, \quad \text{and} \quad L = \gamma_1/a = \cot \beta_1.$$

The normal velocity at the plane $y = 0$ which is associated with these fields is consistent with the opening of the crack, but only for the incident field of negative sign.

For an incident field given as in equations 6 and 7, but with a positive sign, there is no tendency for the crack to open. The appropriate continuity

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conditions may then be used to derive reflection coefficients R^s_{mn} and transmission coefficients T^s_{mn} with

$$(10)\qquad \left\{\begin{aligned} &R^s_{21} = -T^s_{21} = R^0_{21}/2,\\ &R^s_{12} = T^s_{12} = R^0_{12}/2,\\ &R^s_{11} = 1 - T^s_{11} = (1 + R^0_{11})/2\\ &\text{and}\\ &R^s_{22} = -1 + T^s_{22} = (R^0_{22} - 1)/2.\end{aligned}\right.$$

4. Formulation for the incident tensile field

For a skew plane wave incident at the edge of the plane semi-infinite crack, there is interest only in the region *behind* the point of intersection of the wave front and the edge. There are two cones of discontinuity with vertex at this point, and in Fig. 1 we represent the field structure in any transverse section across the cones, for the case when the incident tensile field is an arbitrary combination of shear and dilatation fields defined in equations 6 and 7.

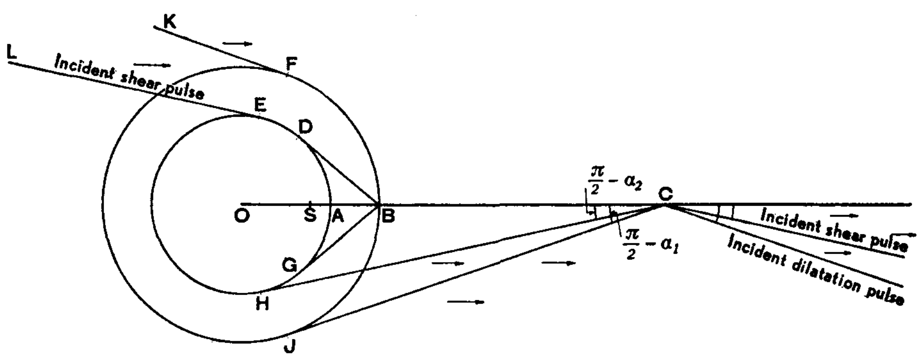

Fig. 1. Field structure for the crack under tensile excitation.

We may use the notation of Part I, with the specific constants

$$a = c_1 \sec \beta_1 = c_2 \sec \beta_2,\ \gamma_1 = c_1/\sin \beta_1,\ \gamma_2 = c_2/\sin \beta_2,$$
$$M = \gamma_2/\gamma_1 = \sec \alpha_1 \cos \alpha_2,\ \text{and}\ L = \cot \beta_1.$$

Then in Figure 1, the point of intersection of the incident pulse system with the crack is $s = \gamma_1 \sec \alpha_1 = \gamma_2 \sec \alpha_2$; the reflection process in the vicinity of this point is exactly that of a pulse system travelling along an infinite crack. We therefore know that the reflection coefficients R^0_{mn} will define the scattered dilatation field outside the circle $s = \gamma_1$, and the scattered shear field outside the circle $s = \gamma_2$ with the exception of the two trian-

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gular head wave regions, ABD where

$$\psi = ih(u_2 + \theta) + kg(u_2 + \theta), \quad \text{and} \quad ABG$$
$$\text{where} \quad \psi = in(u_2 - \theta) + km(u_2 - \theta)$$

The four arbitrary functions h, g, n and m are constant on the characteristic tangent planes $u_2 \pm \theta = \text{constant}$. Within $s = \gamma_1$, the potential ϕ is harmonic in the variables v_1 and θ, and within $s = \gamma_2$ the vector components A and C are harmonic in the variables v_2 and θ. The variables v are defined in equation A.5.

We introduce the complex potentials W^ϕ, W^A and W^C as in equations A.6 and A.7, and we use the mapping $\zeta_1 = \operatorname{sech}(v_1 + i\theta)$, $\zeta_2 = \operatorname{sech}(v_2 + i\theta)$, which transform the interiors of the circles $s = \gamma$, $0 < \theta < 2\pi$, into the whole of the complex ζ-plane, cut on the whole real axis except for the segment $-1 < \zeta < 0$. We note that for points on the crack $\zeta_1 = M\zeta_2$.

