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Distribution of the number of prime factors with a given multiplicity

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Abstract. Given an integer $k \geq 2$, let $\omega_k(n)$ denote the number of primes that divide n with multiplicity exactly k. We compute the density $e_{k,m}$ of those integers n for which $\omega_k(n) = m$ for every integer $m \geq 0$. We also show that the generating function $\sum_{m=0}^{\infty} e_{k,m} z^m$ is an entire function that can be written in the form $\prod_p \big(1 + (p-1)(z-1)/p^{k+1}\big)$; from this representation we show how to both numerically calculate the $e_{k,m}$ to high precision and provide an asymptotic upper bound for the $e_{k,m}$. We further show how to generalize these results to all additive functions of the form $\sum_{j=2}^{\infty} a_j \omega_j(n)$; when $a_j = j-1$ this recovers a classical result of Rényi on the distribution of $\Omega(n) - \omega(n)$.

1 Introduction

Let $\omega(n)$ be the number of distinct prime factors of a positive integer n, and let $\Omega(n)$ be the number of prime factors of n counted with multiplicity. Average behaviours of such arithmetic functions are understood via their summatory functions. It is known [7] (see also [8, Theorems 427–430]) that

$$\begin{aligned} \sum_{n \leqslant x} \omega(n) &= x \log\log x + bx + O\left(\frac{x}{\log x}\right) \\ \sum_{n \leqslant x} \Omega(n) &= x \log\log x + \left(b + \sum_p \frac{1}{p(p-1)}\right)x + O\left(\frac{x}{\log x}\right); \end{aligned} \tag{1.1}$$

here the constant b is defined by

$$b = \gamma_0 + \sum_p \sum_{j=2}^{\infty} \frac{1}{jp^j} \tag{1.2}$$

where γ_0 denotes the Euler–Mascheroni constant. (In this paper, $\sum_p$ and $\prod_p$ always denote sums and products running over all prime numbers.) The celebrated Erdős–Kac theorem tells us that both $\omega(n)$ and $\Omega(n)$ can be normalized to have Gaussian limiting distribution functions.

By the asymptotic formulas (1.1), the difference $\Omega(n) - \omega(n)$ has an average value, namely the constant

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n \leqslant x} \big(\Omega(n) - \omega(n)\big) = \sum_p \frac{1}{p(p-1)},$$

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which provides motivation to study the frequency of each possible value of $\Omega(n) - \omega(n)$. For any integer $m \geqslant 0$, define

$$\mathcal{N}_m(x) = \{n \leqslant x : \Omega(n) - \omega(n) = m\}.$$

Rényi [10] (see also [9, Section 2.4]) proved that the (natural) densities

$$d_m = \lim_{x\to\infty} \frac{\#\mathcal{N}_m(x)}{x} = \frac{6}{\pi^2} \sum_{\substack{f\in\mathcal{F} \\ \Omega(f)-\omega(f)=m}} \frac{1}{f} \prod_{p\mid f} \left(1 + \frac{1}{p}\right)^{-1} \tag{1.3}$$

exist for every $m \geqslant 0$, where $\mathcal{F}$ is the set of powerful numbers (the set of positive integers all of whose prime factors have multiplicity $\geqslant 2$). Furthermore, he showed that these densities have the generating function

$$\sum_{m=0}^{\infty} d_m z^m = \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \frac{1}{p-z}\right) \qquad (|z| < 2). \tag{1.4}$$

(Note the special case $d_0 = \prod_p (1 - \frac{1}{p})(1 + \frac{1}{p}) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$ for the density of squarefree numbers, which can also be confirmed by realizing that the sum in equation (1.3) contains only the single term $f = 1$ when $m = 0$.) In particular, the smaller function $\Omega(n) - \omega(n)$ already has a (discrete) limiting distribution function, without needing normalization in the way that the larger functions $\omega(n)$ and $\Omega(n)$ individually do.

As a refinement of the function $\omega(n)$, Liu and the first author introduced the functions

$$\omega_k(n) = \sum_{p^k \| n} 1$$

for each integer $k \geqslant 1$, so that $\omega_k(n)$ counts the number of prime factors of n with multiplicity k and thus $\omega(n) = \sum_{k=1}^{\infty} \omega_k(n)$. They showed [2] that

$$\sum_{n\leqslant x} \omega_1(n) = x \log\log x + \left(b - \sum_p \frac{1}{p^2}\right)x + O\left(\frac{x}{\log x}\right)$$

where b is the constant from equation (1.2), while

$$\sum_{n\leqslant x} \omega_k(n) = x \sum_p \frac{p-1}{p^{k+1}} + O\left(x^{(k+1)/(3k-1)} \log^2 x\right) \qquad (k \geqslant 2). \tag{1.5}$$

They also showed that the larger function $\omega_1(n)$ has a Gaussian limiting distribution function after being normalized in the same way as the classical $\omega(n)$ and $\Omega(n)$. However, since equation (1.5) shows that $\omega_k(n)$ has an average value for each $k \geqslant 2$, we might expect these smaller functions to have limiting distributions without needing to be normalized.

In this paper, we obtain the limiting distribution for the functions $\omega_k(n)$ for $k \geqslant 2$, analogous to the results of Rényi described above. For integers $m \geqslant 0$, define

$$\mathcal{N}_{k,m}(x) = \{n \leqslant x : \omega_k(n) = m\} \tag{1.6}$$

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to be the set of positive integers $n \leqslant x$ with exactly m prime factors of multiplicity k. Our main result establishes the existence of the densities

$$e_{k,m} = \lim_{x\to\infty} \frac{\#\mathcal{N}_{k,m}(x)}{x}$$

and gives an exact formula for the $e_{k,m}$ involving an infinite sum.

Theorem 1.1 *Uniformly for all integers $k \geqslant 2$ and $m \geqslant 0$,*

$$\#\mathcal{N}_{k,m}(x) = e_{k,m}x + O(x^{1/2}\log x)$$

with

$$e_{k,m} = \frac{6}{\pi^2} \sum_{\substack{f\in\mathcal{F} \\ \omega_k(f)=m}} \frac{1}{f} \prod_{p\mid f}\left(1+\frac{1}{p}\right)^{-1}.$$

Remark 1.2 Note that the $e_{k,m}$ are all nonnegative, and we can check that they do sum to 1:

$$\sum_{m=0}^{\infty} e_{k,m} = \sum_{m=0}^{\infty} \frac{6}{\pi^2} \sum_{\substack{f\in\mathcal{F} \\ \omega_k(f)=m}} \frac{1}{f} \prod_{p\mid f}\left(1+\frac{1}{p}\right)^{-1} = \frac{6}{\pi^2} \sum_{f\in\mathcal{F}} \frac{1}{f} \prod_{p\mid f}\left(1+\frac{1}{p}\right)^{-1}.$$

Since the summand is a multiplicative function of f, as is the indicator function of $\mathcal{F}$, the right-hand side equals its Euler product

$$\frac{6}{\pi^2} \prod_p \left(1 + 0 + \left(1+\frac{1}{p}\right)^{-1}\left(\frac{1}{p^2}+\frac{1}{p^3}+\cdots\right)\right) = \frac{6}{\pi^2}\prod_p\left(1-\frac{1}{p^2}\right)^{-1} = 1.$$

The same remark applies to the densities in equation (1.10) below.

Moreover, we obtain an identity analogous to equation (1.4) for the generating function of the densities $e_{k,m}$ for fixed $k \geqslant 2$, from which we can derive an upper bound for the densities $e_{k,m}$ when $k \geqslant 2$ is fixed and $m \to \infty$.

Theorem 1.3 *Let $k \geqslant 2$ be an integer. For all $z \in \mathbb{C}$ with $|z| \leqslant 1$,*

$$\sum_{m=0}^{\infty} e_{k,m} z^m = \prod_p \left(1 + \frac{(p-1)(z-1)}{p^{k+1}}\right). \tag{1.7}$$

Corollary 1.4 *For each fixed $k \geqslant 2$, we have $e_{k,m} \leqslant m^{-(k-o(1))m}$ as $m \to \infty$.*

Remark 1.5 The proof of the upper bound in Corollary 1.4 (see Section 3) shows that for each $k \geqslant 2$, the bound is attained for infinitely many m; it would be interesting to try to show that $e_{k,m} = m^{-(k-o(1))m}$ for all k and m. Moreover, the corollary and its proof show that both sides of equation (1.7) converge to entire functions, and thus Theorem 1.3 actually holds for all $z \in \mathbb{C}$ by uniqueness of analytic continuation. The same remarks apply to the generating functions in Corollary 1.14 and the upper bounds in Corollary 1.15 below.

Table 1: Some values of $e_{k,m}$.

	$m = 0$	$m = 1$	$m = 2$	$m = 3$
$k = 2$	0.748535831	0.226618489	0.023701061	0.001117529
$k = 3$	0.904708927	0.092831692	0.002440388	0.000018941
$k = 4$	0.959088654	0.040585047	0.000325821	0.000000477
$k = 5$	0.981363751	0.018587581	0.000048654	0.000000014

Some numerical values of $e_{k,m}$ are given in Table 1. The numbers in the first column corresponding to $m = 0$ are increasing as k increases, whereas the numbers in other columns are decreasing. This behaviour stems from the fact that the case $m = 0$ indicates the nonexistence of prime factors with multiplicity k, which becomes more probable as k increases. (Note also that each number in the first column exceeds $\frac{6}{\pi^2} \approx 0.608$, since every squarefree number n certainly has $\omega_k(n) = 0$ for all $k \geqslant 2$.) On the other hand, for $m \geqslant 1$, the criterion $\omega_k(n) = m$ indicates the existence of prime factors with multiplicity k, which becomes less probable as k increases. Details of the calculations of these values are given in Section 4, although we do note here that the calculations use the generating function in Theorem 1.3 rather than the formula for $e_{k,m}$ in Theorem 1.1.

A consequence of Theorem 1.1 and Remark 1.2 is that $\omega_k(n)$ has a limiting distribution, which is the same as the distribution of the nonnegative integer-valued random variable X_k that takes the value m with probability $e_{k,m}$. While it is straightforward to calculate the expectation and variance of this limiting distribution via the expressions

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leqslant x} \omega_k(n) \qquad \text{and} \qquad \lim_{x\to\infty} \frac{1}{x} \sum_{n\leqslant x} \omega_k(n)^2 - \left(\lim_{x\to\infty} \frac{1}{x} \sum_{n\leqslant x} \omega_k(n) \right)^2,$$

we can observe that the generating function from Theorem 1.3 provides a quick way to obtain the answers with no further input from number theory.

Corollary 1.6 *The limiting distribution of $\omega_k(n)$ has expectation $\sum_p \frac{p-1}{p^{k+1}}$ and variance $\sum_p \frac{p-1}{p^{k+1}}\left(1 - \frac{p-1}{p^{k+1}}\right)$.*

Remark 1.7 Not surprisingly, these quantities are the expectation and variance of the sum of infinitely many Bernoulli random variables B_p, indexed by primes p, where B_p takes the value 1 with probability $(p-1)/p^{k+1}$ (the density of those integers exactly divisible by p^k).

These quantities are easy to calculate to reasonably high precision (see Section 4 for details); we record some numerical values in Table 2. The reader can confirm that the listed expectations are in good agreement with the quantities $0e_{k,0} + 1e_{k,1} + 2e_{k,2} + 3e_{k,3}$ as calculated from Table 1.