The incident field certainly results in a uniform opening of the crack on the segment BC, but the question of whether the crack remains open or shut on the segment OB is not yet determined. The following is a consistency argument in which the displacement, following assumptions on the state of the crack on OB is examined.

We may begin with the statement that the very edge of the crack remains closed. But it is not *only* the edge which remains closed, since in this case, that of a fully open crack, we could go directly to Miles's results [4] in which we find that the calculated normal displacement results in certain parts of the lower side of the crack being above the upper side. (Miles only discusses the case of the disturbing fronts parallel to the edge, so that $\beta = \pi/2$, but the analysis is quite easily extended to the case of skew-incidence). Therefore we suppose the crack to be closed in a segment $0 < s < P$, and to be open on the remaining segments $s > P$. The quantity P is the velocity with which the crack will be expected to close itself.

We can examine the consequences of the linkage between the harmonic functions ϕ, A and C at the crack for $s < \gamma_2$.

For the open section of the crack, the conditions of vanishing stress, and of solenoidal vector potential, lead precisely in the manner of equations A.26 to A.31 to the statements that

$$(11) \quad \frac{dW^\phi}{d\zeta_1} = \frac{(1+L^2\zeta_1^2)[\zeta_1^2(1-L^2M^2)-2M^2]iP_2 - 2P_3M(1+L^2\zeta_1^2)(1-\zeta_1^2/M^2)^{\frac{1}{2}}}{M^2R(\zeta_1)},$$

$$(12) \quad \frac{dW^C}{d\zeta_2} = \frac{2M(1+L^2M^2\zeta_2^2)(1-M^2\zeta_2^2)^{\frac{1}{2}}P_2 + i[\zeta_2^2(1-L^2M^2)-2]P_3}{R(M\zeta_2)}$$

and

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$$\frac{\mathrm{d}W^A}{\mathrm{d}\zeta_2} = -LM\zeta_2\frac{\mathrm{d}W^C}{\mathrm{d}\zeta_2}, \tag{13}$$

with

$$\begin{aligned} R(M\zeta_2) = {} & [\zeta_2^2(1-L^2M^2)-2]^2 \\ & -4(1+L^2M^2\zeta_2^2)(1-\zeta_2^2)^{\frac{1}{2}}(1-M^2\zeta_2^2)^{\frac{1}{2}}. \end{aligned} \tag{14}$$

Equation 13 refers to a uniform relation between the complex potentials W^A and W^C in the whole complex plane, and hence need not be referred to again. The functions P_2 and P_3 are real functions of ζ in the region $P/\gamma_2 < \zeta_2 < 1$, which corresponds to the segment $P < s < \gamma_2$ of the crack, and the conditions in the head wave regions, derived as in equations A.23 to A.25 are satisfied if the functions P_2 and P_3 are real for $1 < \zeta_2 < 1/M$.

For the closed segment of the crack we can develop similar results for the *change* in the complex derivatives across the crack; but to ensure continuity of normal velocity we are left with the results that

$$\left.\frac{\mathrm{d}W^\phi}{\mathrm{d}\zeta_1}\right]_{\theta=2\pi}^{\theta=0} = \frac{-2(1+L^2\zeta_1^2)(1-\zeta_1^2/M^2)^{\frac{1}{2}}T_3}{MR(\zeta_1)}, \tag{15}$$

and

$$\left.\frac{\mathrm{d}W^C}{\mathrm{d}\zeta_2}\right]_{\theta=2\pi}^{\theta=0} = \frac{i[\zeta_2^2(1-L^2M^2)-2]T_3}{R(M\zeta_2)}, \tag{16}$$

where T_3 is a real function of ζ_1 for $0 < \zeta_1 < P/\gamma_1$.