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Table 2: Statistics of the limiting distribution of $\omega_k(n)$.

	expectation of $\omega_k(n)$	variance of $\omega_k(n)$
$k = 2$	0.277484775	0.254931583
$k = 3$	0.097769500	0.093205673
$k = 4$	0.041238122	0.040192048
$k = 5$	0.018684931	0.018433195

1.1 Generalizations

It turns out that our proof of Theorem 1.1 goes through for a far larger class of additive functions than just the $\omega_k(n)$. Given any sequence $A = (a_1, a_2, a_3, \dots)$ of complex numbers, define the additive function

$$\omega_A(n) = \sum_{j=1}^{\infty} a_j \omega_j(n), \tag{1.8}$$

which is of course a finite sum for each integer n.

Remark 1.8 This definition generalizes all the examples we have seen so far:

- if $a_j = 1$ always then $\omega_A(n) = \omega(n)$;
- if $a_j = j$ always then $\omega_A(n) = \Omega(n)$;
- if $a_j = j - 1$ always then $\omega_A(n) = \Omega(n) - \omega(n)$;
- for a fixed positive integer k, if $a_k = 1$ while $a_j = 0$ for $j \neq k$, then $\omega_A(n) = \omega_k(n)$.

When $a_1 \neq 0$, classical techniques show that the large function $\frac{1}{a_1}\omega_A(n)$ has the same Gaussian limiting distribution as $\omega(n)$ and $\Omega(n)$ when properly normalized (at least if the a_j do not grow too quickly). Therefore we restrict our attention to the smaller functions $\omega_A(n)$ where $a_1 = 0$, which we expect to have limiting distributions without needing normalization.

For $m \in \mathbb{C}$, define

$$\mathcal{N}_{A,m}(x) = \{n \leqslant x : \omega_A(n) = m\}.$$

Our next result, which generalizes both equation (1.3) and Theorem 1.1, establishes the existence of the densities

$$e_{A,m} = \lim_{x\to\infty} \frac{\#\mathcal{N}_{A,m}(x)}{x} \tag{1.9}$$

and provides an exact formula for them.

Theorem 1.9 *Uniformly for all sequences $A = (0, a_2, a_3, \dots)$ of complex numbers with $a_1 = 0$ and for all $m \in \mathbb{C}$,*

$$\#\mathcal{N}_{A,m}(x) = e_{A,m}x + O(x^{1/2}\log x)$$

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with

$$e_{A,m} = \frac{6}{\pi^2} \sum_{\substack{f \in \mathcal{F} \\ \omega_A(f)=m}} \frac{1}{f} \prod_{p \mid f} \left(1 + \frac{1}{p}\right)^{-1}. \tag{1.10}$$

If we now restrict to the case where the a_j (and thus all values of $\omega_A(n)$) are nonnegative integers, it once again makes sense to consider generating functions. Our next result generalizes both equation (1.4) and Theorem 1.3 in light of Remark 1.8.

Theorem 1.10 *Let $A = (0, a_2, a_3, \dots)$ be a sequence of nonnegative integers. For all $z \in \mathbb{C}$ with $|z| \leqslant 1$,*

$$\sum_{m=0}^{\infty} e_{A,m} z^m = \prod_p \left(1 - \frac{1}{p^2} + \sum_{j=2}^{\infty} z^{a_j} \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right)\right). \tag{1.11}$$

Remark 1.11 The Erdős–Wintner theorem [3, 4] (see also [11, Chapter III.4]) implies the existence of a limiting distribution for $\omega_A(n)$, as well as a formula for its characteristic function that is related to equation (1.11); indeed that approach works for any additive function f with $f(p) = 0$ for all primes p. On the other hand, our elementary approach shares the advantages of Rényi's [10] of giving formulas for the densities $e_{A,m}$ and the means to compute their numerical values and asymptotic size.

Again Theorem 1.10 shows that $\omega_A(n)$ has a limiting distribution when the a_j are nonnegative integers, and we can therefore generalize Corollary 1.6; we record only the expectation for simplicity.

Corollary 1.12 *Let $A = (0, a_2, a_3, \dots)$ be a sequence of nonnegative integers. The limiting distribution of $\omega_A(n)$ has expectation $\sum_p \sum_{j=2}^{\infty} a_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right)$.*

Remark 1.13 It is certainly possible for this expectation to be infinite, as the example $A = (0, 2, 4, 8, 16, \dots)$ shows. In such cases $\frac{1}{x} \sum_{n \leqslant x} \omega_A(n)$ grows too quickly for the mean value of $\omega_A(n)$ to exist. Note, however, that Theorems 1.9 and 1.10 hold no matter how quickly the sequence A might grow.

We examine three specific examples of such sequences for the purposes of illustration: set $S = (0, 1, 1, \dots)$ and $E = (0, 1, 0, 1, \dots)$ and $O = (0, 0, 1, 0, 1, 0, 1, \dots)$. Then the corresponding omega functions are

$$\omega_S(n) = \sum_{j \geqslant 2} \omega_j(n) \quad \text{and} \quad \omega_E(n) = \sum_{\substack{j \geqslant 2 \\ j \text{ even}}} \omega_j(n) \quad \text{and} \quad \omega_O(n) = \sum_{\substack{j \geqslant 3 \\ j \text{ odd}}} \omega_j(n)$$

which count, respectively, the number of primes dividing the powerful part of n (that is, the number of primes dividing n at least twice), the number of primes dividing n with even multiplicity, and the number of primes dividing n with odd multiplicity exceeding 1. For integers $m \geqslant 0$, let $e_{S,m}$ and $e_{E,m}$ and $e_{O,m}$ be the corresponding densities defined in equation (1.9). An easy calculation of the right-hand side of

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equation (1.11) in these cases (for which each factor becomes a geometric series) yields the following generating functions:

Corollary 1.14 *For all $z \in \mathbb{C}$ with $|z| \leqslant 1$,*

$$\sum_{m=0}^{\infty} e_{S,m} z^m = \prod_p \left(1 + \frac{z-1}{p^2}\right)$$
$$\sum_{m=0}^{\infty} e_{E,m} z^m = \prod_p \left(1 + \frac{z-1}{p(p+1)}\right)$$
$$\sum_{m=0}^{\infty} e_{O,m} z^m = \prod_p \left(1 + \frac{z-1}{p^2(p+1)}\right).$$

Corollary 1.15 *We have $e_{S,m} \leqslant m^{-(2-o(1))m}$ and $e_{E,m} \leqslant m^{-(2-o(1))m}$ and $e_{O,m} \leqslant m^{-(3-o(1))m}$ as $m \to \infty$.*

Remark 1.16 One interesting class of functions for which our methods accomplish less than desired are functions of the form $\omega_A(n)$ where A contains integers but not necessarily only nonnegative integers. For example, if $A = (0, 1, -1, 0, 0, \dots)$ then $\omega_A(n) = \omega_2(n) - \omega_3(n)$, while if $A = (0, 1, -1, 1, -1, \dots)$ then $\omega_A(n) = \omega_E(n) - \omega_O(n)$. The target $m = 0$ is natural to investigate, as $\omega_A(n) = 0$ in these two examples translates into $\omega_2(n) = \omega_3(n)$ and $\omega_E(n) = \omega_O(n)$, respectively. While Theorem 1.9 gives a formula for the density of those integers n satisfying each of these equalities, our numerical techniques in Section 4 (which ultimately rely on being able to find the values of the derivatives of the appropriate generating function at $z = 0$) are not able to approach the question of good numerical approximations to these densities.

In Section 2 we establish Theorems 1.9 and 1.10, the formula and generating function for $e_{A,m}$, from which Theorems 1.1 and 1.3 follow as special cases. In Section 3 we deduce Corollaries 1.4 and 1.15 (the decay rates of $e_{k,m}$ and certain variants) from Theorem 1.3 and Corollary 1.14. Finally, in Section 4 we describe the computations leading to the numerical values in Tables 1 and 2, as well as establishing Corollaries 1.6 and 1.12 concerning the expectation and variance of the additive functions under examination.

2 Exact formula and generating function for the densities

We first prove Theorem 1.9, which will also establish the special case that is Theorem 1.1, by following the exposition of Rényi's result (1.3) in [9, Section 2.4]. Recall the notation of equation (1.8), and recall that $\mathcal{F}$ denotes the set of powerful numbers.

Lemma 2.1 *Uniformly for all sequences $A = (a_1, a_2, \dots)$ of complex numbers and all $m \in \mathbb{C}$,*

$$\sum_{\substack{f \leqslant x \\ f \in \mathcal{F} \\ \omega_A(f) = m}} \frac{1}{f^{1/2}} \prod_{p \mid f} (1 - p^{-1/2})^{-1} \ll \log x.$$

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Proof By dropping the condition $\omega_A(f) = m$ and noting that $f \leqslant x$ implies that all prime factors of f are at most x, we have by positivity

$$\sum_{\substack{f\leqslant x\\ f\in\mathcal{F}\\ \omega_A(f)=m}} \frac{1}{f^{1/2}} \prod_{p\mid f}(1-p^{-1/2})^{-1} \leqslant \sum_{\substack{f\leqslant x\\ f\in\mathcal{F}}} \frac{1}{f^{1/2}} \prod_{p\mid f}(1-p^{-1/2})^{-1}$$
$$\leqslant \sum_{\substack{f\in\mathcal{F}\\ p\mid f \implies p\leqslant x}} \frac{1}{f^{1/2}} \prod_{p\mid f}(1-p^{-1/2})^{-1}.$$

The right-hand side has an Euler product whose factors involve geometric series with common ratio $p^{-1/2}$:

$$\sum_{\substack{f\in\mathcal{F}\\ p\mid f \implies p\leqslant x}} \frac{1}{f^{1/2}} \prod_{p\mid f}(1-p^{-1/2})^{-1} = \prod_{p\leqslant x}\left(1+\frac{(1-p^{-1/2})^{-1}}{(p^2)^{1/2}}+\frac{(1-p^{-1/2})^{-1}}{(p^3)^{1/2}}+\cdots\right)$$
$$= \prod_{p\leqslant x}\left(1+\frac{1}{(p^{1/2}-1)^2}\right)$$
$$= \prod_{p\leqslant x}\left(1-\frac{1}{p}\right)^{-1} \prod_{p\leqslant x}\left(1+\frac{2}{p(p^{1/2}-1)}\right);$$

this establishes the lemma, since the first product is asymptotic to a multiple of $\log x$ as shown by Mertens, while the second is a convergent product of the form $\prod_p(1+O(p^{-3/2}))$. ■

Lemma 2.2 *Uniformly for all sequences $A = (a_1, a_2, \dots)$ of complex numbers and all $m \in \mathbb{C}$,*

$$\sum_{\substack{f>x\\ f\in\mathcal{F}\\ \omega_A(f)=m}} \frac{1}{f} \prod_{p\mid f}\left(1+\frac{1}{p}\right)^{-1} \ll x^{-1/2}.$$

Proof Golomb [6] proved that the number of powerful numbers up to y is asymptotic to a constant times $y^{1/2}$. Thus for each integer $r \geqslant 0$,

$$\sum_{\substack{2^r x<f\leqslant 2^{r+1}x\\ f\in\mathcal{F}\\ \omega_A(f)=m}} \frac{1}{f} \prod_{p\mid f}\left(1+\frac{1}{p}\right)^{-1} < \frac{1}{2^r x} \sum_{\substack{2^r x<f\leqslant 2^{r+1}x\\ f\in\mathcal{F}}} 1 \ll \frac{1}{2^r x}(2^{r+1}x)^{1/2} \ll 2^{-r/2}x^{-1/2},$$

and consequently

$$\sum_{\substack{f>x\\ f\in\mathcal{F}\\ \omega_A(f)=m}} \frac{1}{f} \prod_{p\mid f}\left(1+\frac{1}{p}\right)^{-1} = \sum_{r=0}^{\infty} \sum_{\substack{2^r x<f\leqslant 2^{r+1}x\\ f\in\mathcal{F}\\ \omega_A(f)=m}} \frac{1}{f} \prod_{p\mid f}\left(1+\frac{1}{p}\right)^{-1} \ll \sum_{r=0}^{\infty} 2^{-r/2}x^{-1/2} \ll x^{-1/2}.$$