Other conditions on the complex derivatives have been enumerated in the earlier paper but there are a few points of difference. Thus

i) we expect simple poles of $\mathrm{d}W^\phi/\mathrm{d}\zeta_1$ at $\zeta_1 = \sec\alpha_1$ to account for the jumps in ϕ at the points F and J, while $\mathrm{d}W^C/\mathrm{d}\zeta_2$ must have simple poles at $\zeta_2 = \sec\alpha_2$ to account for the jumps in C at E and I.

ii) the absence of applied load at the edge of the crack provides for bounded complex derivatives as $\zeta \to 0$, so that the energy flux near the edge tends to zero.

iii) the simple pole at $\zeta_2 = -\gamma_R/\gamma_2$ associated as in section 5 of Part I with the Rayleigh wave velocity, has no physical justification, and must be annulled, together with the complex behaviour of $R(M\zeta_2)$ in the range $-1/M < \zeta_2 < -1$.

To satisfy the final requirement both P_2 and P_3 must contain a factor

$$\begin{aligned} Q(M\zeta_2) = {} & \left(\zeta_2+\frac{\gamma_R}{\gamma_2}\right) \\ & \exp -\frac{1}{\pi}\int_{-1/M}^{-1}\left\{\arctan\left[\frac{4(1+L^2M^2t^2)(t^2-1)^{\frac{1}{2}}(1-M^2t^2)^{\frac{1}{2}}}{t^2(1-L^2M^2)-2}\right]\right\}\frac{\mathrm{d}t}{t-\zeta_2}. \end{aligned} \tag{17}$$

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This singular integral provides, through a simple application of the Plemelj formula, an exact annulment of the complex behaviour of $R(M\zeta_2)$ when $-1/M < \zeta_2 < -1$.

All conditions are satisfied if we write

$$\frac{dW^{\phi}}{d\zeta_1} = \frac{Q(\zeta_1)\zeta_1^2(1+L^2\zeta_1^2)}{(\zeta_1-\sec\alpha_1)R(\zeta_1)}\left\{iS_2\left[\frac{\zeta_1^2}{M^2}(1-L^2M^2)-2\right]\left[\frac{\zeta_1\gamma_1-P}{(1+\zeta_1)\gamma_1}\right]^{\frac{1}{2}} - \frac{2S_3}{M}\left(\frac{\zeta_1}{M+\zeta_1}\right)^{\frac{1}{2}}\left(1-\frac{\zeta_1^2}{M^2}\right)^{\frac{1}{2}}\right\} \tag{18}$$

and

$$\frac{dW^{C}}{d\zeta_2} = \frac{Q(M\zeta_2)M\zeta_2^2}{(\zeta_2-\sec\alpha_2)R(M\zeta_2)}\left\{2MS_2(1+L^2M^2\zeta_2^2)(1-M^2\zeta_2^2)^{\frac{1}{2}}\left[\frac{M(\zeta_2\gamma_2-P)}{\gamma_2(1+M\zeta_2)}\right]^{\frac{1}{2}} + iS_3[\zeta_2^2(1-L^2M^2)-2]\left(\frac{\zeta_2}{1+\zeta_2}\right)^{\frac{1}{2}}\right\}, \tag{19}$$

where S_1 and S_2 are real functions of ζ which, being bounded at infinity and having no singularities in the finite part of the complex plane, must be constants. The behaviour of the three complex derivatives, given as in equations 13, 18 and 19, as $\zeta \to \infty$ ensures bounded displacements and velocities at the corresponding points $s = \gamma$, $\theta = +\pi$.

The values of the constants S_2 and S_3 may now be found. The form of the complex derivatives is only correct on a certain segment of the crack, but the expressions may be continued round the various branch points to give the appropriate results elsewhere. In particular, the residue of the complex derivatives at simple poles depends on whether we approach the singularity from above or below the real ζ-axis. There is no need to use the results of equation 9 in order to determine the constants, since the residues at $\zeta_1 = \sec\alpha_1$ and at $\zeta_2 = \sec\alpha_2$ provide enough information to define the coefficients R^0_{mn} as well as the constants S_2 and S_3. However, prior knowledge of these reflection coefficients saves work.