■

Proof of Theorem 1.9 Fix a sequence $A = (0, a_2, a_3, \dots)$ of complex numbers and a target $m \in \mathbb{C}$. Every positive integer n can be written uniquely as $n = qf$ where q is squarefree, f is powerful, and $(q, f) = 1$ (indeed, q is the product of the primes dividing n exactly once). In this notation, the condition $\omega_A(n) = m$ is equivalent to $\omega_A(f) = m$ (since $a_1 = 0$), and thus

$$\#\mathcal{N}_{A,m}(x) = \sum_{\substack{f \leqslant x \\ f \in \mathcal{F} \\ \omega_A(f) = m}} \sum_{\substack{q \leqslant x/f \\ (q,f)=1}} \mu^2(q). \tag{2.1}$$

To estimate the inner sum above, we use [9, Lemma 2.17] which says that for any $y \geqslant 1$ and any positive integer f,

$$\sum_{\substack{n \leqslant y \\ (n,f)=1}} \mu^2(n) = \frac{6}{\pi^2} y \prod_{p \mid f} \left(1 + \frac{1}{p}\right)^{-1} + O\left(y^{1/2} \prod_{p \mid f} (1 - p^{-1/2})^{-1}\right).$$

Inserting this asymptotic formula into equation (2.1) yields

$$\begin{aligned}
\#\mathcal{N}_{A,m}(x) &= \frac{6}{\pi^2} x \sum_{\substack{f \leqslant x \\ f \in \mathcal{F} \\ \omega_A(f) = m}} \frac{1}{f} \prod_{p \mid f} \left(1 + \frac{1}{p}\right)^{-1} + O\left(x^{1/2} \sum_{\substack{f \leqslant x \\ f \in \mathcal{F} \\ \omega_A(f) = m}} \frac{1}{f^{1/2}} \prod_{p \mid f} (1 - p^{-1/2})^{-1}\right) \\
&= \frac{6}{\pi^2} x \left(\sum_{\substack{f \in \mathcal{F} \\ \omega_A(f) = m}} \frac{1}{f} \prod_{p \mid f} \left(1 + \frac{1}{p}\right)^{-1} - \sum_{\substack{f > x \\ f \in \mathcal{F} \\ \omega_A(f) = m}} \frac{1}{f} \prod_{p \mid f} \left(1 + \frac{1}{p}\right)^{-1} \right) + O(x^{1/2} \log x) \\
&= \frac{6}{\pi^2} x \left(\sum_{\substack{f \in \mathcal{F} \\ \omega_A(f) = m}} \frac{1}{f} \prod_{p \mid f} \left(1 + \frac{1}{p}\right)^{-1} + O(x^{-1/2}) \right) + O(x^{1/2} \log x)
\end{aligned}$$

by Lemmas 2.1 and 2.2, which completes the proof of the theorem. ■

With Theorem 1.9 now established, it is a simple matter to prove Theorem 1.10, which will also establish the special case that is Theorem 1.3.

Proof of Theorem 1.10 Fix a sequence $A = (0, a_2, a_3, \dots)$ of nonnegative integers. Note that $\sum_{m=0}^{\infty} e_{A,m} = 1$ (by the argument in Remark 1.2), and therefore $\sum_{m=0}^{\infty} e_{A,m} z^m$ converges absolutely for any complex number z with $|z| \leqslant 1$. By Theorem 1.9,

$$\sum_{m=0}^{\infty} e_{A,m} z^m = \frac{6}{\pi^2} \sum_{m=0}^{\infty} z^m \sum_{\substack{f \in \mathcal{F} \\ \omega_A(f) = m}} \frac{1}{f} \prod_{p \mid f} \left(1 + \frac{1}{p}\right)^{-1} = \frac{6}{\pi^2} \sum_{f \in \mathcal{F}} \frac{z^{\omega_A(f)}}{f} \prod_{p \mid f} \left(1 + \frac{1}{p}\right)^{-1}.$$

Since $z^{\omega_A(f)}/f = \prod_{p^j \| f} z^{\omega_A(p^j)}/p^j = \prod_{p^j \| f} z^{a_j}/p^j$, the right-hand side equals its Euler product

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$$\frac{6}{\pi^2}\sum_{f\in\mathcal{F}}\frac{z^{\omega_A(f)}}{f}\prod_{p\mid f}\Big(1+\frac{1}{p}\Big)^{-1}=\frac{6}{\pi^2}\prod_p\bigg(1+\Big(1+\frac{1}{p}\Big)^{-1}\sum_{j=2}^{\infty}\frac{z^{a_j}}{p^j}\bigg)$$
$$=\prod_p\Big(1-\frac{1}{p^2}\Big)\bigg(1+\Big(1+\frac{1}{p}\Big)^{-1}\sum_{j=2}^{\infty}\frac{z^{a_j}}{p^j}\bigg)$$
$$=\prod_p\bigg(1-\frac{1}{p^2}+\Big(1-\frac{1}{p}\Big)\sum_{j=2}^{\infty}\frac{z^{a_j}}{p^j}\bigg),$$

which is equal to the right-hand side of equation (1.11), thus establishing the theorem. ∎

3 Decay rates of the densities

In this section, we deduce Corollary 1.4 from Theorem 1.3 and Corollary 1.15 from Theorem 1.10. The key step is to give a proposition establishing the rate of growth of infinite products such as those appearing in Theorems 1.3 and 1.10, which we do after the following simple lemma for the prime-counting function $\pi(y)$ and its logarithmically weighted version $\theta(y)$.

Lemma 3.1 *$\pi(y)\log y-\theta(y)\sim y/\log y$ as $y\to\infty$.*

Proof By the prime number theorem,

$$\pi(y)\log y-\theta(y)=\big(\mathrm{li}(y)+O(ye^{-c\sqrt{\log y}})\big)\log y-\big(y+O(ye^{-c\sqrt{\log y}})\big)$$
$$=\bigg(\Big(\frac{y}{\log y}+\frac{y}{\log^2 y}+O\Big(\frac{y}{\log^3 y}\Big)\Big)+O\Big(\frac{y}{\log^3 y}\Big)\bigg)\log y-\bigg(y+O\Big(\frac{y}{\log^2 y}\Big)\bigg)$$
$$=\frac{y}{\log y}+O\Big(\frac{y}{\log^2 y}\Big).$$
∎

Proposition 3.2 *Fix a real number $\kappa>1$, and let $R(p)$ be a positive function defined on primes p such that $R(p)\sim p^{-\kappa}$ as $p\to\infty$. Define the function*

$$P(x)=\prod\nolimits_p\big(1+R(p)x\big).$$

Then $\log P(x)\asymp x^{1/\kappa}/\log x$ as $x\to\infty$.

Proof All implicit constants in this proof may depend on $R(p)$ and κ. Choose p_0 so that $\frac{1}{2}p^{-\kappa}<R(p)<2p^{-\kappa}$ for all $p>p_0$. We write

$$\log P(x)=\sum_{p\leqslant p_0}\log\big(1+R(p)x\big)+\sum_{p_0<p\leqslant x^{1/\kappa}}\log\big(1+R(p)x\big)+\sum_{p>x^{1/\kappa}}\log\big(1+R(p)x\big)$$
$$=O(\log x)+\sum_{p_0<p\leqslant x^{1/\kappa}}\log\big(1+R(p)x\big)+\sum_{p>x^{1/\kappa}}\log\big(1+R(p)x\big),\tag{3.1}$$

since the number of terms in the first sum, and the largest value of $R(p)$ appearing in that sum, are both bounded in terms of the function R.

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In the first sum in equation (3.1),

$$\tfrac{1}{2}p^{-\kappa}x < R(p)x < 1 + R(p)x < (x^{1/\kappa}/p)^{\kappa} + 2p^{-\kappa}x = 3p^{-\kappa}x.$$

Therefore

$$\sum_{p_0 < p \leqslant x^{1/\kappa}} \log(\tfrac{1}{2}p^{-\kappa}x) \leqslant \sum_{p_0 < p \leqslant x^{1/\kappa}} \log\big(1 + R(p)x\big) \leqslant \sum_{p_0 < p \leqslant x^{1/\kappa}} \log(3p^{-\kappa}x).$$

The right-hand inequality is the same as

$$\begin{aligned}\sum_{p_0 < p \leqslant x^{1/\kappa}} \log\big(1 + R(p)x\big) &\leqslant (\log x + \log 3)\big(\pi(x^{1/k}) - \pi(p_0)\big) - \kappa\big(\theta(x^{1/k}) - \theta(p_0)\big) \\ &= \kappa\big(\pi(x^{1/\kappa})\log(x^{1/\kappa}) - \theta(x^{1/\kappa})\big) + \pi(x^{1/\kappa})\log 3 + O(\log x) \\ &\sim (\kappa + \log 3)\frac{x^{1/\kappa}}{\log(x^{1/\kappa})} = \kappa(\kappa + \log 3)\frac{x^{1/\kappa}}{\log x}\end{aligned}$$

by Lemma 3.1. By the same calculation with $\log\frac{1}{2}$ in place of $\log 3$,

$$\sum_{p_0 < p \leqslant x^{1/\kappa}} \log\big(1 + R(p)x\big) \geqslant \kappa(\kappa - \log 2)\frac{x^{1/\kappa}}{\log x}$$

(note that $\kappa - \log 2 > 1 - \log 2$ is bounded away from 0). We conclude that

$$\sum_{p_0 < p \leqslant x^{1/\kappa}} \log\big(1 + R(p)x\big) \asymp \frac{x^{1/\kappa}}{\log x}. \tag{3.2}$$

In the second sum in equation (3.1),

$$0 \leqslant \log(1 + R(p)x) \leqslant R(p)x < 2p^{-\kappa}x,$$

and thus by partial summation,

$$\begin{aligned}0 \leqslant \sum_{p > x^{1/\kappa}} \log\big(1 + R(p)x\big) \leqslant \sum_{p > x^{1/\kappa}} 2p^{-\kappa}x &= 2x\int_{x^{1/\kappa}}^{\infty} t^{-\kappa}\, d\pi(t) \\ &= 2x\left(\pi(t)t^{-\kappa}\Big|_{x^{1/\kappa}}^{\infty} + \int_{x^{1/\kappa}}^{\infty} \kappa t^{-\kappa-1}\pi(t)\, dt\right).\end{aligned}$$

The boundary term is well defined (since $\kappa > 1$) and negative, and thus by the prime number theorem,

$$\begin{aligned}0 \leqslant \sum_{p > x^{1/\kappa}} \log\big(1 + R(p)x\big) &\ll x\left(0 + \int_{x^{1/\kappa}}^{\infty} t^{-\kappa-1}\frac{t}{\log t}\, dt\right) \\ &\ll \frac{x}{\log x}\int_{x^{1/\kappa}}^{\infty} t^{-\kappa}\, dt = \frac{x}{\log x}\frac{(x^{1/\kappa})^{1-\kappa}}{\kappa - 1} \ll \frac{x^{1/\kappa}}{\log x}.\end{aligned}$$

The proposition now follows by combining these inequalities with equations (3.1) and (3.2). ■

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All that is left is to connect the rates of growth of the generating functions in Theorems 1.3 and 1.10 to the decay rate of their Maclaurin coefficients. We use the following classical information about entire functions [1, Definition 2.1.1 and Theorem 2.2.2]:

Definition 3.3 An entire function $f(z)$ is said to be *of order* ρ if

$$\limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r} = \rho$$

where $M_f(r) = \max_{|z|=r} |f(z)|$. It is *of finite order* if it is of order ρ for some $\rho \in \mathbb{R}$.