When the incident dilatation field is a step function of amplitude $-A_1$, travelling with a shear field whose z-component is a step-function of amplitude $-A_2$, the residue of $dW^C/d\zeta_2$ at $\zeta_2 = \sec\alpha_2$ is $A_2/\pi i$ or $(A_1R^0_{12} + A_2R^0_{22})/\pi i$ according as the pole is approached from above or below. The results

$$[1+L^2\sec^2\alpha_1]S_2 = \frac{(2\gamma_1)^{\frac{1}{2}}\cos\alpha_1/2\cos^2\alpha_1 R(\sec\alpha_1)}{\pi(\gamma_1-P\cos\alpha_1)^{\frac{1}{2}}Q(\sec\alpha_1)}\left\{\frac{(L^2M^2+\cos 2\alpha_2)}{\cos^2\alpha_2(1+L^2\sec^2\alpha_1)}A_1 + \tan\alpha_2(1+L^2\sec^2\alpha_1)A_2\right\} \tag{20}$$

and

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$$S_3 = \frac{2^{\frac{1}{2}} M \cos\alpha_2/2\cos^2\alpha_1 R(\sec\alpha_1)}{\pi Q(\sec\alpha_1)} \left\{ \frac{2\tan\alpha_1}{1+L^2\sec^2\alpha_1} A_1 - \frac{(L^2M^2+\cos 2\alpha_2)A_2}{\cos^2\alpha_2} \right\}. \tag{21}$$

The equations 18 and 19 may be used to calculate the normal displacements at the crack. For $0 < s < P$ we have already imposed and satisfied the condition of continuous normal velocity, but if the velocity P with which the crack closes is greater than the transverse Rayleigh velocity γ_R, the complex potentials still have a simple pole at $\zeta_1 = \gamma_R/\gamma_1$ or $\zeta_2 = \gamma_R/\gamma_2$. This pole results in a step discontinuity in normal *displacement*, Y, travelling with the Rayleigh velocity. The expression for the normal velocity $\partial y/\partial t$ is given by the formula

$$r\frac{\partial Y}{\partial t} = Rl\left\{ i\zeta_1(1-\zeta_1^2)^{\frac{1}{2}}\frac{dW^\phi}{d\zeta_1} - \zeta_2(1+L^2M^2\zeta_2^2)\frac{dW^C}{d\zeta^2}\right\}$$

which reduces to the form

$$r\frac{\partial Y}{\partial t} = Rl\left\{\frac{\zeta_1^5 Q(\zeta_1)}{(\zeta_1-\sec\alpha)R(\zeta_1)M^3}\left[\frac{-S_2(M^2-\zeta_1^2)^{\frac{1}{2}}}{(1+L^2\sec^2\alpha_1)}\left[\frac{\zeta_1\gamma_1-P}{\gamma_1(1+\zeta_1)}\right]^{\frac{1}{2}} + \frac{iS_3}{2}\left[\frac{\zeta_1}{M+\zeta_1}\right]^{\frac{1}{2}}\left[2-\frac{\zeta_1^2}{M^2}(1-L^2M^2)\right]\right]\right\} \tag{22}$$

for all values of ζ_1 in the range $0 < \zeta_1 < 1$. To find the jump in Y at $s = \gamma_R$, we evaluate the residue of the right-hand side of this equation; we see that when $P > \gamma_R$ the part of the discontinuity in Y which depends on S_3 is continuous across the crack, whereas the part of this discontinuity which depends on S_2 is antisymmetric across the crack when $\gamma_R < s < P < \gamma_2$. The sign of S_2 ensures the *opening* of the crack for $s < \gamma_R$, contrary to hypothesis. If $\gamma_1 > P > \gamma_2$, the same formula for the normal velocity may be derived, and again the hypothesis of a closed crack is shown wrong in the range $\gamma_R < s < \gamma_2$. Thus $P \leqq \gamma_R$. But if with $P < \gamma_R$, we now consider the neighbourhood of the Rayleigh pole which now corresponds to a point on the *open* section of crack, the term in S_2 contributes an *inward* logarithmic singularity in displacement, travelling with the Rayleigh velocity. And although the crack is open in this region, the singularity ensures that the surfaces must perform the feat of passing through each other. The only possibility which remains is that the Rayleigh singularity $s = \gamma_R$ is the point which separates the closed from the open section of crack, and this is the only self-consistent hypothesis available.

Thus the opened crack closes itself with the transverse velocity of Rayleigh waves, and the complex potentials associated with the incident pulses

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$$\phi = -A_1 U\left(t - \frac{z}{a} - \frac{x \cos \alpha_1 + y \sin \alpha_1}{\gamma_1}\right),$$

$$\psi = -A_2(\boldsymbol{k} - i\boldsymbol{L} \sec \alpha_1) U\left(t - \frac{z}{a} - \frac{x \cos \alpha_2 + y \sin \alpha_2}{\gamma_2}\right),$$

are given by equations 13, and 18 through 21, with the specific value $P = \gamma_R$.

The velocity potentials in the head wave still remain to be determined, but their derivatives follow easily from the continuity of $\partial\psi/\partial\theta$ across the circle $s = \gamma_2$, and from the known functional behaviour in these regions.

5. Formulation for the incident compressive field

The case when we have the incident field

$$\phi = +A_1 U\left(t - \frac{z}{a} - \frac{x \cos \alpha_1 + y \sin \alpha_1}{\gamma_1}\right),$$

$$\psi = +A_2(\boldsymbol{k} - i\boldsymbol{L} \sec \alpha_1) U\left(t - \frac{z}{a} - \frac{x \cos \alpha_2 + y \sin \alpha_2}{\gamma_2}\right),$$

is more straightforward than that considered in the previous section. In the first place the incident field is now compressive, the crack stays closed outside the circle $s = \gamma_1$, and there is no reason to have it open within this circle. Secondly, with the crack uniformly closed we can separately examine the combinations of the symmetric part of ψ with the antisymmetric part of ϕ, and then the alternate combination. With the aid of the known reflection and transmission coefficients R^s_{mn} and T^s_{mn} we can check the form of the potentials on the circles $s = \gamma$; we see that on $s = \gamma_1$, the potential ϕ is antisymmetric about $\theta = \pi$, so that $\partial\phi/\partial s$ must vanish on $\theta = \pi$ when $s < \gamma_1$. Likewise the vector potential ψ is symmetric about $\theta = \pi$ on the circle $s = \gamma_2$, so that $\partial\psi/\partial\theta$ vanishes for $\theta = \pi$, $s < \gamma_2$.

These results make necessary the imposing of the extra conditions that $\mathrm{d}W^C/\mathrm{d}\zeta_1$ be imaginary for $-1 = \zeta_1 < 0$, and that $\mathrm{d}W^C/\mathrm{d}\zeta_2$ be real for $-1 < \zeta_2 < 0$. These conditions are satisfied directly by the equations 18 and 19 with $P = 0$, provided that S_2 is also zero, this in order to satisfy the new symmetry conditions.

The remaining constant S_3 is found by evaluating the residue of the complex derivatives at $\zeta_1 = \sec \alpha_1$; it takes *precisely* the open crack value given in equation 21. This equality is a direct consequence of the simple relations (equation 10) between the open and the closed crack scattering coefficients.

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6. Results for the field incident along the crack

In this section the initial situation is that of a combination of plane shear and dilatation fronts travelling in a general direction away from the crack and satisfying the appropriate boundary and continuity conditions at the crack. The formulation of the problem is the same as that already given, except that the incident field combination is travelling in the negative instead of the positive x-direction.

The transverse field structure behind the point of intersection of the pulse fronts and the edge of the crack is shown in figure 2. One restriction is necessary as long as we restrict attention to conical motions. To avoid the nonconical situation associated with totally reflected dilatation waves the angle α_2 is restricted to the range $0 < \alpha_2 < \operatorname{arcsec} 1/m \sec \alpha_1$. With this proviso we turn directly to the equations 18 and 19 for general results to be satisfied by the complex derivatives. The only change required in these equations is a sign change in the quantities $\sec \alpha_1$ and $\sec \alpha_2$, so that the simple pole of the complex potentials is on the negative ζ-axis.