Lemma 3.4 *Let $f(z) = \sum_{m=0}^{\infty} b_m z^m$ be an entire function. The function $f(z)$ is of finite order if and only if*

$$\mu = \limsup_{\substack{m\to\infty \\ b_m\neq 0}} \frac{m\log m}{\log(1/|b_m|)}$$

is finite, and in this case $f(z)$ is of order μ.

Proof of Corollary 1.4 Set

$$P(x) = \prod_p \left(1 + \frac{p-1}{p^{k+1}}x\right) \quad \text{and} \quad Q(z) = \sum_{m=0}^{\infty} e_{k,m} z^m = \prod_p \left(1 + \frac{p-1}{p^{k+1}}(z-1)\right).$$

When $|z| = r$, note that

$$|Q(z)| \leqslant \prod_p \left(1 + \frac{p-1}{p^{k+1}}(|z|+1)\right) = P(r+1);$$

thus by Proposition 3.2 with $\kappa = k$ and $R(p) = (p-1)/p^{k+1}$,

$$\log|Q(z)| \ll \frac{(r+1)^{1/k}}{\log(r+1)} \ll \frac{r^{1/k}}{\log r}.$$

On the other hand, when $z = r > 3$ is real, then

$$\log|Q(r)| = \log P(r-1) \gg \frac{(r-1)^{1/k}}{\log(r-1)} \gg \frac{r^{1/k}}{\log r}$$

again by Proposition 3.2. Together these last estimates show that $\log M_Q(r) \asymp r^{1/k}/\log r$, which implies that

$$\limsup_{r\to\infty} \frac{\log\log M_Q(r)}{\log r} = \limsup_{r\to\infty} \frac{\log(r^{1/k}) - \log\log r + O(1)}{\log r} = \frac{1}{k}.$$

In particular, $Q(z)$ has order $\frac{1}{k}$ by Definition 3.3; consequently, by Lemma 3.4,

$$\limsup_{\substack{m\to\infty \\ e_{k,m}\neq 0}} \frac{m\log m}{\log(1/|e_{k,m}|)} = \frac{1}{k}.$$

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We know that $e_{k,m} > 0$ by Theorem 1.1, and so

$$\frac{m \log m}{\log(1/e_{k,m})} \leqslant \frac{1}{k} + o(1)$$

with asymptotic equality for infinitely many m; we conclude that

$$e_{k,m} \leqslant m^{-(k-o(1))m}$$

which completes the proof of the corollary. ∎

Proof of Corollary 1.14 The proof is the same as the proof of Corollary 1.4, except that $Q(z)$ is changed to each of the three products

$$\sum_{m=0}^{\infty} e_{S,m} z^m = \prod_p \left(1 + \frac{z-1}{p^2}\right)$$
$$\sum_{m=0}^{\infty} e_{E,m} z^m = \prod_p \left(1 + \frac{z-1}{p(p+1)}\right)$$
$$\sum_{m=0}^{\infty} e_{O,m} z^m = \prod_p \left(1 + \frac{z-1}{p^2(p+1)}\right)$$

in turn, with corresponding modifications to $P(x)$ and $R(p)$; instead of with $\kappa = k$, the appeal to Proposition 3.2 is made with $\kappa = 2$ in the first two cases and $\kappa = 3$ in the last case, and the rest of the proof goes through in exactly the same way. ∎

4 Numerical calculations of densities, expectations, and variances

We now describe how we used the generating functions in Theorem 1.3 to facilitate the calculation of the densities $e_{k,m}$ in Table 1 to the indicated high level of precision. Our approach is based on observations of Marcus Lai (private communication).

Proposition 4.1 *Let $P(z)$ be any function with Maclaurin series*

$$P(z) = \sum_{n=0}^{\infty} C(n) z^n,$$

so that $C(n) = \frac{1}{n!} P^{(n)}(0)$ for every $n \geq 0$. Define

$$S(0,z) = \log P(z) \quad \text{and} \quad S(n,z) = \frac{d^n}{dz^n} S(0,z)$$

for all $n \geq 1$; and define $S(n) = S(n,0)$ and $\tilde{S}(n) = \frac{1}{(n-1)!} S(n)$. Then for any $n \geq 1$,

$$P^{(n)}(z) = \sum_{k=0}^{n-1} \binom{n-1}{k} P^{(k)}(z) S(n-k, z).$$

In particular, for $n \geq 1$,

$$C(n) = \frac{1}{n} \sum_{k=0}^{n-1} C(k) \tilde{S}(n-k), \tag{4.1}$$

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so that for example

$$C(1) = C(0)\tilde{S}(1) = P(0)\tilde{S}(1),$$
$$C(2) = \frac{1}{2}\big(C(0)\tilde{S}(2) + C(1)\tilde{S}(1)\big) = \frac{P(0)}{2}\big(\tilde{S}(1)^2 + \tilde{S}(2)\big),$$
$$C(3) = \frac{1}{3}\big(C(0)\tilde{S}(3) + C(1)\tilde{S}(2) + C(2)\tilde{S}(1)\big) = \frac{P(0)}{6}\big(\tilde{S}(1)^3 + 3\tilde{S}(1)\tilde{S}(2) + 2\tilde{S}(3)\big).$$

Proof We first verify that

$$P'(z) = P(z)\frac{P'(z)}{P(z)} = P(z)\frac{d}{dz}\log P(z) = P(z)\frac{d}{dz}S(0,z) = P(z)S(1,z),$$

which is the case $n = 1$ of the first identity. The general case of the first identity now follows from using the product rule $n-1$ times in a row on this initial identity $P'(z) = P(z)S(1,z)$. The second identity follows by plugging in $z = 0$ into the first identity and recalling that $C(n) = \frac{1}{n!}P^{(n)}(0)$. ∎