For the incident tensile field with

$$\phi = -A_1 U\left(t - \frac{z}{a} + \frac{x\cos\alpha_1 - y\sin\alpha_1}{\gamma_1}\right) - (A_1 R^0_{11} + A_2 R^0_{21}) U\left(t - \frac{z}{a} + \frac{x\cos\alpha_1 + y\sin\alpha_1}{\gamma_1}\right),$$

$$\psi = -[k + iL\sec\alpha_1]\left\{A_2 U\left(t - \frac{z}{a} + \frac{x\cos\alpha_2 - y\sin\alpha_2}{\gamma_2}\right) + (A_2 R^0_{22} + A_1 R^0_{12}) U\left(t - \frac{z}{a} + \frac{x\cos\alpha_2 + y\sin\alpha_2}{\gamma_2}\right)\right\},$$

we fix constants S_2 and S_3 by evaluating the residues at $\zeta_2 = \sec\alpha_2$ of the derivative $\mathrm{d}W^C/\mathrm{d}\zeta_2$; the residue approaching the pole from above is $-A_2/\pi i$ while that approaching the pole from below is $-(A_2 R^0_{22} + A_1 R^0_{12})/\pi i$. We have results very similar to those of equations 20 and 21; taking note that $R(\sec\alpha_1) = R(-\sec\alpha_1)$, that $Q(-\sec\alpha_1)$ as defined from equation 17 is negative, the quantity S_2 retains its positive definite form and this plays an important part in the argument for putting $P = \gamma_R$. Thus

$$(1 + L^2\sec^2\alpha_1) S_2 = \frac{-(2\gamma_1)^{\frac{1}{2}}\cos\alpha_{\frac{1}{2}}\cos^2\alpha_1 R(\sec\alpha_1)}{\pi(\gamma_1 + \gamma_R\cos\alpha_1)^{\frac{1}{2}} Q(-\sec\alpha_1)} \left\{\frac{(L^2M^2 + \cos 2\alpha_2)}{\cos^2\alpha_2(1 + L^2\sec^2\alpha_1)} A_1 + A_2\tan\alpha_2(1 + L^2\sec^2\alpha_2)\right\}, \tag{23}$$

and

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$$
\begin{aligned}
S_3 = {} & \frac{-2^{\frac{1}{2}} M \cos \alpha_{\frac{1}{2}} \cos^2 \alpha_1 R(\sec \alpha_1)}{\pi Q(-\sec \alpha_1)} \\
& \left\{ \frac{2 \tan \alpha_1}{1 + L^2 \sec^2 \alpha_1} A_1 - \left(\frac{L^2 M^2 + \cos 2\alpha_2}{\cos^2 \alpha_2} \right) A_2 \right\},
\end{aligned} \tag{24}
$$

are the constants to be used in equation 18 and 19 with P.

When the incident field is compressive, the arguments of section 5 hold, with again the result that equations 18 and 19 with $P = 0$, $S_2 = 0$ and with S_3 given exactly as in equation 24, define the complex derivatives.

7. Extention of results

The problem discussed in this paper is a non-linear one, since the state of the crack, and therefore the form of the continuity conditions to be applied, is a function of the dependent variables themselves. The solution for incident step-function fields is only possible because we can make self-consistent assumptions about the state of the crack which depend only on the nature, compressive or tensile, of the incident field. The most important remaining point is to discuss whether the solutions derived in previous sections of this paper may be utilized in cases of more general incident field.

The case of an incident compressive field may be disposed of most easily. Since, by applying Duhamel's principle, we may regard any function as the superposition of a sequence of step-functions, and since there is no evidence that a compressive pulse is capable of opening the crack, the linear superposition of scattered fields is justifiable.

The same is true for an incident tensile field if it does not meet the crack before arrival at the edge. The point here is that whatever the previous history of the crack, the edge is always closed, and each step-function component has to contend with the same conditions on arrival at the edge. Nor does it matter what the previous history of the crack is along its line of intersection with a particular pulse; a change of state at any part of the crack does not involve a finite separation of surfaces as long as we are concerned with first order deformation theory, so that a given incident field determines for itself what the local state of the crack shall be.

The situation for which we do not have available information involves the superposition of incident tensile pulses which travel along the crack before arriving at the edge. Although we have a clear solution for a single pulse any subsequent pulse will be scattered not only at the edge of the crack, but also at the line, moving with the velocity of Rayleigh waves where there is a definite change of state in the crack. To superimpose results for the individual pulses is not enough, we need to know the scattering effect of a moving change of state on an incident field and we also need to know the

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scattering effect of the crack edge on this scattered field. The method of this paper is not adequate to produce this information, but one based on the study of singular waves is available and may give information.

Coda: Application to the brittle fracture problem

One way in which the analysis described in this paper may be extended economically lies in the discussion of a brittle fracture problem. An already existing, stress-free semi-infinite plane crack is, as before, an obstacle to an incident tensile step-function field. The field singularities which are to be expected in the normal elasticity theory at the edge of the crack are taken to cause brittle fracture, and the resulting crack is taken to be smooth, and to be extending itself in the plane of the original crack with a constant transverse velocity γ. This final condition is one which implies that the mean strain-energy density at the moving edge is zero.

The first statement to be made is that the same conditions hold which led to the earlier conclusion that the original crack closes itself with the velocity of Rayleigh waves.

As far as the new crack is concerned we have two practical possibilities which affect the analysis. The constant velocity of propagation γ lies either in the range $0 < \gamma < \gamma_R$ or $\gamma \geqq \gamma_R$.

If it is in the former range, there is no possibility for the new crack to be anything but closed (otherwise we have to introduce branch points into the expressions for potential, and this results in contradictions). Then in equation 18 we have to replace, in the product containing S_3, the factor $(\zeta_1)^{\frac{1}{2}}$ by the factor $(\zeta_1 + \gamma/\gamma_1)^{\frac{1}{2}}$. This leaves a corresponding change to be made in equation 19, and eventually in the expression for S_3 itself [in equation 21, the right-hand side of the equation must be multiplied by $(1+\gamma/\gamma_1 \cos\alpha)^{\frac{1}{2}}$].

The reason for rejecting this possibility follows the calculation of the strain energy density near the edge of the crack. If the crack were being extended by the application of a singular load, e.g. by pulling an infinite wire through the material we could accept a non-zero energy density, but the problem here is one with the source of energy at infinity. Likewise if the velocity of the crack were not constant we could expect the energy of deformation in the vicinity of the crack edge to be non-zero. For a given value of γ the strain energy density reduces to the form

$$\frac{4\mu[D_3(\gamma)]^2}{\gamma_1^2\gamma_2\tau\gamma}\left\{\left(1+\frac{L^2\gamma^2}{\gamma_1^2}\right)\left(1-\frac{\gamma^2}{M^2\gamma_2^2}\right)^{\frac{1}{2}}-\left[1-\frac{\gamma^2}{2M^2\gamma_2^2}\,(1-L^2M^2)\right]\right\} = \frac{4\mu[D_3(\gamma)]^2C(\gamma)}{\gamma_1^2\gamma_2\gamma\tau},$$

where $D_3(\gamma)$ is a function of γ defined in conformity with equation 11 by the ratio $P_3(\zeta_1)/R(\zeta_1)$ for the value $\zeta_1 = \gamma/\gamma_1$.

This can only vanish if $\gamma = 0$. (The equation $C(\gamma) = 0$ appears to have two other pairs of roots besides a double root at the origin, but these are introduced in the process of squaring.)

When $\gamma_2 > \gamma > \gamma_R$ there is not the need in equation 17 to introduce the factor $(\zeta_2 + \gamma_R/\gamma_2)$, but to preserve the form of the solutions 18 and 19 as $\zeta_2 \to \infty$ we must replace this factor by ζ_2; and again we must have the radical $(\zeta_1 + \gamma/\gamma_1)^{\frac{1}{2}}$ present in the S_3 terms of both equations 18 and 19. A local examination of displacements in the neighbourhood of the $\zeta_1 = \gamma_R/\gamma_1$ gives contradictory results. In fact to show that $\gamma = \gamma_R$ is the only self-consistent non-zero possibility involves precisely the same stages of argument as those used in Section 4 to show that the existing crack closes itself with the velocity of Rayleigh waves.

Without any further analysis therefore we are able to say that the only possible unforced velocity of brittle fracture is itself equal to the velocity of free surface waves.

Previous work on the dynamics of crack propagation is restricted to the papers of Yoffe [6] and Craggs [7], and in both these papers the motion concerned is forced by the application of a load to the surfaces of an existing crack. In such a system a given load is linked with a steady velocity of extension which is less than the Rayleigh velocity. The limiting case in the situation escribed by Craggs, is also associated with the Rayleigh velocity, but this case, one in which the external forces do no work because their limiting value is zero, is one which tells nothing about the transient nature of crack propagation. The point of the discussion here is both that a crack is eventually propagating in an unforced manner and that a reasonable excuse for the initiation of this process has been provided.

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