We apply this recursive formula (with subscripts inserted throughout the notation for clarity) with $C(m) = e_{k,m}$, so that

$$P_k(z) = \sum_{m=0}^{\infty} e_{k,m}z^m = \prod_p\left(1 + \frac{(p-1)(z-1)}{p^{k+1}}\right) \tag{4.2}$$

by Theorem 1.3. We compute

$$S_k(0,z) = \sum_p \log\left(1 + \frac{(p-1)(z-1)}{p^{k+1}}\right)$$
$$S_k(1,z) = \frac{d}{dz}\sum_p \log\left(1 + \frac{(p-1)(z-1)}{p^{k+1}}\right) = \sum_p\left(z + \frac{p^{k+1}}{p-1} - 1\right)^{-1} \tag{4.3}$$
$$S_k(n,z) = \frac{d^{n-1}}{dz^{n-1}}\sum_p\left(z + \frac{p^{k+1}}{p-1} - 1\right)^{-1} = \sum_p(-1)^{n-1}(n-1)!\left(z + \frac{p^{k+1}}{p-1} - 1\right)^{-n},$$

so that

$$\tilde{S}_k(n) = (-1)^{n-1}\sum_p\left(\frac{p^{k+1}}{p-1} - 1\right)^{-n}. \tag{4.4}$$

Therefore equation (4.1) becomes

$$e_{k,m} = \frac{1}{m}\sum_{j=0}^{m-1} e_{k,j}\tilde{S}_k(m-j),$$

and in particular we have

$$e_{k,0} = \prod_p\left(1 - \frac{p-1}{p^{k+1}}\right),$$
$$e_{k,1} = e_{k,0}\tilde{S}_k(1),$$

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$$e_{k,2} = \frac{e_{k,0}}{2}\big(\tilde{S}_k(1)^2 + \tilde{S}_k(2)\big),$$

$$e_{k,3} = \frac{e_{k,0}}{6}\big(\tilde{S}_k(1)^3 + 3\tilde{S}_k(1)\tilde{S}_k 2) + 2\tilde{S}_k(3)\big).$$

Remark 4.2 We coded these formulas, and the one in equation (4.4), into SageMath and calculated approximations to them where we truncated the infinite product and sums to run over primes $p \leqslant 10^7$, resulting in the densities appearing in Table 1 (the cases $k = 2, 3, 4, 5$ and $m = 0, 1, 2, 3$). While we do not include a formal analysis of the error arising from these truncations, we have listed the densities to nine decimal places to display our confidence in that level of precision.

Finally, we extract the expectations and variances of various limiting distributions from their generating functions by relating those quantities to derivatives of their generating functions.

Proof of Corollary 1.6 Let X_k be the discrete random variable whose distribution is the same as the limiting distribution of $\omega_k(n)$; then the generating function of this distribution is the function $P_k(z)$ in equation (4.2). Proposition 4.1 and equations (4.2)–(4.3) tell us that

$$P_k'(1) = P_k(1)S_k(1,1) = 1 \cdot \sum_p \frac{p-1}{p^{k+1}} = \sum_p \frac{p-1}{p^{k+1}}$$

$$P_k''(1) = P_k'(1)S_k(1,1) + P_k(1)S_k(2,1)$$

$$= \sum_p \frac{p-1}{p^{k+1}} \cdot \sum_p \frac{p-1}{p^{k+1}} + 1\left(-\sum_p \left(\frac{p-1}{p^{k+1}}\right)^2\right) = \left(\sum_p \frac{p-1}{p^{k+1}}\right)^2 - \sum_p \left(\frac{p-1}{p^{k+1}}\right)^2$$

But now by standard results from probability [5, Chapter XI, Theorems 2–3],

$$\mathbb{E}[X_k] = P_k'(1) = \sum_p \frac{p-1}{p^{k+1}}$$

$$\sigma^2[X_k] = P_k''(1) + P_k'(1) - P_k'(1)^2 = \sum_p \frac{p-1}{p^{k+1}} - \sum_p \left(\frac{p-1}{p^{k+1}}\right)^2$$

which is equivalent to the statement of the corollary. ∎

Remark 4.3 As before, we used SageMath to calculate truncations of these infinite sums, running over primes $p \leqslant 10^7$, to generate the approximate expectations and variances listed in Table 2 for $k = 2, 3, 4, 5$.

Proof of Corollary 1.12 Let X_A be the random variable whose distribution is the same as the limiting distribution of $\omega_A(n)$. Using the same approach starting from the generating function (1.10), we see that

$$\begin{aligned}
\mathbb{E}[X_A] = P_A'(1) &= \frac{d}{dz}\prod_p\left(1-\frac{1}{p^2}+\sum_{j=2}^{\infty}z^{a_j}\left(\frac{1}{p^j}-\frac{1}{p^{j+1}}\right)\right)\bigg|_{z=1}\\
&= \left\{\prod_p\left(1-\frac{1}{p^2}+\sum_{j=2}^{\infty}z^{a_j}\left(\frac{1}{p^j}-\frac{1}{p^{j+1}}\right)\right)\sum_p\frac{d}{dz}\log\left(1-\frac{1}{p^2}+\sum_{j=2}^{\infty}z^{a_j}\left(\frac{1}{p^j}-\frac{1}{p^{j+1}}\right)\right)\right\}\bigg|_{z=1}\\
&= 1\sum_p\left(0+\sum_{j=2}^{\infty}a_jz^{a_j-1}\left(\frac{1}{p^j}-\frac{1}{p^{j+1}}\right)\right)\bigg/\left(1-\frac{1}{p^2}+\sum_{j=2}^{\infty}z^{a_j}\left(\frac{1}{p^j}-\frac{1}{p^{j+1}}\right)\right)\bigg|_{z=1}\\
&= \sum_p\sum_{j=2}^{\infty}a_j\left(\frac{1}{p^j}-\frac{1}{p^{j+1}}\right)\bigg/1
\end{aligned}$$

as claimed. ∎

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